# PRIME DIVISORS OF $a^n - b^n$

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*Received: 2/3/23, Accepted: 4/19/23, Published: 6/2/23*

**Abstract**

Let $P(m)$ denote the greatest prime factor of an integer $m > 1$. It has been known since the 1900s that $P_n := P(a^n - b^n) > n + 1$ for integers $a > b > 0$ and $n > 2$. A conjecture of Stewart (1977) states that $P_n \gg \phi(n)^2$ where the implied constant is absolute. He (2013) later proved that $P_n \gg_{a,b} n^{1+\frac{1}{104 \log\log n}}$. Earlier, Murty and Wong (2002) had shown that the usual *abc*-conjecture implies that $P_n \gg_{a,b,\varepsilon} n^{2-\varepsilon}$. Recently, Murty and Séguin (2019) formulated a conjecture concerning the $p$-adic valuation of $a^f - 1$ where $p \nmid a$, and $f$ is the order of $a$ in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$. Conditional on their conjecture, they confirmed the conjecture of Stewart in the case that $b = 1$ with the implied constant depending on $a$. We prove that a milder *abc*-conjecture implies that $P_n \gg (n/\tau(n))^2$ where $\tau(n)$ is the number of distinct positive divisors of $n$, and crucially, the implied constant is independent of $a$ and $b$. This is an improvement over the result of Murty and Wong. Furthermore, as a simple consequence, Stewart's conjecture follows in the case that $n$ is prime, thereby refining the result of Murty and Séguin. Additionally, we obtain a distribution result for the prime factors of $\gcd(n, \Phi_n(a,b))$, generalizing a similar result of Murty and Séguin.

## 1. Introduction

Let $a$, $b$ be integers with $a > b > 0$. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ of positive integers defined by

$$u_n = u_n(a,b) := a^n - b^n, \tag{1}$$

and its associated sequence $(P(u_n))_{n\in\mathbb{N}}$, where $P(m)$ denotes the greatest prime factor of an integer $m > 1$. It has long been known that $P(u_n) \to \infty$ with $n$. However, it is generally believed that $P(u_n)$ grows rapidly with $n$. Erdős [2] conjectured that $P(u_n)/n \to \infty$ with $n$ in the case that $a = 2$ and $b = 1$. In the same spirit, one is naturally led to conjecture that $P(u_n)/n \to \infty$ with $n$ for arbitrary integers $a$, $b$ with $a > b > 0$. Stewart [10] confirmed the conjecture for the set of

DOI: 10.5281/zenodo.7997984

integers $n$ having at most $\kappa \log\log n$ distinct prime factors for a given $\kappa$ satisfying $0 < \kappa < 1/\log 2$. Subsequently, he [11] extended his results to general Lucas and Lehmer sequences and proposed the following conjecture.

**Conjecture 1.** There is an effectively computable absolute positive constant $C$ such that

$$P(u_n) > C\phi(n)^2 \tag{2}$$

for every $n > 2$, where $\phi$ denotes the Euler totient function.

Murty and Wong [8] proved that the *abc*-conjecture of Masser and Oesterlé implies that for a given $\varepsilon > 0$, one has

$$P(u_n) \gg n^{2-\varepsilon},$$

where the implied constant depends on $a$, $b$ and $\varepsilon$. Murata and Pomerance [6] proved that subject to the generalized Riemann hypothesis, for almost all integers $n$, one has

$$P(2^n - 1) > \frac{n^{4/3}}{\log\log n}.$$

Stewart [12] provided the first unconditional result in this direction by proving that there is a constant $N_0 > 0$ depending only on $\omega(ab)$, where $\omega(m)$ denotes the number of distinct prime factors of an integer $m > 1$, such that for every $n > N_0$, one has

$$P(u_n) > n^{1+\frac{1}{104\log\log n}},$$

thereby completely resolving the conjecture of Erdős.

For a positive integer $m$ and a prime $p$, let $\nu_p(m)$ denote the largest exponent of $p$ such that $p^{\nu_p(m)} \mid m$. Further, for an integer $a$ with $p \nmid a$, let $f_p(a)$ denote the order of $a$ in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$. In a recent article, Murty and Séguin [9] formulated the conjecture that given an integer $a > 1$, there is a constant $\kappa > 1$ (depending on $a$) such that

$$\nu_p(a^{f_p(a)} - 1) \le \kappa$$

for every prime $p \nmid a$. Conditional on their conjecture, Murty and Séguin resolved Conjecture 1 in the particular case that $b = 1$, with the constant $C$ in Equation (2) depending on $a$.

The present article aims to prove that a weaker *abc*-hypothesis implies that $P(u_n) \gg n^2/\tau(n)^2$, where $\tau(n)$ denotes the number of distinct positive divisors of a positive integer $n$ and where the implied constant is absolute. For a given positive integer $m > 1$, its *radical* $\operatorname{rad}(m)$ is defined as

$$\operatorname{rad}(m) := \prod_{\substack{p \mid m \\ p-\text{prime}}} p.$$

**Conjecture 2.** (The quasi *abc*-conjecture) There is an absolute constant $\kappa > 1$ such that if $a$, $b$ and $c$ are pairwise relatively prime positive integers satisfying $a + b = c$, then

$$c < (\mathrm{rad}(abc))^{\kappa}.$$

A conjecture of Granville and Tucker [3] suggests that $\kappa = 2$. We note that for our purposes, a weaker hypothesis than the one in Conjecture 2 would suffice. We will discuss this next. Let $k > 1$ be a given integer. By the fundamental theorem of arithmetic, there are unique positive integers $U$ and $V$ such that

$$u_n = UV^{k+1} \tag{3}$$

where, every prime divisor $p$ of $U$ satisfies $\nu_p(U) \le k$. We refer to $U$ as the $(k+1)$-*free* part of $u_n$. Observe that if $k \ge \kappa$ where $\kappa$ is the constant appearing in Conjecture 2, then Conjecture 2 implies that

$$UV^{k+1} = u_n < a^n < (\mathrm{rad}(abUV))^{\kappa} < a^{2\kappa} U^{\kappa} V^{\kappa}.$$

It follows that

$$V \le V^{k-\kappa+1} < a^{2\kappa} U^{\kappa-1} \le (aU)^{2\kappa}. \tag{4}$$

The estimate in Equation (4) is all that is required to prove our main result (Theorem 1 below). We record the inequality in Equation (4) for ease of future reference.

**Hypothesis 1.** There is an absolute constant $\lambda > 2$ such that for every integer $k \ge \lambda$ if $u_n$ is given by Equation (3), then $V < (aU)^{\lambda}$.

Our main result is the following.

**Theorem 1.** *Let $\lambda$ be the constant appearing in Hypothesis 1. For arbitrary integers $a$ and $b$ with $a > b > 0$, let $u_n$ be as defined in Equation* (1)*. Then subject to Hypothesis 1, there is an effectively computable absolute constant $n_0 > 1$ such that for every integer $n > n_0$, one has*

$$P(u_n) > C\frac{n^2}{\tau(n)^2}, \tag{5}$$

*where $C$ can be taken to be $C = 0.002\lambda^{-5}$.*

Set $c_0 = \max\{n^2/\tau(n)^2 : n \le n_0\}$ where $n_0$ is the constant appearing in Theorem 1. Then trivially, one has

$$P(u_n) > c_0^{-1}\frac{n^2}{\tau(n)^2}$$

for all $n \le n_0$. Thus, setting $C_0 = \min\{C, 1/c_0\}$ where $C$ is as stated in Theorem 1, we have

**Corollary 1.** *For arbitrary integers $a$ and $b$ with $a > b > 0$, let $u_n$ be as defined in Equation* (1)*. Then subject to Hypothesis 1, there is an effectively computable absolute constant $C_0 > 0$ such that for every integer $n > 2$, one has*

$$P(u_n) > C_0 \frac{n^2}{\tau(n)^2}.$$

Consequently, Conjecture 1 follows whenever $n$ is prime.

**Corollary 2.** *Let $p > 2$ be a prime, and for arbitrary integers $a$ and $b$ with $a > b > 0$, let $u_p$ be as defined in Equation* (1)*. Then subject to Hypothesis 1, there is an effectively computable absolute constant $C' > 0$ such that*

$$P(u_p) > C' p^2.$$

It is well-known (Theorem 317, [4]) that for every $\delta > 0$, one has

$$\tau(n) < 2^{(1+\delta)\log n/\log\log n}$$

for $n \gg_\delta 1$. Accordingly, we have the following.

**Corollary 3.** *For arbitrary integers $a$ and $b$ with $a > b > 0$, let $u_n$ be as defined in Equation* (1)*. Then subject to Hypothesis 1, for every $\delta > 0$, there is an effectively computable constant $C_\delta > 0$ such that for every integer $n > 2$, one has*

$$P(u_n) > C_\delta \frac{n^2}{4^{(1+\delta)\log n/\log\log n}}. \tag{6}$$

Perhaps it is worth highlighting the key aspects where our results improve upon the best-known conditional lower bound to date on $P(u_n)$ due to Murty and Wong [8] mentioned earlier. To begin with, the underlying hypothesis (Conjecture 2) of Theorem 1 is weaker than the usual *abc*-conjecture. Secondly, since for every $\varepsilon > 0$ and $\delta > 0$,

$$4^{(1+\delta)\log n/\log\log n} = o(n^\varepsilon),$$

the lower bound on $P(u_n)$ in Equation (6) is considerably sharper than the one due to Murty and Wong. Thirdly, the implied constant $C_\delta$ appearing in Corollary 1, is independent of the integers $a$ and $b$. The last condition is an essential requirement in Conjecture 1. If $n$ is prime, the constant $C'$ appearing in Corollary 2 is absolute.

For a positive integer $n$, set $\zeta_n := e^{2\pi i/n}$. The $n$th cyclotomic polynomial $\Phi_n(x)$ is defined as

$$\Phi_n(x) = \prod_{\substack{0<j<n \\ \gcd(j,n)=1}} (x - \zeta_n^j).$$

It is well known that $\Phi_n(x) \in \mathbb{Z}[x]$ is a monic polynomial with $\deg \Phi_n = \phi(n)$. The $n$th *homogenized* cyclotomic polynomial $\Phi_n(x, y)$ is defined by

$$\Phi_n(x, y) = y^{\phi(n)} \Phi_n\left(\frac{x}{y}\right).$$

By a standard result on the factorization of $x^n - y^n$, one has

$$a^n - b^n = \prod_{d|n} \Phi_d(a,b).$$

For $d \mid n$, set

$$v_d = |\Phi_d(a,b)|. \tag{7}$$

Observe that $v_n \mid u_n$ for all $n$, so that $P(u_n) \geq P(v_n)$. In most of the past work cited thus far, the authors have obtained a lower bound on $P(v_n)$, which is trivially a lower bound on $P(u_n)$. We shall adopt a slightly different strategy in that we consider the prime factors of $v_{d_n}$ for a certain large divisor $d_n$ of $n$. These details are discussed in the next section.

In proving Theorem 1, we will need information on the prime factors of $v_d$ for $d \mid n$. These are summarized in the following.

**Lemma 1** ([11])**.** *Let $a$ and $b$ be integers with $a > b > 0$ and $\gcd(a,b) = 1$, and let $v_d$ be defined as in Equation (7). Then*

$$v_d = p_d^{\delta_d} N \tag{8}$$

*where $p_1 = 1$, $\delta_1 = 1$, and for every $d > 1$,*

$$p_d = P\left(\frac{d}{\gcd(3,d)}\right), \quad \delta_d \in \{0,1\},$$

*and either $N = 1$, or every prime factor $p$ of $N$ satisfies $p \equiv 1 \pmod{d}$.*

In the special case that $b = 1$, Murty and Séguin (see Theorem 1.2, [9]) established that for some $\theta \in (0,1)$,

$$\sum_{n \leq x} \delta_n \log p_n = O(x^\theta).$$

The last estimate implies that $\delta_n = 0$ more often than not. By Abel's summation formula, one readily deduces from the last estimate above that

$$\sum_{n=1}^{\infty} \frac{\delta_n \log p_n}{n} \ll 1.$$

We will prove that the last result holds in general.

**Theorem 2.** *We have*

$$\sum_{n=1}^{\infty} \frac{\delta_n \log p_n}{n} \ll 1$$

*where $\delta_n$ and $p_n$ are defined as in Lemma 1.*

## 2. Proofs

Throughout, we will assume that $n > 2$. Further, we will assume without loss of any generality that $\gcd(a, b) = 1$. Let $\lambda > 2$ be as defined in Hypothesis 1. We may and do further suppose that $\lambda$ is an integer. Let integers $U$ and $V$ be as defined in Equation (3) with $k = \lambda$. For each $d \mid n$, let $U_d$ denote the $(\lambda+1)$-free part of $v_d$, and let $V_d$ be the positive integer such that

$$v_d = U_d V_d^{\lambda+1}$$

where $v_d$ is as defined in Equation (7). From $u_n = \prod_{d|n} v_d$, we have that

$$UV^{\lambda+1} = \prod_{d|n} U_d \prod_{d|n} V_d^{\lambda+1}.$$

Since

$$U \leq \prod_{d|n} U_d,$$

and hence,

$$V \geq \prod_{d|n} V_d.$$

Let $d_n \mid n$ be such that $U_{d_n}$ is maximal. That is, $U_d \leq U_{d_n}$ for every $d \mid n$. Thus,

$$U \leq U_{d_n}^{\tau(n)}. \tag{9}$$

We have the following estimate on the size of $d_n$ conditional on Hypothesis 1.

**Lemma 2.** *Subject to Hypothesis 1, we have for $n > 1$ that*

$$\phi(d_n) > C_1 \frac{n}{\tau(n)}, \tag{10}$$

*where $C_1$ can be taken to be $C_1 = 1/6(\lambda^2 + \lambda + 1)$.*

*Proof.* By an easy induction argument, we have

$$\log u_n > \frac{n}{2} \log a. \tag{11}$$

On the other hand,

$$\log u_n = \log U + (\lambda + 1) \log V. \tag{12}$$

Now, Hypothesis 1 implies that $V < (aU)^\lambda$. So, from Equation (9) and Equation (12), we deduce that

$$\begin{aligned} \log u_n &< \lambda(\lambda+1) \log a + (\lambda^2 + \lambda + 1) \log U \qquad (13) \\ &\leq \lambda(\lambda+1) \log a + \tau(n)(\lambda^2 + \lambda + 1) \log U_{d_n}. \end{aligned}$$

Next, by the triangle inequality, for every $x > 0$, one has

$$|\Phi_{d_n}(x)| \le (1+x)^{\phi(d_n)}.$$

Setting $x = a/b$ above, we get

$$U_{d_n} \le |\Phi_{d_n}(a,b)| \le (a+b)^{\phi(d_n)} < a^{2\phi(d_n)}.$$

We now deduce from Equation (13) that

$$\log u_n < 3(\lambda^2+\lambda+1)\tau(n)\phi(d_n)\log a. \tag{14}$$

Finally, comparing Equation (11) and Equation (14), we obtain

$$\frac{n}{2} < 3(\lambda^2+\lambda+1)\tau(n)\phi(d_n).$$

The lemma follows. □

In proving Theorem 1, we will need an upper bound on $\log U_{d_n}$ in terms of $P(u_n)$. For this purpose, we will appeal to the following version of the Brun-Titchmarsh inequality due to Montgomery and Vaughan (see Theorem 2, [5]). For $x > 0$ and positive integers $\ell$ and $r$ with $\gcd(\ell, r) = 1$, let $\pi(x, \ell, r)$ denote the number of primes $p \le x$ satisfying $p \equiv r \pmod{\ell}$.

**Lemma 3** ([5])**.** *For $0 < \ell < x$, one has*

$$\pi(x,\ell,r) < \frac{2x}{\phi(\ell)\log(x/\ell)}.$$

*Proof of Theorem 1.* From Hypothesis 1 and Equation (9), we have

$$\begin{aligned} \frac{n}{2}\log a < \log u_n &= \log U + (\lambda+1)\log V \qquad (15)\\ &< \log U + \lambda(\lambda+1)\log aU \\ &= \lambda(\lambda+1)\log a + \tau(n)(\lambda^2+\lambda+1)\log U_{d_n}. \end{aligned}$$

Using Lemma 1,

$$\log U_{d_n} < \log n + \lambda \sum_{\substack{p\le P_n \\ p\equiv 1 \pmod{d_n}}} \log p \tag{16}$$

where $P_n = \max\{en, P(u_n)\}$. Moreover, from Lemma 3 and the trivial bound $d_n \le n$, one has

$$\sum_{\substack{p\le P_n \\ p\equiv 1 \pmod{d_n}}} \log p \le \frac{2P_n\log P_n}{\phi(d_n)\log(P_n/d_n)} \le \frac{2P_n\log P_n}{\phi(d_n)\log(P_n/n)}. \tag{17}$$

From Equation (15), Equation (16) and Equation (17), we obtain

$$\frac{n}{2}\log a < C_2 \log a + C_2\tau(n)\log n + \frac{2C_2\tau(n)P_n \log P_n}{\phi(d_n)\log(P_n/n)} \tag{18}$$

where $C_2 = \lambda(\lambda^2+\lambda+1)$. Since $a \geq 2$, using the well-known estimate that $\tau(n) \leq 2\sqrt{n}$, we have from Equation (10) and Equation (18) that

$$\frac{n}{3} < \frac{2C_2\tau(n)P_n \log P_n}{\phi(d_n)\log(P_n/n)} < \frac{2C_3\tau(n)^2 P_n \log P_n}{n\log(P_n/n)} \tag{19}$$

for $n \gg 1$, and where $C_3 = C_2/C_1$. Since $P_n \geq en$, we get from Equation (19) that

$$\frac{n^2}{6C_3\tau(n)^2} < P_n \log P_n. \tag{20}$$

Thus, for $n \gg 1$, one has

$$P_n > \frac{n^2}{12C_3\tau(n)^2\log\left(\frac{n^2}{6C_3\tau(n)^2}\right)} > n^{3/2}.$$

It follows that $\log P_n < 3\log(P_n/n)$. Using this estimate in Equation (19), we obtain

$$P_n > \frac{1}{18C_3}\frac{n^2}{\tau(n)^2}.$$

The expression on the right-hand side above is $> en$ for $n \gg 1$. So, $P_n = P(u_n)$ for $n \gg 1$. The theorem now follows by observing that

$$C_3 = 6\lambda(\lambda^2+\lambda+1)^2 < 24\lambda^5$$

since $\lambda > 1$. $\square$

We next turn to the proof of Theorem 2. We begin by recalling a well-known result concerning the resultant of cyclotomic polynomials.

**Lemma 4** ([1])**.** *Let $m$ and $n$ be integers with $m > n > 1$. If $m/n$ is not a power of a prime, then there are polynomials $u(x)$ and $v(x)$ in $\mathbb{Z}[x]$ such that*

$$u(x)\Phi_m(x) + v(x)\Phi_n(x) = 1.$$

*On the other hand, if $m = p^k n$ where $p$ is a prime, then there are polynomials $u(x)$ and $v(x)$ in $\mathbb{Z}[x]$ such that*

$$u(x)\Phi_m(x) + v(x)\Phi_n(x) = p^{\phi(n)}.$$

In particular, Lemma 4 implies that if there is a prime $p$ such that

$$p \mid \gcd(\Phi_m(c), \Phi_n(c))$$

for some integer $c$, then $m = p^k n$ for some positive integer $k$.

To prove Theorem 2, we need a precise description of positive integers $n$ for which $\delta_n = 1$. This is the content of the next result.

**Proposition 1.** *For a positive integer $n > 1$, let $p_n$ and $\delta_n$ be as in Lemma 1. Further, let $m = n/p_n^{\nu_{p_n}(n)}$. If $\delta_n = 1$, then $p_n \equiv 1 \pmod{m}$.*

*Proof.* We let $p$ denote $p_n$ for brevity. Suppose that $\delta_n = 1$ for some $n > 1$. Since $\gcd(a, b) = 1$, there is a unique $c \in (\mathbb{Z}/p\mathbb{Z})^\times$ such that $ab^{-1} \equiv c \pmod{p}$. Let $f = f_p(c)$ so that $p$ is a *primitive* divisor of $\Phi_f(c)$ (that is, $p \nmid \Phi_d(c)$ for every $d < f$). Thus,

$$p \mid \Phi_f(c), \quad p \equiv 1 \pmod{f}. \tag{21}$$

Also, $p \mid \Phi_n(a, b)$ implies that $p \mid \Phi_n(c)$. That is,

$$p \mid \gcd(\Phi_f(c), \Phi_n(c)).$$

From the remark following Lemma 4, we deduce that $f = n/p^k$ for some positive integer $k$. Next, observe that $c^n \equiv 1 \pmod{p}$ since $p \mid \Phi_n(c)$. Now, using Fermat's little theorem, we deduce that

$$c^m \equiv c^n \equiv 1 \pmod{p}.$$

So, $f \mid m$. It follows that $k = \nu_p(n)$, and as such, $f = m$. The proposition follows by observing from Equation (21) that $p \equiv 1 \pmod{f}$. $\square$

For a pair of relatively prime integers $a$ and $b$ with $a > b > 0$, and a prime $p \nmid ab$, let $f_p$ denote the smallest positive integer such that

$$a^{f_p} \equiv b^{f_p} \pmod{p}. \tag{22}$$

The proof of Theorem 2 rests upon the following result, which is an adaptation of a result of similar flavour from [7] (see Inequality (3), [7]).

**Proposition 2.** *For $f_p$ defined above, we have*

$$\sum_{p \nmid ab} \frac{\log p}{(p-1) f_p} \ll 1.$$

We need the following lemma to prove Proposition 2.

**Lemma 5.** *For $n \gg 1$, one has*

$$\sum_{p \mid n} \frac{\log p}{p-1} \leq 4 \log \log n.$$

*Proof.* By Corollary 2.3, Inequality (14) in [7], for $n \gg 1$, one has

$$\sum_{p|n} \frac{\log p}{p} \le 2 \log\log n.$$

Therefore, for $n \gg 1$,

$$\sum_{p|n} \frac{\log p}{p-1} = \sum_{p|n} \frac{p}{p-1}\frac{\log p}{p} \le 2\sum_{p|n} \frac{\log p}{p} \le 4\log\log n.$$

□

*Proof of Proposition 2.* For $x > 0$, define

$$A(x) := \prod_{f \le x} (a^f - b^f).$$

It is easily seen that

$$A(x) < \prod_{f \le x} a^f < a^{x^2}.$$

Thus,

$$\log\log A(x) < 2\log x + \log\log a < 3\log x \tag{23}$$

for $x \gg 1$. For an integer $f > 0$, let

$$\delta(f) := \sum_{f_p = f} \frac{\log p}{p-1},$$

and for $x > 0$, let

$$\Delta(x) := \sum_{f \le x} \delta(f).$$

Observe that for $f \le x$, the fact that $f_p = f$ implies that $p \mid A(x)$. Thus, from Lemma 5 and Equation (23), we obtain

$$\Delta(x) \le \sum_{p|A(x)} \frac{\log p}{p-1} \le 12\log x. \tag{24}$$

Noting that $p \equiv 1 \pmod{f_p}$, we have by the Abel summation formula that

$$\begin{aligned}
\sum_{\substack{p \le x \\ p \nmid ab}} \frac{\log p}{(p-1)f_p} &\le \sum_{f \le x} \frac{\delta(f)}{f} \\
&= \frac{\Delta(x)}{x} + \int_1^x \frac{\Delta(t)}{t^2} dt + O(1) \\
&\le \frac{12\log x}{x} + 12\int_1^x \frac{\log t}{t^2} dt + O(1) = O(1)
\end{aligned}$$

for $x \gg 1$. The proposition follows. □

*Proof of Theorem 2.* Let

$$S := \sum_{n=1}^{\infty} \frac{\delta_n \log p_n}{n},$$

where $\delta_n$ and $p_n$ are as stated in the theorem. For a positive integer $n$, let $k_n = \nu_{p_n}(n)$, and let $m_n = n/p_n^{k_n}$. From Proposition 1, $\delta_n = 1$ implies that $p_n \equiv 1 \pmod{m_n}$. Furthermore, $m_n$ is the smallest positive integer such that

$$a^{m_n} \equiv b^{m_n} \pmod{p_n}.$$

We deduce that $\delta_n = 1$ implies that $f_{p_n} = m_n$. Also, since $\gcd(a, b) = 1$, we have $p_n \nmid ab$ if $\delta_n = 1$. Thus,

$$S \le \sum_{p \nmid ab} \sum_{k=1}^{\infty} \frac{\log p}{p^k f_p} = \sum_{p \nmid ab} \frac{\log p}{(p-1) f_p} \ll 1 \tag{25}$$

by Proposition 2, thereby proving the theorem. $\square$

**Acknowledgement.** The author thanks the anonymous referee for their valuable comments and corrections.

## References

[1] T. Apostol, Resultants of cyclotomic polynomials, *Proc. Amer. Math. Soc.* **24** (1970), 457-462.

[2] P. Erdős, Some recent advances and current problems in number theory, *in: Lectures on Modern Mathematics*, Vol. III, Wiley, New York, 1965, pp. 196-244.

[3] A. Granville and T. J. Tucker, It's as easy as abc, *Notices Amer. Math. Soc.* **49** (2002), 1224-1231.

[4] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Sixth edition. Revised by D. R. Heath-Brown and J. H. Silverman. With a foreword by Andrew Wiles. Oxford University Press, Oxford, 2008.

[5] H. L. Montgomery and R. C. Vaughan, The large sieve, *Mathematika* **20** (1973), 119-134.

[6] L. Murata and C. Pomerance, On the largest prime factor of a Mersenne number, *in: Number Theory, in: CRM Proc. Lecture Notes*, Vol. 36, Amer. Math. Soc., Providence, RI, 2004, pp. 209-218.

[7] M. R. Murty, M. Rosen and J. H. Silverman, Variations on a theme of Romanoff, *Internat. J. Math.* **7** (1996), 373-391.

[8] M. R. Murty and S. Wong, The *ABC* conjecture and prime divisors of the Lucas and Lehmer sequences, *in: Number Theory for the Millennium*, III, Urbana, IL, 2000, A K Peters, Natick, MA, 2002, pp. 43-54.

[9] M. R. Murty and F. Séguin, Prime divisors of sparse values of cyclotomic polynomials and Wieferich primes, *J. Number Theory* **201** (2019), 1-22.

[10] C. L. Stewart, The greatest prime factor of $a^n - b^n$, *Acta Arith.* **26** (1975), 427-433.

[11] C. L. Stewart, On divisors of Fermat, Fibonacci, Lucas and Lehmer numbers, *Proc. Lond. Math. Soc. (3)* **35** (1977), 425-447.

[12] C. L. Stewart, On divisors of Lucas and Lehmer numbers, *Acta Math.* **211** (2013), 291-394.