WANG-SUN FORMULA IN $\overrightarrow{GL}(\mathbb{Z}/2k\mathbb{Z})$

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Received: 8/2/22, Revised: 1/22/23, Accepted: 5/3/23, Published: 6/2/23

Abstract

Wang and Sun proved a certain summatory formula involving derangements and primitive roots of unity. This study establishes such a formula but for the particular case of the set of affine derangements in $\overrightarrow{GL}(\mathbb{Z}/2k\mathbb{Z})$ and its subset of involutive affine derangements in particular; in this last case its value is relatively simple and it is related to even unitary divisors of k.

1. Introduction

In [6], for $n > 1$ an odd integer and ζ a n-th primitive root of unity, Wang and Sun proved that

$$\sum_{\pi \in D(n-1)} \operatorname{sign}(\pi) \prod_{j=1}^{n-1} \frac{1+\zeta^{j-\pi(j)}}{1-\zeta^{j-\pi(j)}} = (-1)^{\frac{n-1}{2}} \frac{((n-2)!!)^2}{n},$$

where $D(n-1)$ is the set of all derangements within S_{n-1}.

Here the viewpoint of mathematical music theory is adopted, with a focus on the general affine linear group

$$\overrightarrow{GL}(\mathbb{Z}/2k\mathbb{Z}) := \{e^u.v\}_{u\in\mathbb{Z}/2k\mathbb{Z}, v\in\mathbb{Z}/2k\mathbb{Z}^\times},$$

where an element $e^u.v \in \overrightarrow{GL}(\mathbb{Z}/2k\mathbb{Z})$ maps $x \in \mathbb{Z}/2k\mathbb{Z}$ to

$$e^u.v(x) := vx + u.$$

The present author is interested not only in plain derangements, but in those derangements which are involutive. Affine involutive derangements are called *quasipolarities*, and its set within $\overrightarrow{GL}(\mathbb{Z}/2k\mathbb{Z})$ is denoted by Q_k. These derangements are important because they relate consonances (K) and dissonances (D) in $2k$-tone

DOI: 10.5281/zenodo.7997988

equal temperaments. More precisely, if $\mathbb{Z}/2k\mathbb{Z} = K \sqcup D$, q is a quasipolarity and $q(K) = D$, then q is the unique affine automorphism with that property and it is called the *polarity* of the dichotomy (K/D) (see [3] for details).

The structure of the article is as follows. Section 2 establishes formulas for the sum of quasipolarities. Then, using code in Maxima, some values of the sum for derangements in $\overrightarrow{GL}(\mathbb{Z}/2k\mathbb{Z})$ are calculated in Section 3. Finally, Section 4 mentions some important remarks.

Notation from [4] for the divisibility relation ($\backslash$), coprimality ($\perp$), as well as Iverson brackets (if P is a predicate, then $[P] = 1$ whenever P is true, and $[P] = 0$ otherwise), are used throughout the paper.

2. The Case of Quasipolarities

First, note that all quasipolarities have the same sign as permutations, so we can disregard it in the original Wang–Sun formula.

Theorem 1. *If k is odd and ζ is a $2k$-th primitive root of unity, then*

$$\sum_{\pi \in Q_k} \prod_{j=0}^{n-1} \frac{1 + \zeta^{j-\pi(j)}}{1 - \zeta^{j-\pi(j)}} = 0.$$

Proof. Whenever $j - \pi(j) = k$, we have

$$1 + \zeta^{j-\pi(j)} = 1 + \zeta^k = 1 - 1 = 0,$$

and thus it suffices to prove that this happens for any quasipolarity $\pi = e^u.v \in Q_k$ for some $0 \le j \le 2k - 1$. From [1, Theorem 3.1], we know that

$$u = \sigma(v)q + \frac{2k}{\tau(v)}, \tag{1}$$

where $\sigma(v) = \gcd(v - 1, 2k)$, $\tau(v) = \gcd(v + 1, 2k)$ and q is any integer. Now, we want to show that there is a j such that

$$j - \pi(j) = j - e^u.v(j) = (1 - v)j - u \equiv k \pmod{2k},$$

which is possible if and only if $\gcd(1 - v, 2k)\backslash(k + u)$. In other words, if and only if $\sigma(v)\backslash(k + u)$. Using Equation (1), this can be rewritten as

$$\sigma(v)\backslash\left(k + \sigma(v)q + \frac{2k}{\tau(v)}\right)$$

for some integer q. Thus, if the congruence

$$\sigma(v)x \equiv k - \frac{2k}{\tau(v)} \pmod{2k}$$

is shown to have a solution for x, then it is done. Indeed, $\sigma(v) = 2I_1$ and $\tau(v) = 2I_2$ for some I_1, I_2 odd and coprime divisors of k. The linear Diophantine equation

$$2x + 2\left(\frac{k}{I_1}\right)y = \left(\frac{k}{I_1} - \frac{k}{I_1 I_2}\right)$$

has a solution since the greatest common divisor of the coefficients on the left is 2 and the number on the right is even. Hence, if (x, y) is a solution of the preceding equation, then

$$2I_1 x + \frac{k}{I_2} - k = \sigma(v)x + \frac{2k}{\tau(v)} - k = -2ky,$$

which shows that x is the required solution. □

k	3	4	5	6	7	8	9	10	11	12
S	0	4	0	8	0	8	0	12	0	16

Table 1: Results of the evaluation of the sum S of Theorem 1 for some values of k (not only odd values).

While this result was relatively easy to conjecture considering a few values of the sum for some k (as they can be seen in Table 1), it is less easy to make a guess for the sum when k is even, but the following seems reasonable.

Conjecture 1. If ζ is a $2k$-th primitive root of unity, then

$$\sum_{\pi \in Q_k} \prod_{j=0}^{n-1} \frac{1 + \zeta^{j-\pi(j)}}{1 - \zeta^{j-\pi(j)}} = \sum_{\pi \in Q_k} \prod_{j=0}^{n-1} [j - \pi(j) \neq k].$$

The search of the sequence 4, 8, 8, 12 in the On-Line Encyclopedia of Integer Sequences (OEIS) [5] yields among its first results A054785, which is the difference of the sum of divisors of $2n$ and n. The values match up to $k = 16$, but they differ at $k = 18$, where the former[1] is 20 and the later is 26. Nonetheless, the results from [2] lead us in the right direction. We need some definitions first.

Definition 1. A divisor d of n is said to be *unitary* if $d \perp n/d$. If d is a unitary divisor of n then, in symbol, we write $d \backslash\backslash n$. Moreover, the sum of unitary divisors function is denoted by

$$s_1^*(n) = \sum_{d \backslash\backslash n} d.$$

In [2] it is proved that

$$|Q_k| = s_1^*(k).$$

The following theorem states Conjecture 1 more precisely.

[1]This is the sequence of the sums of even unitary divisors of $2n$ and it was added by Amiram Eldar to the OEIS on 28 January 2023 as entry A360156 [5].

Theorem 2. *If ζ is a $2k$-th primitive root of unity, then*

$$\sum_{\pi\in Q_k}\prod_{j=0}^{n-1}\frac{1+\zeta^{j-\pi(j)}}{1-\zeta^{j-\pi(j)}} = [2\backslash k](|Q_{2k}|-|Q_k|) = [2\backslash k](s_1^*(2k)-s_1^*(k)).$$

Proof. The proof for the case when k is odd has been considered already. Now, suppose k is even. If $\pi = e^u.v \in Q_k$ and $j-\pi(j) \not\equiv k \pmod{2k}$, so that the summand associated to π is not 0, then it is necessary for the congruence

$$(v-1)j \equiv k-u \pmod{2k} \tag{2}$$

to have no solutions for j; this happens if and only if $\sigma(v) \nmid (k-u)$. Again, because of [1, Theorem 3.1], we have $\sigma(v) = \frac{4k}{\tau(v)}$ and $u = \sigma(v)q + \frac{2k}{\tau(v)}$. Hence Equation (2) has no solutions if and only if

$$\sigma(v) \nmid \left(\frac{\sigma(v)\tau(v)}{4} + \sigma(v)q + \frac{\sigma(v)}{2}\right). \tag{3}$$

It is known that $\sigma(v) = 2I_1$ and $\tau(v) = 2I_2$ where $I_1 \perp I_2$. Moreover

$$2k = \frac{\sigma(v)\tau(v)}{2} = \frac{4I_1I_2}{2} = 2I_1I_2$$

thus I_1 and I_2 are unitary divisors of k. By [2, Proposition 2.3], both I_1 and I_2 represent involutions of $\mathbb{Z}/2k\mathbb{Z}$. If I_1 is odd, then $I_2 > 0$ is even and the associated involution defines $\frac{2k}{\sigma(v)} = I_2$ quasipolarities. In particular, $2I_1 = \sigma(v)$ divides k. Furthermore, $\tau(v)$ is divisible by 4 and $\sigma(v)$ does not divide $\sigma(v)/2$, thus Equation (3) holds. Conversely, if Equation (3) is true, then I_2 has to be even, otherwise

$$\begin{aligned}\tfrac{1}{4}\sigma(v)\tau(v) + \tfrac{1}{2}\sigma(v) &= \tfrac{1}{2}\sigma(v)I_2 + \tfrac{1}{2}\sigma(v)\\ &= \frac{1}{2}\sigma(v)(2s+1) + \tfrac{1}{2}\sigma(v)\\ &= \sigma(v)(s+1)\end{aligned}$$

for some integer s, which contradicts Equation (3).

For every $\pi = e^u.v$ with its linear part v represented by an even unitary divisor of k and every $0 \le j \le 2k-1$, there is an ℓ such that $(j-\pi(j)) - (\ell-\pi(\ell)) \equiv k \pmod{2k}$, for this is equivalent to

$$(v-1)\ell \equiv k-(v-1)j \pmod{2k}.$$

Since $\sigma(v)\backslash k$ and $\sigma(v)\backslash(1-v)$, there exists a solution ℓ. Therefore, for a summand indexed by π such that $j-\pi(j) \ne k$ for every $0 \le j \le 2k-1$ and has a factor $1+\zeta^{j-\pi(j)}$ in the denominator, it also has a factor

$$1-\zeta^{\ell-\pi(\ell)} = 1-\zeta^{j-\pi(j)+k} = 1-\zeta^{j-\pi(j)}\zeta^k = 1+\zeta^{j-\pi(j)}$$

in the numerator. It follows that all the factors cancel out and the summand equals 1. Thus, summing up the number of even unitary divisors of k yields $s_1^*(2k) - s_1^*(k)$ because

$$\begin{aligned} s_1^*(2k) - s_1^*(k) &= \sum_{d \,\backslash\backslash\, 2k, d \text{ even}} d + \sum_{d \,\backslash\backslash\, 2k, d \text{ odd}} d - \sum_{d \,\backslash\backslash\, k, d \text{ even}} d - \sum_{d \,\backslash\backslash\, k, d \text{ odd}} d \\ &= 2 \sum_{d \,\backslash\backslash\, k, d \text{ even}} d + \sum_{d \,\backslash\backslash\, k, d \text{ odd}} d - \sum_{d \,\backslash\backslash\, k, d \text{ even}} d - \sum_{d \,\backslash\backslash\, k, d \text{ odd}} d \\ &= \sum_{d \,\backslash\backslash\, k, d \text{ even}} d. \end{aligned}$$

□

3. The Case of the Derangements of the General Affine Group

Let us denote the derangements within $\overrightarrow{GL}(\mathbb{Z}/2k\mathbb{Z})$ with Δ_k. The following code in Maxima calculates

$$S = \sum_{\pi \in \Delta_k} \operatorname{sign}(\pi) \prod_{j=0}^{n-1} \frac{1 + \zeta^{j-\pi(j)}}{1 - \zeta^{j-\pi(j)}}$$

for the particular case of $k = 4$.

```
load("combinatorics");
S:0; kk:4;
zeta: exp(%pi*%i/kk); ZZ:[]; L:[];
for k1:0 thru (2*kk-1) do ZZ:append(ZZ,[k1]);
for u:0 thru (2*kk-1) do
 (for v:1 thru (2*kk-1) step 2 do
  (if (gcd(v^2,2*kk)=1) then
   (
    M:mod(v*ZZ+u-ZZ,2*kk),
    if(not(product(M[l],l,1,2*kk)=0)) then
     L:append(L,[[u,v]])
   )
  )
 )$
for k1:1 thru length(L) do
 (P: (-1)^perm_parity(mod(L[k1][2]*ZZ+L[k1][1],2*kk)+1),
  for k2:0 thru (2*kk-1) do
   P: trigrat(P*(1+zeta^(k2-mod(L[k1][2]*k2+L[k1][1],2*kk)))
   /(1-zeta^(k2-mod(L[k1][2]*k2+L[k1][1],2*kk)))),
  S:ratsimp(S+P)
 )$
```

As far as the author's computer and simplification capacity of Maxima allow, the values of S are calculated for $k = 3, \ldots, 9$ *mutatis mutandis*. The results are contained in Table 2. Except for the sign and the prime factors of k in the denominator, no other pattern is evident and searches in the OEIS do not point yet in a meaningful direction.

k	S
6	1456/27
8	−2300
10	762256/5
12	−10643506432/729
14	13444304416/7
16	−332995177452
18	1450048309488389824/19683

Table 2: Results of the execution of the code.

4. Some Final Remarks

An interesting outcome of Theorem 1, from the musicological viewpoint, is that for $2k$-tone equal temperaments with k odd and any of its quasipolarities there is pair of a consonance and a dissonance which are separated by a *tritone* (which always corresponds to k).

When k is even, the proof of Theorem 2 tells us that the Wang–Sun sum counts quasipolarities represented by even unitary divisors along the direction of [2], so it suggests that the relationship between the two concepts and its musicological implications should be explored more deeply.

For the Wang–Sun sum over all affine derangements we could not prove or conjecture a general formula, and thus we stress the non-triviality of studying it for derangements within subgroups.

Acknowledgment. The author deeply thanks José Hernández Santiago at Universidad Autónoma de Guerrero for his help in completing the proof of Theorem 1.

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