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THE TRIANGLE ALGORITHM FOR BERNOULLI POLYNOMIALS

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Abstract

Algorithms like ones used to generate Pascal's triangle for generating Bernoulli polynomials and their various generalizations are given. It is remarkable that the algorithms for Bernoulli polynomials are natural interpolations of the ones for Bernoulli numbers. The algorithms presented in this paper can be understood as essentially unique, even if the Bernoulli polynomials are generalized in various ways.

1. Triangle Algorithm for Bernoulli Numbers

Two types of *Bernoulli numbers* $\{B_n\}_n$ and $\{C_n\}_n$ are defined by the generating functions [1]

$$\frac{te^t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!} \quad \text{and} \quad \frac{t}{e^t-1}=\sum_{n=0}^{\infty}C_n\frac{t^n}{n!},$$

respectively. In their study on special values at non-positive integers of multiple zeta functions of Euler-Zagier type, S. Akiyama and Y. Tanigawa [1] found a *triangle algorithm* for generating Bernoulli numbers like "Pascal's triangle algorithm" for binomial coefficients.

Akiyama and Tanigawa's triangle algorithm can be written as follows. We denote the m-th number in the n-th row by $b_{n,m}$ which is determined by the *recurrence formula*

$$b_{n+1,m}=(m+1)(b_{n,m}-b_{n,m+1}) \quad (n,m\geq 0), \tag{1}$$

and start with the 0-th row $\{b_{0,m}\}_m=\{\frac{1}{m+1}\}_m=\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots\}$, which we call the *initial sequence* of the algorithm. Then 0-th component $b_{n,0}$ of each row is the

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[1]Note that $B_1=-C_1=\frac{1}{2}$ and $B_n=C_n$ for any non-negative integer $n\neq 1$.

n-th Bernoulli number B_n (see Figure 1) [2]. As a generalization, M. Kaneko [9] replaced the initial sequence $\{\frac{1}{m+1}\}_m$ by $\{\frac{1}{(m+1)^k}\}_m$ for any integer k, and applied the same recurrence formula to obtain poly-Bernoulli numbers, which are treated in the next section. On the other hand, K.-W. Chen [4] replaced the recurrence formula (1) by

$$b_{n+1,m} = mb_{n,m} - (m+1)b_{n,m+1} \quad (n, m \geq 0), \tag{2}$$

and applied $\{\frac{1}{m+1}\}_m$ as the initial sequence to obtain Bernoulli numbers $\{C_n\}_n$ as the resulting sequence (see Figure 2).

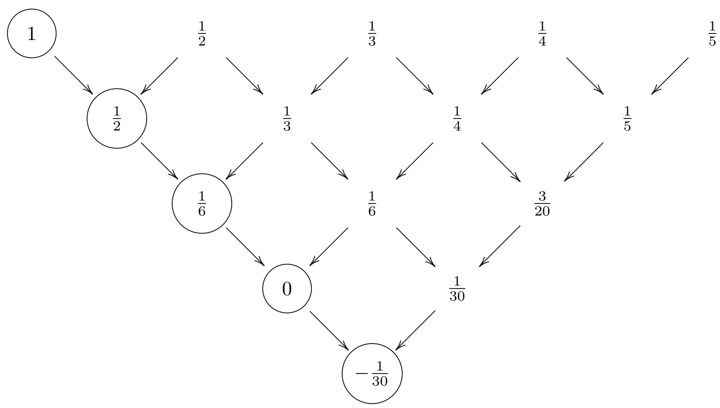

Figure 1: Akiyama and Tanigawa's algorithm for B_n

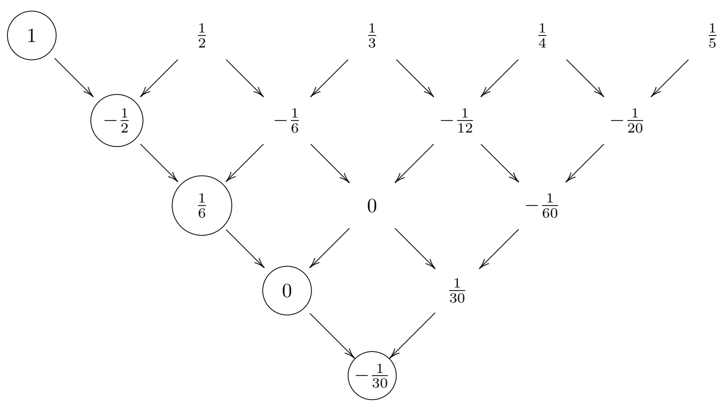

Figure 2: Chen's algorithm for C_n

[2]To be precise, using our notation, [1] states the fact $\{b_{n,0}\}_{n\geq 2} = \{C_n\}_{n\geq 2}$, and if we write this using B_n instead of C_n, the formula is valid for $n = 0$ and 1 as well.

To state our first step, we introduce the definition of *Bernoulli polynomials* $\{B_n(x)\}_n$ as follows:

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

One can easily see that

$$B_n(0) = C_n, \quad B_n(1) = B_n.$$

As our first step, we describe the following algorithm (preliminary version) before considering various generalizations.

Theorem 1. *Setting the initial sequence* $\{b_{0,m}(x)\}_m = \{\frac{1}{m+1}\}_m$ *as before and applying the recurrence formula*

$$b_{n+1,m}(x) = (m+x)b_{n,m}(x) - (m+1)b_{n,m+1}(x) \quad (n, m \geq 0), \tag{3}$$

the resulting sequence $\{b_{n,0}(x)\}_n$ *is* $\{B_n(x)\}_n$.

Miraculously, the above recurrence formula gives a natural interpolation between those of Akiyama and Tanigawa's and Chen's (see Figure 3). In other words, if we set $x = 0$ or $x = 1$ in the formula (3), then it is reduced to the one we have already mentioned for $\{C_n\}_n$ or $\{B_n\}_n$, respectively. Thus, we succeed in obtaining an algorithm, named *a triangle algorithm* for Bernoulli polynomials as a natural generalization of Akiyama and Tanigawa's and Chen's algorithms.

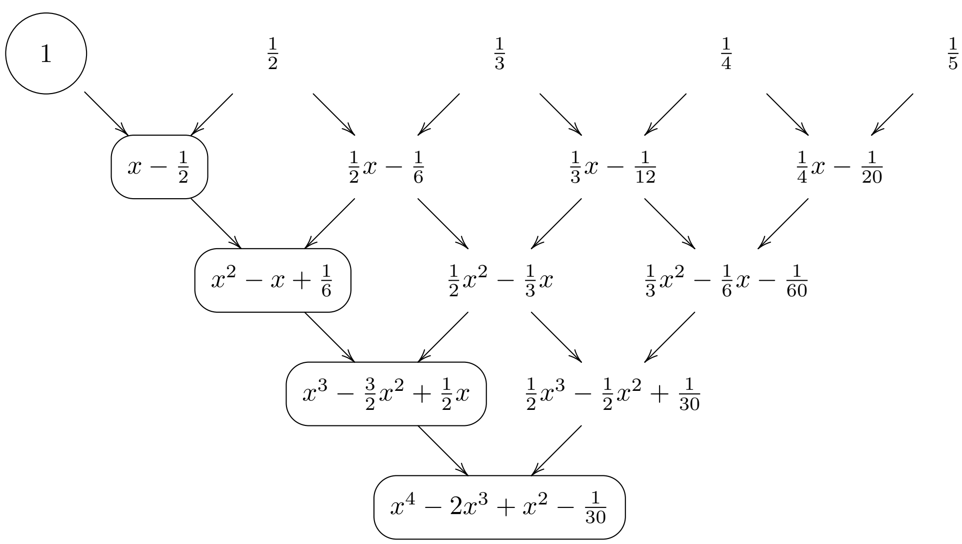

Figure 3: Triangle algorithm for $B_n(x)$ (preliminary version)

In this paper, we discuss more general triangle algorithms for various generalizations of Bernoulli numbers and polynomials, including multi-poly-Bernoulli polynomials. In these discussions, we note that it is more natural to invert the signs in the recurrence formula (3). In fact, by inverting the signs, the recurrence formulas of triangle algorithms for various generalizations of Bernoulli numbers and polynomials are essentially unified.

This paper is organized as follows. In Section 2, we introduce the triangle algorithm for poly-Bernoulli numbers/polynomials. In Section 3, we introduce and show the triangle algorithm for multi-poly-Bernoulli polynomials. It is remarkable that we can naturally obtain the algorithms for Bernoulli and poly-Bernoulli numbers/polynomials and multi-poly-Bernoulli numbers as restricted cases of that for multi-poly-Bernoulli polynomials. In Section 4, we discuss two kinds of variants of multi-poly-Bernoulli numbers. For example, in Section 4.2, we define generalized multi-poly-Bernoulli polynomials like that for the classical generalized Bernoulli polynomials, and discuss their triangle algorithm.

2. Triangle Algorithm for Poly-Bernoulli Numbers and Polynomials

For any integer k, *poly-Bernoulli polynomials* $\{B_n^{(k)}(x)\}_n$ are defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}$$

and two types of *poly-Bernoulli numbers* $\{B_n^{(k)}\}_n$ and $\{C_n^{(k)}\}_n$ are defined by

$$B_n^{(k)}(1)=B_n^{(k)} \quad \text{and} \quad B_n^{(k)}(0)=C_n^{(k)},$$

respectively. In other words, the generating functions for these two series are

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!} \quad \text{and} \quad \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}=\sum_{n=0}^{\infty}C_n^{(k)}\frac{t^n}{n!}.$$

Here, $\mathrm{Li}_k(z)$ is defined as the special case when $\mathbf{k}=(k)$ of the formal power series

$$\mathrm{Li}_{\mathbf{k}}(z):=\sum_{0<m_1<\cdots<m_r}\frac{z^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}$$

for any r-tuple of integers $\mathbf{k}=(k_1,\ldots,k_r)$. One can easily see that $B_n^{(1)}(x)=B_n(x)$ for any non-negative integer n, and thus $\{B_n^{(1)}\}_n$ and $\{C_n^{(1)}\}_n$ coincide with the ordinary Bernoulli numbers $\{B_n\}_n=\{1,\frac{1}{2},\frac{1}{6},\ldots\}$ and $\{C_n\}_n=\{1,-\frac{1}{2},\frac{1}{6},\ldots\}$, respectively.

We introduce a recurrence formula slightly different from the one in the previous section for obtaining poly-Bernoulli polynomials:

$$b_{n+1,m}(x) = (m+1)b_{n,m+1}(x) - (m+1-x)\,b_{n,m}(x) \quad (n, m \geq 0). \tag{4}$$

Also, we replace the initial sequence $\{\frac{1}{m+1}\}_m$ by $\{\frac{1}{(m+1)^k}\}_m$ for any integer k, and apply the algorithm (4) to obtain poly-Bernoulli polynomials $B_n^{(k)}(x)$ (see Figure 4).

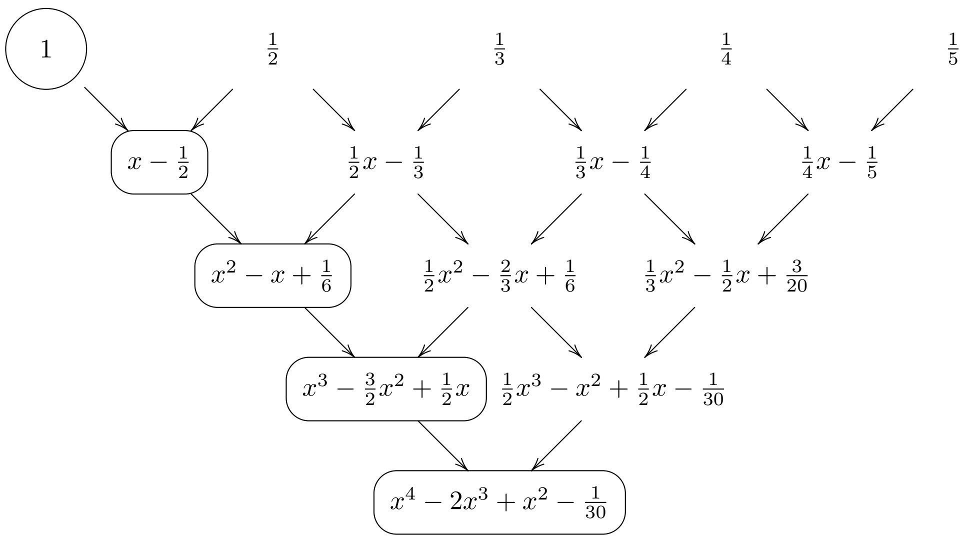

Figure 4: Triangle algorithm for $B_n(x)$

Substituting $x = 0$ and $x = 1$, the recurrence formula (4) becomes

$$b_{n+1,m} = (m+1)(b_{n,m+1} - b_{n,m}) \quad (n, m \geq 0) \tag{5}$$

and

$$b_{n+1,m} = (m+1)b_{n,m+1} - mb_{n,m} \quad (n, m \geq 0), \tag{6}$$

and we obtain $\{C_n^{(k)}\}$ and $\{B_n^{(k)}\}$ as the respective resulting sequences.

It is easy to see that the recurrence formulas (5) and (6) can be obtained by multiplying the right-hand sides of the ones (1) and (2) by -1. the ones (1) and (2) by -1. Thus, the previous algorithms (1) and (2) can each be considered to be a kind of "dual" of the recurrence formulas (5) and (6). Through these inversions of signs, with the well-known property $(-1)^n C_n = B_n$ of the ordinary (classical) Bernoulli numbers, the algorithms (5) and (6) can be understood as recurrence formulas for generating $\{C_n\}_n$ (see Figure 5) and $\{B_n\}_n$ (see Figure 6), respectively.

Remark 1. In [9], M. Kaneko already obtained poly-Bernoulli numbers $(-1)^n C_n^{(k)}$ by applying the recurrence formula (1) to the initial sequence $\{\frac{1}{m+1}\}_m$; however, here we apply its dual recurrence formula (5).

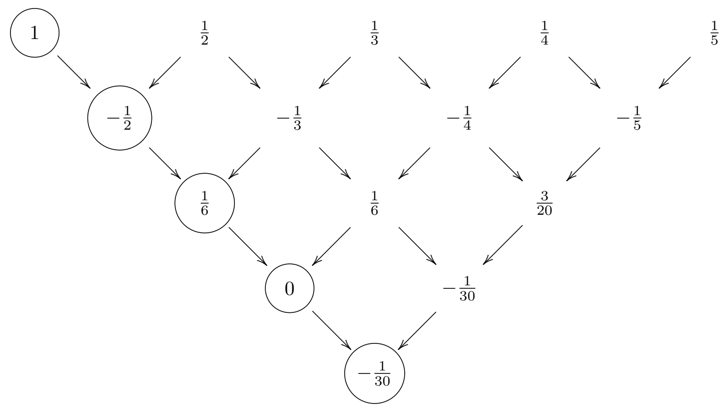

Figure 5: Triangle algorithm for C_n

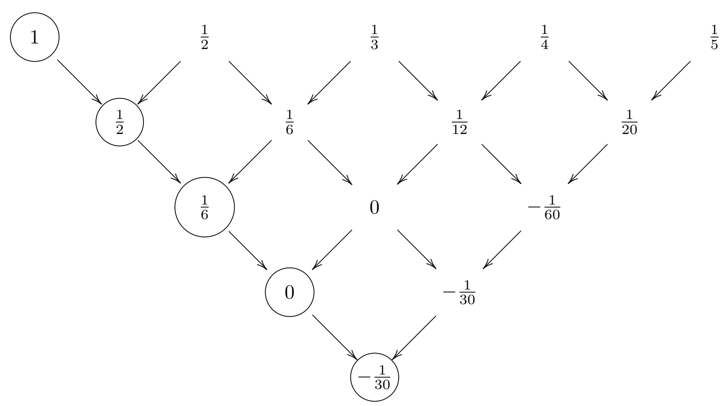

Figure 6: Triangle algorithm for B_n

3. Triangle Algorithm for Multi-poly-Bernoulli Polynomials

In this section, we introduce multi-poly-Bernoulli polynomials as a natural generalization of the classical Bernoulli polynomials, and state their generating algorithm and give a proof.

Throughout this paper, index $\mathbf{k} = (k_1, \ldots, k_r)$ means any r-tuple of integers.

3.1. Algorithm for Multi-poly-Bernoulli Polynomials

We here define multi-poly-Bernoulli numbers/polynomials and present their recurrence formula and triangle algorithm.

Definition 1. For any index $\mathbf{k} = (k_1, \dots, k_r)$ and for any integer l satisfying $1 \le l \le r$, we define *multi-poly-Bernoulli polynomials* $\{B_n^{(\mathbf{k})/l}(x)\}_n$ by the generating function

$$\frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-t})}{(e^t-1)^l} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\mathbf{k})/l}(x) \frac{t^n}{n!},$$

and we define two types of *multi-poly-Bernoulli numbers* $\{B_n^{(\mathbf{k})/l}\}_n$ and $\{C_n^{(\mathbf{k})/l}\}_n$ by

$$B_n^{(\mathbf{k})/l} = B_n^{(\mathbf{k})/l}(l) \quad \text{and} \quad C_n^{(\mathbf{k})/l} = B_n^{(\mathbf{k})/l}(0).$$

In other words, the generating functions for these two types are

$$\frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-t})}{(1-e^{-t})^l} = \sum_{n=0}^{\infty} B_n^{(\mathbf{k})/l} \frac{t^n}{n!} \quad \text{and} \quad \frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-t})}{(e^t-1)^l} = \sum_{n=0}^{\infty} C_n^{(\mathbf{k})/l} \frac{t^n}{n!}.$$

Multi-poly-Bernoulli polynomials $\{B_n^{(\mathbf{k})/l}(x)\}_n$ also appeared in [12] and include various generalizations of Bernoulli numbers/polynomials as follows. By setting $l = 1$, we obtain $\{B_n^{(\mathbf{k})}(x)\}_n$ defined by K. Imatomi [7, Definition 6.1], and moreover we obtain $\{B_n^{(k)}(x)\}_n$ by setting $r = l = 1$. We can obtain $B_n^{(\mathbf{k})/l}$ and $C_n^{(\mathbf{k})/l}$ in the cases when $x = 1$ and 0, respectively. We obtain $B_n^{(\mathbf{k})}$ and $C_n^{(\mathbf{k})}$ defined by Imatomi-Kaneko-E. Takeda [8, (1),(2)] if $l = 1$, and $\mathbb{B}_n^{(\mathbf{k})}$ defined by T. Arakawa-Kaneko [3, p.202 Remarks (ii)] if $l = r$.

The triangle algorithm for $\{B_n^{(\mathbf{k})/l}(x)\}_n$ which we show in this paper can be stated as follows. Suppose that the initial sequence $\{b_{0,m}(x)\}_m$ is given by

$$b_{0,m}(x) = \sum_{0<m_1<\cdots<m_{r-1}<m+l} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} (m+l)^{k_r}} \qquad (m \ge 0), \tag{7}$$

and apply the recurrence formula

$$b_{n+1,m}(x) = (m+1) b_{n,m+1}(x) - (m+l-x)\, b_{n,m}(x) \quad (n, m \ge 0). \tag{8}$$

Then the claim is that we obtain $\{b_{n,0}(x)\}_n = \{B_n^{(\mathbf{k})/(l)}(x)\}_n$ as the resulting sequence.

It is remarkable that the terms in the initial sequence $\{b_{0,m}(x)\}_m$ match the truncated (finite) multiple zeta values (see [6, (2)], [14]).

Now we define *Stirling numbers of the second kind* $\left\{ {n \atop m} \right\}$ for any integers n and m by the recurrence formula

$$\left\{ {n+1 \atop m} \right\} = \left\{ {n \atop m-1} \right\} + m \left\{ {n \atop m} \right\}$$

with the initial conditions $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {n \atop 0} \right\} = \left\{ {0 \atop m} \right\} = 0$ $(n, m \neq 0)$.

The above-mentioned algorithm is based on the identity for $B_n^{(\mathbf{k})/l}(x)$ in the following theorem, whose proof is carried out in Section 3.2.

Theorem 2 (Explicit formula)**.** *For any index* $\mathbf{k} = (k_1, \ldots, k_r)$ *and non-negative integers* l, r*, and* n *with* $1 \leq l \leq r$*, we have*

$$B_n^{(\mathbf{k})/l}(x) = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} (-1)^i \sum_{m=0}^{i} \sum_{0<m_1<\cdots<m_{r-1}<m+l} \frac{(-1)^m \sum_{j=0}^{l} (m+j)! \left\{ {n \atop m+j} \right\} \binom{l}{j}}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} (m+l)^{k_r}}.$$

From Theorem 2, it suffices to show the following theorem, whose proof is given in Section 3.3, to justify the above-mentioned algorithm for $B_n^{(\mathbf{k})/l}(x)$.

Theorem 3. *For given* $\{b_{0,m}(x)\}_m$*, define* $b_{n+1,m}(x)$ $(n, m \geq 0)$ *by the recurrence formula* (8)*. Then, we have*

$$b_{n,0}(x) = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} (-1)^i \sum_{m=0}^{i} (-1)^m \sum_{j=0}^{l} (m+j)! \left\{ {i \atop m+j} \right\} \binom{l}{j} b_{0,m}(x). \tag{9}$$

By specializing in Theorems 2 and 3 to the cases of $x = l$ and $x = 0$, we obtain the algorithms for the multi-poly-Bernoulli numbers $\{B_n^{(\mathbf{k})/l}\}_n$ and $\{C_n^{(\mathbf{k})/l}\}_n$, respectively, as follows.

Example 1. Suppose that the initial sequence $\{b_{0,m}(x)\}_m$ is given as the formula (7). By setting $x = l$ in the formula (8), we obtain the recurrence formula (6) in Section 2. Applying the initial sequence (7) to (9) with $x = l$, we obtain the resulting sequence $\{b_{n,0}(l)\}_n = \{B_n^{(\mathbf{k})/l}\}_n$ from Theorem 2.

On the other hand, by setting $x = 0$ in (8), we obtain

$$b_{n+1,m} = (m+1) b_{n,m+1} - (m+l) b_{n,m} \quad (n, m \geq 0).$$

Applying the initial sequence (7) to (9) with $x = 0$, we obtain $\{b_{n,0}(0)\}_n = \{C_n^{(\mathbf{k})/l}\}_n$in the same way as above.

We can interpret the recurrence formula (8) by using generating functions as follows (see [9]).

Remark 2. Suppose that the ordinary generating function of the initial sequence $\{b_{0,m}(x)\}_m$ is

$$A(x; t) = \sum_{m=0}^{\infty} b_{0,m}(x) t^m$$

and the exponential generating function for the resulting sequence $\{b_{n,0}(x)\}_n$, from the recurrence formula (8), is

$$B(x;t) = \sum_{n=0}^{\infty} b_{n,0}(x) \frac{t^n}{n!}.$$

Then we have

$$B(x;t) = e^{(x-l)t} A(x; 1-e^{-t}),$$

as a natural consequence of the formula (9) and Lemma 1 in the next subsection.

3.2. Proof of Theorem 2

We give the explicit formula for $B_n^{(\mathbf{k})/l}$ and $C_n^{(\mathbf{k})/l}$. The latter is necessary for the proof of Theorem 2.

Theorem 4. *For any r-tuple of integers $\mathbf{k} = (k_1, \ldots, k_r)$ and non-negative integer n, we have*

$$B_n^{(\mathbf{k})/l} = (-1)^n \sum_{m=0}^{n} \sum_{0<m_1<\cdots<m_{r-1}<m+l} \frac{(-1)^m m! \left\{ {n \atop m} \right\}}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} (m+l)^{k_r}},$$

and

$$C_n^{(\mathbf{k})/l} = (-1)^n \sum_{m=0}^{n} \sum_{0<m_1<\cdots<m_{r-1}<m+l} \frac{(-1)^m \sum_{j=0}^{l} (m+j)! \left\{ {n \atop m+j} \right\} \binom{l}{j}}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} (m+l)^{k_r}}.$$

Remark 3. By specializing in Theorem 4, we obtain known formulas as follows. If $l = r$, we obtain the explicit formula for $B_n^{(\mathbf{k})/r}$ proved by Y. Hamahata-H. Masubuchi [5, Theorem 7]. If $l = 1$, we obtain the explicit formula for $B_n^{(\mathbf{k})/1}$ and $C_n^{(\mathbf{k})/1}$ proved by Imatomi-Kaneko-Takeda [8, Theorem 3].

We need the following lemma to prove Theorem 4.

Lemma 1. *For any non-negative integers l and m, we have*

$$e^{lt}(e^t - 1)^m = \sum_{n=m}^{\infty} \sum_{j=0}^{l} (m+j)! \left\{ {n \atop m+j} \right\} \binom{l}{j} \frac{t^n}{n!}. \tag{10}$$

Proof. We prove by induction on l. If $l = 0$, the formula (10) is well known (see [2, Proposition 2.6 (7)] for example). If we assume that the lemma is valid up to

$l < k$, then for $l = k$ we have

$$
\begin{aligned}
&e^{kt}(e^t-1)^m\\
&= e^{(k-1)t}(e^t-1)^{m+1} + e^{(k-1)t}(e^t-1)^m\\
&= \sum_{n=m+1}^{\infty}\sum_{j=0}^{k-1}(m+j+1)!\left\{{n \atop m+j+1}\right\}\binom{k-1}{j}\frac{t^n}{n!}\\
&\quad + \sum_{n=m}^{\infty}\sum_{j=0}^{k-1}(m+j)!\left\{{n \atop m+j}\right\}\binom{k-1}{j}\frac{t^n}{n!}\\
&= \sum_{n=m}^{\infty}\left(\sum_{j=1}^{k}(m+j)!\left\{{n \atop m+j}\right\}\binom{k-1}{j-1} + \sum_{j=0}^{k-1}(m+j)!\left\{{n \atop m+j}\right\}\binom{k-1}{j}\right)\frac{t^n}{n!}\\
&= \sum_{n=m}^{\infty}\sum_{j=0}^{k}(m+j)!\left\{{n \atop m+j}\right\}\binom{k}{j}\frac{t^n}{n!}.
\end{aligned}
$$

Thus the formula (10) is valid for any non-negative integer l by induction. □

Proof of Theorem 4. From the definition of $B_n^{(\mathbf{k})/l}$, we obtain

$$
\sum_{n=0}^{\infty} B_n^{(\mathbf{k})/l}\frac{t^n}{n!} = \sum_{0<m_1<\cdots<m_{r-1}<m+l}\frac{(-1)^m(e^{-t}-1)^m}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(m+l)^{k_r}}.
$$

Taking $l = 0$ and applying Lemma 1 to the right-hand side of the above equality, we obtain

$$
\begin{aligned}
\text{R.H.S.} &= \sum_{0<m_1<\cdots<m_{r-1}<m+l}\frac{(-1)^m m!}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(m+l)^{k_r}}\sum_{n=m}^{\infty}\left\{{n \atop m}\right\}\frac{(-t)^n}{n!}\\
&= \sum_{n=0}^{\infty}\left((-1)^n\sum_{m=0}^{n}\sum_{0<m_1<\cdots<m_{r-1}<m+l}\frac{(-1)^m m!\left\{{n \atop m}\right\}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(m+l)^{k_r}}\right)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ for the two sides, we obtain the first formula in

Theorem 4. Similarly, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_n^{(\mathbf{k})/l} \frac{t^n}{n!} \\
&= \sum_{0<m_1<\cdots<m_{r-1}<m+l} \frac{(-1)^m (e^{-t}-1)^m}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(m+l)^{k_r}} e^{-lt} \\
&= \sum_{0<m_1<\cdots<m_{r-1}<m+l} \frac{(-1)^m}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(m+l)^{k_r}} \sum_{n=m}^{\infty}\sum_{j=0}^{l}(m+j)!\left\{ {n \atop m+j} \right\}\binom{l}{j}\frac{(-t)^n}{n!} \\
&= \sum_{n=0}^{\infty}\left((-1)^n \sum_{m=0}^{n} \sum_{0<m_1<\cdots<m_{r-1}<m+l} \frac{(-1)^m \sum_{j=0}^{l}(m+j)!\left\{ {n \atop m+j} \right\}\binom{l}{j}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(m+l)^{k_r}} \right)\frac{t^n}{n!},
\end{aligned}$$

by the definition of $C_n^{(\mathbf{k})/l}$ and Lemma 1, and we thereby obtain the second formula in Theorem 4. □

Proof of Theorem 2. By comparing the generating functions for $\{B_n^{(\mathbf{k})/l}(x)\}_n$ and $\{C_n^{(\mathbf{k})/l}\}_n$, we easily obtain the equality

$$B_n^{(\mathbf{k})/l}(x) = \sum_{i=0}^{n}\binom{n}{i} C_i^{(\mathbf{k})/l} x^{n-i}$$

for any index $\mathbf{k}$ and non-negative integer n. Applying the explicit formula for $C_n^{(\mathbf{k})/l}$ given in Theorem 4 to the right-hand side of the above equality, we readily obtain Theorem 2. □

3.3. Proof of Theorem 3

The proof of Theorem 3 uses the following lemma.

Lemma 2 (see [2, Proposition 2.6 (4)]). *For any integer n, we have*

$$\left(x\frac{d}{dx}\right)^n = \sum_{m=1}^{n}\left\{ {n \atop m} \right\} x^m \left(\frac{d}{dx}\right)^m .$$

Proof of Theorem 3. We use the generating function

$$g_n(x;t) = \sum_{m=0}^{\infty} b_{n,m}(x) t^m$$

to prove Theorem 3. By the recurrence formula (8), we have

$$
\begin{aligned}
&g_n(x;t)\\
&= \sum_{m=0}^{\infty} \{(m+1)(b_{n-1,m+1}(x) - b_{n-1,m}(x)) + (1-l+x)\, b_{n-1,m}(x)\} t^m\\
&= \frac{d}{dt}\sum_{m=1}^{\infty} b_{n-1,m}(x)t^m - \frac{d}{dt}\sum_{m=0}^{\infty} b_{n-1,m}(x)t^{m+1} + (1-l+x)\sum_{m=0}^{\infty} b_{n-1,m}(x)t^m\\
&= \frac{d}{dt}(g_{n-1}(x;t) - b_{n-1,0}(x)) - \frac{d}{dt}(tg_{n-1}(x;t)) + (1-l+x)\, g_{n-1}(x;t)\\
&= \left(-l + x + (1-t)\frac{d}{dt}\right) g_{n-1}(x;t).
\end{aligned}
$$

Thus, if we set $h_n(x;t) = (t-1)^l g_n(x;t)$, we obtain

$$
\begin{aligned}
h_n(x;t) &= \left(x - (t-1)\frac{d}{dt}\right) h_{n-1}(x;t) \quad (n \ge 1)\\
&= \left(x - (t-1)\frac{d}{dt}\right)^n h_0(x;t)\\
&= \sum_{i=0}^{n} \binom{n}{i} x^{n-i}(-1)^i \left((t-1)\frac{d}{dt}\right)^i h_0(x;t).
\end{aligned}
$$

It follows from Lemma 2, setting x equal to $t-1$, that

$$
\begin{aligned}
&\left((t-1)\frac{d}{dt}\right)^i h_0(x;t)\\
&= \sum_{m=1}^{i} \left\{ {i \atop m} \right\} (t-1)^m \left(\frac{d}{dt}\right)^m h_0(x;t)\\
&= \sum_{m=0}^{i} \left\{ {i \atop m} \right\} (t-1)^m \left(\frac{d}{dt}\right)^m \left((t-1)^l \sum_{p=0}^{\infty} b_{0,p}(x)t^p\right)\\
&= \sum_{m=0}^{i} \left\{ {i \atop m} \right\} (t-1)^m \left(\frac{d}{dt}\right)^m \left(\sum_{p=0}^{\infty}\sum_{j=0}^{l} \binom{l}{j}(-1)^{l-j} b_{0,p-j}(x)t^p\right)\\
&= \sum_{m=0}^{i} \left\{ {i \atop m} \right\} (t-1)^m \sum_{p=m}^{\infty}\sum_{j=0}^{l} \binom{l}{j}(-1)^{l-j} p(p-1)\cdots(p-(m-1)) b_{0,p-j}(x)t^{p-m}.
\end{aligned}
$$

Here $b_{0,i}$ is taken to be 0 for any negative integer i. Setting $t=0$, we obtain

$$(-1)^l b_{n,0}(x) = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}(-1)^i \sum_{m=0}^{i} \left\{ {i \atop m} \right\} (-1)^m m! \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} b_{0,m-j}(x),$$

$$\begin{aligned} b_{n,0}(x) &= \sum_{i=0}^{n} \binom{n}{i} x^{n-i}(-1)^i \sum_{j=0}^{l} \sum_{m=-j}^{i-j} (-1)^m (m+j)! \left\{ {i \atop m+j} \right\} \binom{l}{j} b_{0,m}(x) \\ &= \sum_{i=0}^{n} \binom{n}{i} x^{n-i}(-1)^i \sum_{m=0}^{i} (-1)^m \sum_{j=0}^{l} (m+j)! \left\{ {i \atop m+j} \right\} \binom{l}{j} b_{0,m}(x). \end{aligned}$$

This gives the formula (9). Thus, we complete the proof of Theorem 3. □

Remark 4. The proof of Theorem 1 can be obtained in a similar way to that of Theorem 3 by applying the formula $B_n(x) = (-1)^n B_n(1-x)$ for any non-negative integer n.

4. Applications

In this section, we give the algorithm for t-multi-poly-Bernoulli numbers. Moreover, we define generalized multi-poly-Bernoulli polynomials and discuss an algorithm for them.

4.1. Interpolated Multi-poly-Bernoulli Numbers

In [10, Definition 2.3], the second named author and H. Wayama introduced another interpolation $\{C_n^{(\mathbf{k})}(t)\}_n$ between two types of multi-poly-Bernoulli numbers based on the Landen connection formula. We call $\{C_n^{(\mathbf{k})}(t)\}_n$ *t-multi-poly-Bernoulli numbers*, and define them by

$$C_n^{(\mathbf{k})}(t) = \sum_{\mathbf{k} \preceq \mathbf{k}'} t^{\mathrm{dep}(\mathbf{k}')-r} C_n^{(\mathbf{k}')/1} \quad (n \geq 0),$$

in which we denote by $\mathbf{k} \preceq \mathbf{k}'$ that $\mathbf{k}$ can be obtained from $\mathbf{k}'$ by replacing some commas of $\mathbf{k}'$ by pluses. This definition interpolates $\{C_n^{(\mathbf{k})/1}\}_n = \{C_n^{(\mathbf{k})}(0)\}_n$ and $\{(-1)^{n+r-1} B_n^{(\mathbf{k})/1}\}_n = \{C_n^{(\mathbf{k})}(1)\}_n$ and is suitable for expressing the special values at negative integral points of t-interpolated Arakawa-Kaneko multiple zeta functions (see [10]).

Curiously, we can obtain the numbers by using the recurrence formula of Akiyama and Tanigawa's algorithm (1). In fact, if we apply the initial sequence $\{b_{0,m}(t)\}_m$ obtained by

$$b_{0,m}(t) = \sum_{\mathbf{k} \preceq \mathbf{k}'} t^{\mathrm{dep}(\mathbf{k}')-r} b_{0,m}^{(\mathbf{k}')} \quad (m \geq 0),$$

and

$$b_{0,m}^{(\mathbf{k})} = \sum_{0<m_1<\cdots<m_{r-1}<m+1} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(m+1)^{k_r}},$$

then the resulting sequence becomes $\{C_n^{(\mathbf{k})}(t)\}_n$.

We can show the above by the algorithm for $C_n^{(\mathbf{k})/1}$ (Example 1) and the definition of $C_n^{(\mathbf{k})}(t)$. For example, we obtain the algorithm diagrammed in Figure 7 when $\mathbf{k} = (2, 2)$.

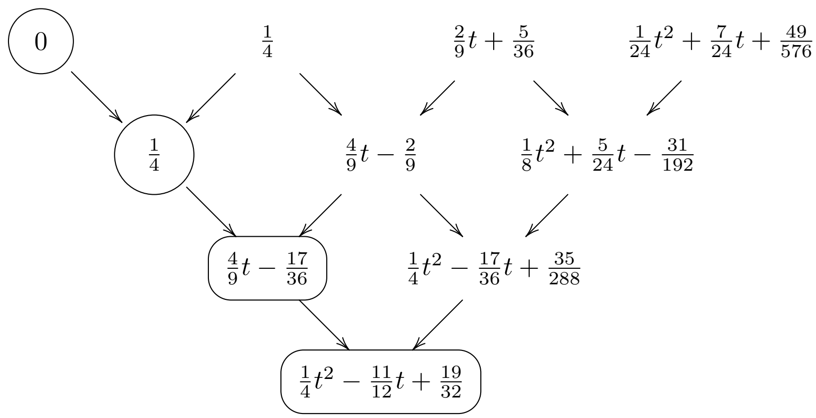

Figure 7: Algorithm for $C_n^{(2,2)}(t)$

4.2. Generalized Multi-poly-Bernoulli Polynomials

We define *generalized multi-poly-Bernoulli polynomials* and give an algorithm for them.

Definition 2. Given a Dirichlet character χ modulo f and an index $\mathbf{k}$, the generalized multi-poly-Bernoulli polynomials $\{B_{n,\chi}^{(\mathbf{k})/l}(x)\}_n$ are defined by

$$\frac{1}{f}\sum_{a=1}^{f} \chi(a)\frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-ft})}{(e^{ft}-1)^l} e^{(a-1+x)t} = \sum_{n=0}^{\infty} B_{n,\chi}^{(\mathbf{k})/l}(x)\frac{t^n}{n!}.$$

The generalized multi-poly-Bernoulli polynomials include the generalized poly-Bernoulli numbers
/polynomials (see [13], [11, (17)]) and multi-poly-Bernoulli polynomials as the cases when $r = x = 1$ and $f = 1$, respectively.

The algorithm for $\{B_{n,\chi}^{(\mathbf{k})/l}(x)\}_n$ is as follows. If we take the initial sequence $\{b_{0,m}(\chi, a; x)\}_m$ obtained from

$$b_{0,m}(\chi, a; x) = \frac{\chi(a)}{f} \sum_{0<m_1<\cdots<m_{r-1}<m+l} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(m+l)^{k_r}} \quad (m \geq 0),$$

and apply the recurrence formula

$$
\begin{aligned}
&b_{n+1,m}(\chi,a;x)\\
&=(m+1)fb_{n,m+1}(\chi,a;x)-\left(m+l-\frac{a-1+x}{f}\right)fb_{n,m}(\chi,a;x)\quad (n,m\geq 0),
\end{aligned}
$$

then the resulting sequence is called the *a-part* of $\{B_{n,\chi}^{(\mathbf{k})}(x)\}_n$. Summing up a-parts from $a=1$ to f and obtain $\{B_{n,\chi}^{(\mathbf{k})}(x)\}_n$. For example, we obtain the algorithm diagrammed in Figure 8 when $l=r=1$ and $\mathbf{k}=(1)$.

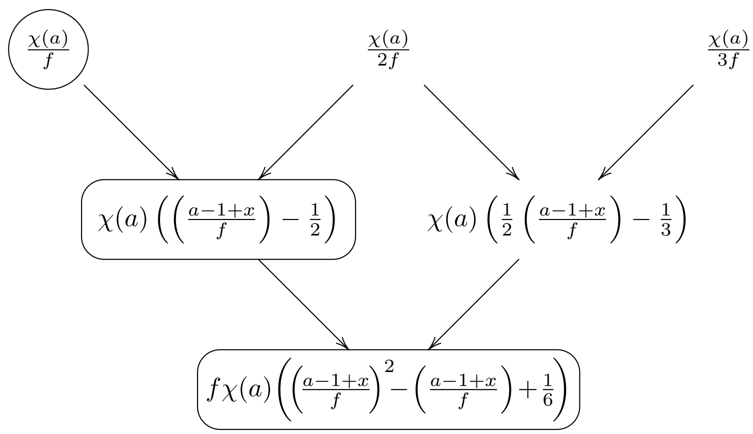

Figure 8: Algorithm for a-part of $\{B_{n,\chi}^{(1)}\}_n=\{B_{n,\chi}\}_n$

This algorithm is based on the explicit formula for $B_{n,\chi}^{(\mathbf{k})/l}(x)$ given in the following theorem.

Theorem 5. *For any Dirichlet character χ modulo f, integers $k_1,\dots,k_r$, positive integer l with $1\leq l\leq r$ and non-negative integer n, we have*

$$
\begin{aligned}
B_{n,\chi}^{(\mathbf{k})/l}(x)&=f^{n-1}\sum_{a=1}^{f}\chi(a)\sum_{i=0}^{n}\binom{n}{i}\left(\frac{a-1+x}{f}\right)^{n-i}\\
&\quad\times(-1)^i\sum_{m=0}^{i}\sum_{0<m_1<\cdots<m_{r-1}<m+l}\frac{(-1)^m\sum_{j=0}^{l}(m+j)!\left\{{n \atop m+j}\right\}\binom{l}{j}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(m+l)^{k_r}}.
\end{aligned}
$$

Proof. Comparing the generating functions for $\{B_{n,\chi}^{(\mathbf{k})/l}(x)\}_n$ and $\{B_{n}^{(\mathbf{k})/l}(x)\}_n$, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\chi}^{(\mathbf{k})/l}(x)\frac{t^n}{n!} &= \frac{1}{f}\sum_{a=1}^{f}\chi(a)\frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-ft})}{(e^{ft}-1)^l}e^{(a-1+x)t}\\ &= \frac{1}{f}\sum_{a=1}^{f}\chi(a)\left(\sum_{n=0}^{\infty}B_n^{(\mathbf{k})/l}\left(\frac{a-1+x}{f}\right)\frac{(ft)^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(f^{n-1}\sum_{a=1}^{f}\chi(a)B_n^{(\mathbf{k})/l}\left(\frac{a-1+x}{f}\right)\right)\frac{t^n}{n!}.\end{aligned}$$

Applying the explicit formula of $B_n^{(\mathbf{k})/l}(x)$, we obtain the required formula. □

Considering Theorem 5, it suffices to show the following theorem to justify the algorithm for $B_{n,\chi}^{(\mathbf{k})/l}(x)$.

Theorem 6. *Given $\{b_{0,m}(\chi,a;x)\}_m$, define $b_{n+1,m}(\chi,a;x)$ $(n,m\geq 0)$ by the recurrence formula*

$$b_{n+1,m}(\chi,a;x) = (m+1)fb_{n,m+1}(\chi,a;x) - \left(m+l-\frac{a-1+x}{f}\right)fb_{n,m}(\chi,a;x). \tag{11}$$

Then, we have

$$\begin{aligned}b_{n,0}(\chi,a;x) = f^n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{a-1+x}{f}\right)^{n-i}\\ \times(-1)^i\sum_{m=0}^{i}(-1)^m\sum_{j=0}^{l}(m+j)!\left\{{i \atop m+j}\right\}\binom{l}{j}b_{0,m}(\chi,a;x).\end{aligned}$$

Proof. We use the generating function

$$g_n(\chi,a;x;t) = \sum_{m=0}^{\infty}b_{n,m}(\chi,a;x)t^m.$$

By recurrence formula (11), we have

$$\begin{aligned}g_n(\chi,a;x;t) &= f\sum_{m=0}^{\infty}\Bigg\{(m+1)(b_{n-1,m+1}(\chi,a;x)-b_{n-1,m}(\chi,a;x))\\ &\qquad+\left(1-l+\frac{a-1+x}{f}\right)b_{n-1,m}(\chi,a;x)\Bigg\}t^m\\ &= f\left(-l+\frac{a-1+x}{f}+(1-t)\frac{d}{dt}\right)g_{n-1}(\chi,a;x;t).\end{aligned}$$

Setting $h_n(\chi,a;x;t)=(t-1)^l g_n(\chi,a;x;t)$, we have

$$\begin{aligned} h_n(\chi,a;x;t) &= f\left(\frac{a-1+x}{f}-(t-1)\frac{d}{dt}\right)h_{n-1}(\chi,a;x;t) \quad (n\geq 1)\\ &= f^n\left(\frac{a-1+x}{f}-(t-1)\frac{d}{dt}\right)^n h_0(\chi,a;x;t)\\ &= f^n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{a-1+x}{f}\right)^{n-i}(-1)^i\left((t-1)\frac{d}{dt}\right)^i h_0(\chi,a;x;t). \end{aligned}$$

Applying Lemma 2 to $((t-1)\frac{d}{dt})^i h_0(\chi,a;x;t)$, we have

$$\begin{aligned} &\left((t-1)\frac{d}{dt}\right)^i h_0(\chi,a;x;t)\\ &=\sum_{m=0}^{i}\left\{ {i \atop m} \right\}(t-1)^m\\ &\quad\times\sum_{p=m}^{\infty}\sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}p(p-1)\cdots(p-(m-1))b_{0,p-j}(\chi,a;x)t^{p-m}. \end{aligned}$$

Here $b_{0,i}$ is taken to be 0 for any negative integer i. Setting $t=0$, we have

$$\begin{aligned} b_{n,0}(\chi,a;x) &= f^n\sum_{i=0}^{n}\binom{n}{i}\left(\frac{a-1+x}{f}\right)^{n-i}\\ &\qquad\times(-1)^i\sum_{m=0}^{i}(-1)^m\sum_{j=0}^{l}(m+j)!\left\{ {i \atop m+j} \right\}\binom{l}{j}b_{0,m}(\chi,a;x). \end{aligned}$$

Thus, we obtain Theorem 6. □

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