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EXTENSIONS OF SURY'S RELATION INVOLVING FIBONACCI k-STEP AND LUCAS k-STEP POLYNOMIALS

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Abstract

Based on the method of generating functions of the sequence of Fibonacci k-step and Lucas k-step polynomials, or on a crucial identity relating Fibonacci k-step and Lucas k-step polynomials, extensions of Sury's relation and the alternating Sury's relation involving Fibonacci k-step and Lucas k-step polynomials are derived, respectively. Extensions of Sury's relation involving Fibonacci-type and Lucas-type polynomials are also obtained. Of course, these relations are generalizations of the well-known Fibonacci-Lucas relation.

1. Introduction

Sury [16] obtained an interesting relation involving Fibonacci and Lucas numbers,

$$2^{n+1}F_{n+1} = 2^0L_0 + 2^1L_1 + \cdots + 2^nL_n,$$

for all positive integers n. We call it the *Fibonacci-Lucas relation* or *Sury's relation involving Fibonacci numbers and Lucas numbers.* However, much earlier, Benjamin and Quinn [2] proved the same relation by using the argument of colored tilings. Proof of the Fibonacci-Lucas relation based on the method of generating function

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was given in [12]. A family of the Fibonacci-Lucas relations by replacing 2 with any positive integer m could be found in [8, 13]. To be precise, it holds that

$$3^{n+1}F_{n+1} = \sum_{i=0}^{n} 3^i L_i + \sum_{i=0}^{n+1} 3^{i-1}F_i,$$

and

$$m^{n+1}F_{n+1} = \sum_{i=0}^{n} m^i L_i + (m-2)\sum_{i=0}^{n+1} m^{i-1}F_i.$$

Indeed, their proofs were based on a crucial identity

$$L_n = F_{n-1} + F_{n+1},\ n \geq 1.$$

More generally, Dafnis, Philippou, and Livieris [7] considered the Fibonacci and the Lucas numbers of order k, and they proved a relation of the same fashion:

$$m^{n+1}F_{n+1}^{(k)} + k - 2 = \sum_{i=0}^{n} m^i \left[L_i^{(k)} + (m-2)F_{i+1}^{(k)} - \sum_{j=3}^{k}(j-2)F_{i-j+1}^{(k)} \right], \qquad (1)$$

where $F_n^{(k)}$ and $L_n^{(k)}$ are the n-th Fibonacci and the n-th Lucas numbers of order k, respectively. (The definition will be given as below.) These results can also be proved by the argument of colored tilings (see [2, 7, 14]).

On the other hand, Martinjak and Prodinger [14] proved the alternating Sury's relation involving Fibonacci numbers and Lucas numbers,

$$(-1)^n F_{n+1} = \sum_{i=0}^{n} (-1)^i r^{n-i} \left[L_{i+1} + (r-2)F_i\right],$$

for any integer $r \geq 2$. Indeed, this relation holds when $r = 1$ as is easily checked. In addition, Bhatnagar [3] proved the Sury's and the alternating Sury's relation for the case in which r is an indeterminate (real or complex) by using Euler's telescoping lemma.

From now on, let $k \geq 2$ be a fixed positive integer and $r \neq 0$ be an indeterminate. We define *the sequence of Fibonacci k-step polynomials* $\{F_n^{(k)}(x)\}_{n\geq -k+1}$ (or *the sequence of Fibonacci polynomials of order k*, or *k-bonacci polynomials sequence*) as follows:

$$F_n^{(k)}(x) = 0,\ \text{for}\ -k+1 \leq n \leq 0,$$

and

$$F_1^{(k)}(x) = 1, F_n^{(k)}(x) = \sum_{i=1}^{k} x^{k-i} F_{n-i}^{(k)}(x) \text{ for } n \geq 2.$$

For example, when $k = 3$ it reduces to the tribonacci polynomials $T_n(x)$, which are defined by

$$T_{-1}(x) = T_0(x) = 0,\ T_1(x) = 1,$$

and

$$T_n(x) = x^2 T_{n-1}(x) + x T_{n-2}(x) + T_{n-3}(x),\ \text{for } n \geq 2.$$

The tribonaaci polynomials were originally studied in an article by Hoggatt and Bicknell [9] in 1973.

Similarly, *the sequence of Lucas k-step polynomials* $\{L_n^{(k)}(x)\}_{n\geq 0}$ (or *the sequence of Lucas polynomials of order k*) is defined as

$$L_0^{(k)}(x) = k \text{ (a constant polynomial)},\ L_1^{(k)}(x) = x^{k-1},$$

and

$$L_n^{(k)}(x) = \begin{cases} nx^{k-n} + \sum_{j=1}^{n-1} x^{k-j} L_{n-j}^{(k)}(x), & 2 \leq n \leq k; \\ \sum_{j=1}^{k} x^{k-j} L_{n-j}^{(k)}(x), & n \geq k+1. \end{cases}$$

The Fibonacci k-step polynomials $F_n^{(k)}(x)$ are generalizations of the "regular" Fibonacci polynomials, which were studied by Catalan and Jacobsthal in 1883. And the Lucas k-step polynomials $L_n^{(k)}(x)$ are generalizations of the "regular" Lucas polynomials, originally studied by Bicknell [4] in 1970. Indeed, when $k = 2$ these become the regular Fibonacci and the regular Lucas polynomials and we should write $F_n^{(2)}(x) := F_n(x)$, and $L_n^{(2)}(x) := L_n(x)$, respectively. The Fibonacci and Lucas polynomials have been extensively studied in the books of Koshy [10, 11].

We notice that, from the definition,

$$F_2^{(k)}(x) = x^{k-1},$$

and

$$L_2^{(k)}(x) = 2x^{k-2} + x^{k-1} L_1^{(k)}(x) = x^{2k-2} + 2x^{k-2}.$$

Also, by taking $x = 1$, $F_n^{(k)}(1) := F_n^{(k)}$ and $L_n^{(k)}(1) := L_n^{(k)}$ are *the n-th Fibonacci* and *the n-th Lucas numbers of order k*, respectively.

We now present our main results in this paper.

Theorem 1. *For any positive integer n, we have the extension of Sury's relation involving Fibonacci k-step and Lucas k-step polynomials:*

$$\begin{aligned} r^{n+1} x^{k-1} F_{n+1}^{(k)}(x) + k - 2 = \sum_{i=0}^{n} r^i \Big[& L_i^{(k)}(x) + (rx^{k-1} - 2) F_{i+1}^{(k)}(x) \\ & - \sum_{j=3}^{k} (j-2) x^{k-j} F_{i-j+1}^{(k)}(x) \Big], \end{aligned} \tag{2}$$

and the extension of alternating Sury's relation involving Fibonacci k-step and Lucas k-step polynomials:

$$(-1)^n x^{k-1} F_{n+1}^{(k)}(x) = \sum_{i=0}^{n} (-1)^i r^{n-i} \Big[L_{i+1}^{(k)}(x) + x^{k-2}(rx-2) F_i^{(k)}(x) - \sum_{j=3}^{k} j x^{k-j} F_{i-j+2}^{(k)}(x) \Big], \tag{3}$$

where the summation $\sum_{j=a}^{b} *$ *is zero if* $b < a$.

In [15], Philippou, Georghiou, and Philippou introduced *the sequence of Fibonacci-type polynomials of order* k, denoted by $\{f_n^{(k)}(x)\}_{n\geq 0}$. The definition is similar to the sequence of Fibonacci k-step polynomials. Define $f_0^{(k)}(x) = 0$, $f_1^{(k)}(x) = 1$, and

$$f_n^{(k)}(x) = \begin{cases} x \sum_{j=1}^{n-1} f_{n-j}^{(k)}(x), & 2 \leq n \leq k; \\ x \sum_{j=1}^{k} f_{n-j}^{(k)}(x), & n \geq k+1. \end{cases}$$

Later, Charalambides [5] introduced *the sequence of Lucas-type polynomials* $\{\ell_n^{(k)}(x)\}_{n\geq 0}$ which was defined as follows. Let $\ell_0^{(k)}(x) = k$ be a constant polynomial, $\ell_1^{(k)}(x) = x$ and

$$\ell_n^{(k)}(x) = \begin{cases} x \left[n + \sum_{j=1}^{n-1} \ell_{n-j}^{(k)}(x) \right], & 2 \leq n \leq k; \\ x \sum_{j=1}^{k} \ell_{n-j}^{(k)}(x), & n \geq k+1. \end{cases}$$

There exists a crucial identity relating to Fibonacci-type and Lucas-type polynomials of order k (Equation (6) in Section 4) and the extension of Sury's relation involving Fibonacci-type and Lucas-type polynomials of order k (Theorem 3).

The rest of this paper is organized as follows. A crucial identity relating to Fibonacci k-step and Lucas k-step polynomials is presented in Section 2. Also, we derive the generating functions of the sequence of Fibonacci k-step and Lucas k-step polynomials, respectively. Proofs of our main results are given in Section 3. Some remarks and conclusions are included in the final section.

2. Preliminaries

Let $F^{(k)}(x;y) = \sum_{n\geq 0} F_n^{(k)}(x) y^n$ be the generating function of the sequence of Fibonacci k-step polynomials. Similarly, we set the generating function of the sequence of Lucas k-step polynomials to be

$$L^{(k)}(x;y)=\sum_{n\geq 0}L_n^{(k)}(x)y^n=L_0^{(k)}(x)+L_1^{(k)}(x)y+L_2^{(k)}(x)y^2+\cdots.$$

Then it is easy to obtain the following lemma.

Lemma 1. *The generating functions of the sequence of Fibonacci k-step polynomial $F_n^{(k)}(x)$ and Lucas k-step polynomial $L_n^{(k)}(x)$ are given by*

$$F^{(k)}(x;y)=\frac{y}{1-\sum\limits_{j=1}^{k}x^{k-j}y^j}\ \text{and}\ L^{(k)}(x;y)=\frac{k-\sum\limits_{j=1}^{k}(k-j)x^{k-j}y^j}{1-\sum\limits_{j=1}^{k}x^{k-j}y^j},$$

respectively.

Proof. Notice that

$$L^{(k)}(x;y)-L_0^{(k)}(x)-L_1^{(k)}(x)y-\cdots-L_k^{(k)}(x)y^k=\sum_{n\geq k+1}L_n^{(k)}(x)y^n.$$

According to the definition of $L_n^{(k)}(x)$, the right-hand side can be written as

$$\begin{aligned}
&\sum_{n\geq k+1}\left(\sum_{j=1}^{k}x^{k-j}L_{n-j}^{(k)}(x)\right)y^n=\sum_{n\geq k+1}\left(x^{k-1}L_{n-1}^{(k)}(x)+\cdots+x^0L_{n-k}^{(k)}(x)\right)y^n\\
&=\ x^{k-1}y\left(L^{(k)}(x;y)-L_0^{(k)}(x)-\cdots-L_{k-1}^{(k)}(x)y^{k-1}\right)\\
&\quad+x^{k-2}y^2\left(L^{(k)}(x;y)-L_0^{(k)}(x)-\cdots-L_{k-2}^{(k)}(x)y^{k-2}\right)\\
&\quad\ \vdots\\
&\quad+x^0y^k\left(L^{(k)}(x;y)-L_0^{(k)}(x)\right).
\end{aligned}$$

Therefore, we find

$$\begin{aligned}
\left(1-\sum_{j=1}^{k}x^{k-j}y^j\right)L^{(k)}(x;y)&=L_0^{(k)}(x)+\left(L_1^{(k)}(x)-x^{k-1}L_0^{(k)}(x)\right)y\\
&+\left(L_2^{(k)}(x)-x^{k-1}L_1^{(k)}(x)-x^{k-2}L_0^{(k)}(x)\right)y^2+\cdots\\
&+\left(L_k^{(k)}(x)-x^{k-1}L_{k-1}^{(k)}(x)-x^{k-2}L_{k-2}^{(k)}(x)-\cdots-L_0^{(k)}(x)\right)y^k.
\end{aligned}$$

So the second generating function now follows. We omit the proof of the first conclusion since it can be obtained in a similar way. □

A crucial identity relating to Fibonacci k-step and Lucas k-step polynomials is given in the following lemma. See also the identity (2.21) in [5].

Lemma 2. *Let* $\{F_n^{(k)}(x)\}_{n\geq -k+1}$ *and* $\{L_n^{(k)}(x)\}_{n\geq 0}$ *be the Fibonacci* k*-step and the Lucas* k*-step polynomials sequence, respectively. Then we have for all* $n \geq 1$*,*

$$L_n^{(k)}(x) = \sum_{j=1}^{k} j x^{k-j} F_{n-j+1}^{(k)}(x). \tag{4}$$

Proof. Let $G^{(k)}(x;y) = \sum_{n\geq 0} F_{n+1}^{(k)}(x) y^n$ and $H^{(k)}(x;y) = \sum_{i=1}^{k}(i-k)x^{k-i}y^i$. Then by Lemma 1 we have

$$L^{(k)}(x;y) = \left(k + H^{(k)}(x;y)\right) G^{(k)}(x;y), \tag{5}$$

since $G^{(k)}(x;y) = F^{(k)}(x;y)/y = \left(1 - \sum_{j=1}^{k} x^{k-j} y^j\right)^{-1}$. This implies that, by comparing the coefficient y^n on both sides of Equation (5),

$$L_n^{(k)}(x) = kF_{n+1}^{(k)}(x) + \sum_{j=1}^{\min\{n,k\}} (j-k)x^{k-j} F_{n-j+1}^{(k)}(x).$$

If $n < k$, then we have

$$\begin{aligned}\sum_{j=1}^{k}(j-k)x^{k-j}F_{n-j+1}^{(k)}(x) &= \sum_{j=1}^{n}(j-k)x^{k-j}F_{n-j+1}^{(k)}(x)\\ &\quad + \sum_{j=n+1}^{k}(j-k)x^{k-j}F_{n-j+1}^{(k)}(x).\end{aligned}$$

The above second term vanishes since $F_n^{(k)}(x) = 0$ if $-k+1 \leq n \leq 0$, and hence

$$\sum_{j=1}^{\min\{n,k\}} (j-k)x^{k-j}F_{n-j+1}^{(k)}(x) = \sum_{j=1}^{k}(j-k)x^{k-j}F_{n-j+1}^{(k)}(x).$$

By the definition of Fibonacci k-step polynomials,

$$F_{n+1}^{(k)}(x) = \sum_{j=1}^{k} x^{k-j} F_{n+1-j}^{(k)}(x),$$

we conclude that Equation (4) holds for all $n \geq 1$. $\square$

3. Proofs of Main Results

We are now ready to prove Theorem 1 by the generating function approach.

Proof of Theorem 1. Consider the generating function

$$T^{(k)}(x;y) := \sum_{n\geq 0}\left[r^{n+1}x^{k-1}F_{n+1}^{(k)}(x) + k - 2 - \sum_{i=0}^{n} r^i L_i^{(k)}(x)\right] y^n.$$

Note that the coefficient of y^n in $T^{(k)}(x;y)$ is clearly

$$r^{n+1}x^{k-1}F_{n+1}^{(k)}(x) + k - 2 - \sum_{i=0}^{n} r^i L_i^{(k)}(x).$$

Now we compute this coefficient in another way. From the definition of the generating function $F^{(k)}(x;y)$, we find

$$\sum_{n\geq 0} r^{n+1}x^{k-1}F_{n+1}^{(k)}(x)y^n = \frac{x^{k-1}}{y}\sum_{n\geq 0} F_{n+1}^{(k)}(x)(ry)^{n+1} = \frac{x^{k-1}F^{(k)}(x;ry)}{y}.$$

Similarly we have

$$T^{(k)}(x;y) = \frac{x^{k-1}F^{(k)}(x;ry)}{y} + \frac{k-2}{1-y} - \frac{L^{(k)}(x;ry)}{1-y}.$$

In light of Lemma 1, this becomes

$$\begin{aligned} T^{(k)}(x;y) &= \frac{rx^{k-1}(1-y) + (k-2)\left(1 - \sum\limits_{j=1}^{k} x^{k-j}(ry)^j\right) - k + \sum\limits_{j=1}^{k}(k-j)x^{k-j}(ry)^j}{(1-y)\left(1 - \sum\limits_{j=1}^{k} x^{k-j}(ry)^j\right)} \\ &= \frac{(rx^{k-1}-2) - \sum\limits_{j=3}^{k}(j-2)x^{k-j}(ry)^j}{(1-y)\left(1 - \sum\limits_{j=1}^{k} x^{k-j}(ry)^j\right)}. \end{aligned}$$

Notice that

$$\begin{aligned} \frac{(rx^{k-1}-2)}{(1-y)\left(1 - \sum\limits_{j=1}^{k} x^{k-j}(ry)^j\right)} &= (rx^{k-1}-2)\cdot\frac{1}{ry(1-y)}\cdot\frac{ry}{1 - \sum\limits_{j=1}^{k} x^{k-j}(ry)^j} \\ &= (rx^{k-1}-2)\sum_{n\geq 0}\left(\sum_{i=0}^{n} r^i F_{i+1}^{(k)}(x)\right) y^n. \end{aligned}$$

Also, we have

$$\frac{\sum_{j=3}^{k}(j-2)x^{k-j}(ry)^j}{(1-y)\left(1-\sum_{j=1}^{k}x^{k-j}(ry)^j\right)} = \sum_{j=3}^{k}(j-2)x^{k-j}(ry)^{j-1}\frac{F^{(k)}(x;ry)}{1-y}$$

$$= \sum_{n\geq 0}\left(\sum_{j=3}^{k}(j-2)x^{k-j}\sum_{i=0}^{n}r^iF_{i-j+1}^{(k)}(x)\right)y^n.$$

Putting this all together, we compare with the coefficient of y^n in $T^{(k)}(x;y)$ to obtain the desired Equation (2).

To prove (3), consider the generating function

$$V^{(k)}(x;y) := \sum_{n\geq 0}\left[(-1)^n x^{k-1}F^{(k)}n+1(x) - \sum_{i=0}^{n}(-1)^i r^{n-i}L_{i+1}^{(k)}(x)\right]y^n,$$

for which the right-hand side is simply equal to

$$-\frac{x^{k-1}F^{(k)}(x;-y)}{y} - \frac{L^{(k)}(x;-y)-L_0^{(k)}(x)}{(-y)(1-ry)}.$$

Thus, by Lemma 1,

$$V^{(k)}(x;y) = \frac{L^{(k)}(x;-y)-k-(1-ry)x^{k-1}F^{(k)}(x;-y)}{y(1-ry)}$$

$$= \frac{(2-rx)x^{k-2}y^2}{y(1-ry)\left(1-\sum_{j=1}^{k}x^{k-j}(-y)^j\right)} + \frac{\sum_{j=3}^{k}jx^{k-j}(-y)^j}{y(1-ry)\left(1-\sum_{j=1}^{k}x^{k-j}(-y)^j\right)}.$$

Notice that

$$\frac{(2-rx)x^{k-2}y^2}{y(1-ry)\left(1-\sum_{j=1}^{k}x^{k-j}(-y)^j\right)} = (rx-2)x^{k-2}\frac{F^{(k)}(x;-y)}{1-ry},$$

and from this, we obtain that the coefficient of y^n is equal to

$$(rx-2)x^{k-2}\sum_{i=0}^{n}(-1)^i r^{n-i}F_i^{(k)}(x).$$

The second term becomes

$$\frac{\sum_{j=3}^{k} jx^{k-j}(-y)^j}{y(1-ry)\left(1-\sum_{j=1}^{k} x^{k-j}(-y)^j\right)} = -\sum_{j=3}^{k} jx^{k-j}(-y)^{j-2} \cdot \frac{F^{(k)}(x;-y)}{1-ry}.$$

Thus the coefficient of y^n in the series expansion is

$$-\sum_{j=3}^{k} jx^{k-j} \sum_{i=0}^{n} (-1)^i r^{n-i} F^{(k)}_{i-j+2}(x).$$

Altogether, by comparing the coefficient of y^n in $V^{(k)}(x;y)$ in two different ways, the desired Equation (3) follows. □

The case $k = 2$ of Theorem 1 reduces to the following corollary.

Corollary 1. *For any positive integer n, we have Sury's relation involving Fibonacci and Lucas polynomials:*

$$r^{n+1} xF_{n+1}(x) = \sum_{i=0}^{n} r^i \left[L_i(x) + (rx-2)F_{i+1}(x)\right],$$

and the alternating Sury's relation involving Fibonacci and Lucas polynomials:

$$(-1)^n xF_{n+1}(x) = \sum_{i=0}^{n} (-1)^i r^{n-i} \left[L_{i+1}(x) + (rx-2)F_i(x)\right].$$

If we replace x with 2 in the first equation of Corollary 1, we get

$$2r^{n+2} F_{n+1}(2) = \sum_{i=0}^{n} r^i \left[L_i(2) + (2r-2)F_{i+1}(2)\right],$$

and since $F_n(2)$ is the familiar Pell number P_n and likewise $L_n(2)$ is the familiar Pell-Lucas number Q_n, this becomes

$$r^{n+1} P_{n+1} = \frac{1}{2} \sum_{i=0}^{n} r^i \left[2(r-1)P_{i+1} + Q_i\right].$$

See also Equation (15) in [1]. Since $Q_n = P_{n+1} + P_{n-1}$, the above equation is equivalent to

$$r^{n+1} P_{n+1} = \sum_{i=0}^{n} r^i \left[P_i + (r-2)P_{i+1} + Q_i\right].$$

Now we replace x with 2 in the second equation of Corollary 1 to get an alternating relation involving Pell and Pell-Lucas numbers,

$$(-1)^n P_{n+1} = \frac{1}{2}\sum_{i=0}^{n}(-1)^i r^{n-i}\left[Q_{i+1} + 2(r-1)P_i\right].$$

So in particular, we have

$$(-1)^n P_{n+1} = \frac{1}{2}\sum_{i=0}^{n}(-1)^i Q_{i+1}.$$

If we take $x = 1$ in Corollary 1, we recover two well-known relations involving Fibonacci and Lucas numbers [3]. The first author proved a more general relation (under the consideration $k = 2$) for the sequence of the W-polynomials and the w-polynomials; see [6].

If we set $x = 1$ in (2) and (3), then we get Equation (1) obtained in [7] and a new identity, respectively.

Corollary 2. *For any positive integer n, we have Sury's relation involving Fibonacci and Lucas numbers of order k:*

$$r^{n+1}F_{n+1}^{(k)} + k - 2 = \sum_{i=0}^{n} r^i \left[L_i^{(k)} + (r-2)F_{i+1}^{(k)} - \sum_{j=3}^{k}(j-2)F_{i-j+1}^{(k)}\right],$$

and the alternating Sury's relation involving Fibonacci and Lucas numbers of order k:

$$(-1)^n F_{n+1}^{(k)} = \sum_{i=0}^{n}(-1)^i r^{n-i}\left[L_{i+1}^{(k)} + (r-2)F_i^{(k)} - \sum_{j=3}^{k} jF_{i-j+2}^{(k)}\right].$$

Actually, Equation (2) is equivalent to Equation (3) through Lemma 2.

Theorem 2. *Equations (2) and (3) listed in Theorem 1 are equivalent.*

Proof. Assume that $r \neq 0$. Our proof strategy is to substitute r for $-1/r$ and then use Lemma 2 to obtain the equivalence of (2) and (3).

Substituting r for $-1/r$, we have

$$\left(-\frac{1}{r}\right)^{n+1} x^{k-1}F_{n+1}^{(k)}(x) + k - 2 = \sum_{i=0}^{n}\left(-\frac{1}{r}\right)^i \Big[L_i^{(k)}(x) + \left(-\frac{x^{k-1}}{r} - 2\right)F_{i+1}^{(k)}(x) - \sum_{j=3}^{k}(j-2)x^{k-j}F_{i-j+1}^{(k)}(x)\Big],$$

or

$$(-1)^n x^{k-1} F_{n+1}^{(k)}(x) = (k-2)r^{n+1} + \sum_{i=0}^{n} (-1)^{i+1} r^{n+1-i} \Big[L_i^{(k)}(x) + \left(-\frac{x^{k-1}}{r} - 2 \right) F_{i+1}^{(k)}(x) - \sum_{j=3}^{k} (j-2) x^{k-j} F_{i-j+1}^{(k)}(x) \Big].$$

After a series of indices shifting and computation, we can rewrite the above right-hand side as

$$\sum_{i=0}^{n} (-1)^i r^{n-i} \left[L_{i+1}^{(k)}(x) + x^{k-2}(rx-2) F_i^{(k)}(x) - \sum_{j=3}^{k} j x^{k-j} F_{i-j+2}^{(k)}(x) \right] + (-1)^{n+1} L_{n+1}^{(k)}(x) + (-1)^{n+1} x^{k-1} F_{n+1}^{(k)}(x) + 2(-1)^n F_{n+2}^{(k)}(x) + (-1)^n \sum_{j=3}^{k} (j-2) x^{k-j} F_{n-j+2}^{(k)}(x).$$

Now, by Lemma 2, we have

$$(-1)^{n+1} L_{n+1}^{(k)}(x) = (-1)^{n+1} \sum_{j=1}^{k} j x^{k-j} F_{n-j+2}^{(k)}(x).$$

Therefore, we obtain that the last few terms vanish. That is to say

$$(-1)^{n+1} L_{n+1}^{(k)}(x) + (-1)^{n+1} x^{k-1} F_{n+1}^{(k)}(x) + 2(-1)^n F_{n+2}^{(k)}(x) + (-1)^n \sum_{j=3}^{k} (j-2) x^{k-j} F_{n-j+2}^{(k)}(x) = 0.$$

Hence the proof that (2) implies (3) is done. And the proof of the reverse direction is similar. □

4. Remarks and Conclusions

In Section 2 we follow the approach of the generating function, however, one can prove Lemma 2 directly by using induction on n. Here is another proof of Lemma 2.

Second proof of Lemma 2. Let $k \geq 2$ be a fixed positive integer. First of all, we show that Equation (4) holds when $n \leq k$. The initial case $n = 1$ holds trivially. So we assume that when $n \leq m$, Equation (4) holds for some positive integer $m < k$. Now, we have $m+1 \leq k$ and by definition

$$L_{m+1}^{(k)}(x) = (m+1)x^{k-m-1} + \sum_{i=1}^{m} x^{k-i} L_{m+1-i}^{(k)}(x).$$

By the inductive hypothesis, the summation can be rewritten as

$$\begin{aligned}
\sum_{i=1}^{m} x^{k-i} L_{m+1-i}^{(k)}(x) &= \sum_{i=1}^{m} x^{k-i} \sum_{j=1}^{k} j x^{k-j} F_{m-i-j+2}^{(k)}(x) \\
&= \sum_{i=1}^{m} x^{k-i} \sum_{j=1}^{m} j x^{k-j} F_{m-i-j+2}^{(k)}(x) \\
&= \sum_{j=1}^{m} j x^{k-j} \sum_{i=1}^{m} x^{k-i} F_{m-i-j+2}^{(k)}(x) \\
&= \sum_{j=1}^{m} j x^{k-j} \left(F_{m-j+2}^{(k)}(x) - \sum_{i=m+1}^{k} x^{k-i} F_{m-i-j+2}^{(k)}(x) \right) \\
&= \sum_{j=1}^{m} j x^{k-j} F_{m-j+2}^{(k)}(x).
\end{aligned}$$

Hence

$$L_{m+1}^{(k)}(x) = (m+1)x^{k-m-1} + \sum_{j=1}^{m} j x^{k-j} F_{m-j+2}^{(k)}(x) = \sum_{j=1}^{m+1} j x^{k-j} F_{m-j+2}^{(k)}(x),$$

and Equation (4) holds for $n \leq k$ by induction.

We now obtain

$$\begin{aligned}
L_{k+1}^{(k)}(x) &= \sum_{i=1}^{k} x^{k-i} L_{k+1-i}^{(k)}(x) = \sum_{i=1}^{k} x^{k-i} \sum_{j=1}^{k} j x^{k-j} F_{k-i-j+2}^{(k)}(x) \\
&= \sum_{j=1}^{k} j x^{k-j} \sum_{i=1}^{k} x^{k-i} F_{k-i-j+2}^{(k)}(x) = \sum_{j=1}^{k} j x^{k-j} F_{k-j+2}^{(k)}(x).
\end{aligned}$$

So Equation (4) holds for $n = k+1$.

Suppose that Equation (4) holds for some positive integer m which is greater than $k+1$. Then we have

$$\begin{aligned}
L_{m+1}^{(k)}(x) &= \sum_{i=1}^{k} x^{k-i} L_{m+1-i}^{(k)}(x) = \sum_{i=1}^{k} x^{k-i} \sum_{j=1}^{k} j x^{k-j} F_{m-i-j+2}^{(k)}(x) \\
&= \sum_{j=1}^{k} j x^{k-j} \sum_{i=1}^{k} x^{k-i} F_{m-i-j+2}^{(k)}(x) = \sum_{j=1}^{k} j x^{k-j} F_{m-j+2}^{(k)}(x).
\end{aligned}$$

Thus by induction, we have proved that Equation (4) holds for all $n \geq 1$ and all $k \geq 2$. □

We use only the result in Lemma 2 to give a rather easier proof of Theorem 1.

Second proof of Theorem 1. In light of Lemma 2, the inner sum of the right-hand side (2) for $i \geq 1$ is equal to

$$\begin{aligned}
&L_i^{(k)}(x) + (rx^{k-1} - 2)F_{i+1}^{(k)}(x) - \sum_{j=3}^{k}(j-2)x^{k-j}F_{i-j+1}^{(k)}(x) \\
&\quad = \sum_{j=1}^{k} jx^{k-j}F_{i-j+1}^{(k)}(x) + (rx^{k-1} - 2)F_{i+1}^{(k)}(x) - \sum_{j=3}^{k}(j-2)x^{k-j}F_{i-j+1}^{(k)}(x) \\
&\quad = (rx^{k-1} - 2)F_{i+1}^{(k)}(x) + 2\sum_{j=1}^{k} x^{k-j}F_{i-j+1}^{(k)}(x) - x^{k-1}F_i^{(k)}(x) \\
&\quad = x^{k-1}\left(rF_{i+1}^{(k)}(x) - F_i^{(k)}(x)\right).
\end{aligned}$$

Therefore we obtain that

$$\begin{aligned}
&\sum_{i=0}^{n} r^i \left[L_i^{(k)}(x) + (rx^{k-1} - 2)F_{i+1}^{(k)}(x) - \sum_{j=3}^{k}(j-2)x^{k-j}F_{i-j+1}^{(k)}(x)\right] \\
&\qquad = rx^{k-1} + k - 2 + \sum_{i=1}^{n} r^i x^{k-1}\left(rF_{i+1}^{(k)}(x) - F_i^{(k)}(x)\right) \\
&\qquad = rx^{k-1} + k - 2 + x^{k-1}\sum_{i=1}^{n}\left[r^{i+1}F_{i+1}^{(k)}(x) - r^iF_i^{(k)}(x)\right] \\
&\qquad = rx^{k-1} + k - 2 + x^{k-1}\left[r^{n+1}F_{n+1}^{(k)}(x) - rF_1^{(k)}(x)\right] \\
&\qquad = r^{n+1}x^{k-1}F_{n+1}^{(k)}(x) + k - 2.
\end{aligned}$$

Hence Equation (2) follows.

For the extension of alternating Sury's relation (3), note that the inner sum is equal to

$$\begin{aligned}
&L_{i+1}^{(k)}(x) + x^{k-2}(rx-2)F_i^{(k)}(x) - \sum_{j=3}^{k} jx^{k-j}F_{i-j+2}^{(k)}(x) \\
&\quad = x^{k-1}F_{i+1}^{(k)}(x) + 2x^{k-2}F_i^{(k)}(x) + x^{k-2}(rx-2)F_i^{(k)}(x) \\
&\quad = x^{k-1}\left[F_{i+1}^{(k)}(x) + rF_i^{(k)}(x)\right].
\end{aligned}$$

Once again we have used the result in Lemma 2. It follows that

$$\sum_{i=0}^{n}(-1)^i r^{n-i}\left[L_{i+1}^{(k)}(x)+x^{k-2}(rx-2)F_i^{(k)}(x)-\sum_{j=3}^{k} jx^{k-j}F_{i-j+2}^{(k)}(x)\right]$$
$$=r^n x^{k-1}+x^{k-1}\sum_{i=1}^{n}\left[(-1)^i r^{n-i}F_{i+1}^{(k)}(x)+(-1)^i r^{n-i+1}F_i^{(k)}(x)\right]$$
$$=r^n x^{k-1}+x^{k-1}\left[(-1)^n F_{n+1}^{(k)}(x)-r^n F_1^{(k)}(x)\right]=(-1)^n x^{k-1}F_{n+1}^{(k)}(x).$$

□

We remark that Equation (2) is equivalent to the following:

$$x^{k-1}\left(r^n F_{n+1}^{(k)}(x)-1\right)=\sum_{i=1}^{n} r^{i-1}\Big[L_i^{(k)}(x)+(rx^{k-1}-2)F_{i+1}^{(k)}(x)$$
$$-\sum_{j=3}^{k}(j-2)x^{k-j}F_{i-j+1}^{(k)}(x)\Big].$$

This implies that the polynomials $r^n F_{n+1}^{(k)}(x)-1$ divide the above right-hand side for all $r\neq 0$ and $n\geq 1$.

Recall the definitions of Fibonacci-type polynomials of order k, $f_n^{(k)}(x)$, and Lucas-type polynomials of order k, $\ell_n^{(k)}(x)$ (on page 4). Expansions in terms of binomial coefficients, generating functions, properties, and connections between these two types of polynomials sequence could be found in [5, 15]. Some applications in combinatorics and probability are also given in [5, 15]. We only summarize a few results here but without proof.

The generating functions of the sequence of Fibonacci-type and Lucas-type polynomials of order k are

$$f^{(k)}(x;y)=\sum_{n\geq 0} f_n^{(k)}(x)y^n=\frac{y}{1-x\sum\limits_{j=1}^{k} y^j},$$

and

$$\ell^{(k)}(x;y)=\sum_{n\geq 0}\ell_n^{(k)}(x)y^n=\frac{k-x\sum\limits_{j=1}^{k}(k-j)y^j}{1-x\sum\limits_{j=1}^{k} y^j},$$

respectively. For $n\geq 1$, we have a crucial identity relating to Fibonacci-type and Lucas-type polynomials of order k,

$$\ell_n^{(k)}(x)=x\sum_{j=1}^{\min\{n,k\}} jf_{n-j+1}^{(k)}(x). \tag{6}$$

In addition, for $n \geq 1$, we have

$$F_n^{(k)}(x) = x^{-n+1} f_n^{(k)}(x^k), \text{ and } L_n^{(k)}(x) = x^{-n} \ell_n^{(k)}(x^k).$$

We also note that $L_n^{(k)} = L_n^{(k)}(1) = \ell_n^{(k)}(1)$ and $F_n^{(k)} = F_n^{(k)}(1) = f_n^{(k)}(1)$, which are the n-th Lucas and the n-th Fibonacci number of order k, respectively. According to the same method in this paper, it is not hard to present two additional extensions of Sury's relation involving Fibonacci-type and Lucas-type polynomials of order k, as presented in the following theorem.

Theorem 3. *For any positive integer n, we have the extension of Sury's relation involving Fibonacci-type and Lucas-type polynomials of order k:*

$$\begin{aligned} r^{n+1} x f_{n+1}^{(k)}(x) + k - 2 = \sum_{i=0}^{n} r^i \Big[& \ell_i^{(k)}(x) + (rx-2) f_{i+1}^{(k)}(x) \\ & - x \sum_{j=3}^{k} (j-2) f_{i-j+1}^{(k)}(x) \Big], \end{aligned} \tag{7}$$

and the extension of alternating Sury's relation involving Fibonacci-type and Lucas-type polynomials of order k:

$$(-1)^n x f_{n+1}^{(k)}(x) = \sum_{i=0}^{n} (-1)^i r^{n-i} \left[\ell_{i+1}^{(k)}(x) + x(r-2) f_i^{(k)}(x) - \sum_{j=3}^{k} jx f_{i-j+2}^{(k)}(x) \right], \tag{8}$$

where the summation $\sum_{j=a}^{b} *$ is zero if $b < a$ and $f_{-m}^{(k)} = 0$ for any positive integer m. Moreover, it can be seen from Equation (6) that Equations (7) and (8) are equivalent.

Finally, we remark that by setting $x = 1$ in (7) and (8), the results coincide with Corollary 2.

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