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A NOTE ON THE NUMBER OF REPRESENTATIONS OF A POSITIVE INTEGER AS A SUM OF GENERALIZED POLYGONAL NUMBERS

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Abstract

Recently, Jha has found identities that connect certain sums over the divisors of n to the number of representations of n as a sum of squares and triangular numbers. In this note, we state a generalized result that gives such relations for s-gonal numbers for any integer $s \geq 3$.

1. Introduction

Jha [3, 4] obtained two identities that connect certain sums over the divisors of n to the number of representations of n as sums of squares and sums of triangular numbers, respectively. Our objective is to show that these results can be generalized to the number of representations of n as a sum of any specific generalized polygonal number. We also obtain some corollaries, including Jha's results (Identities (3) and (8), below). In this section, we introduce two definitions and related tools to use in the later sections.

Definition 1 *For an integer $s \geq 3$, the generalized n^{th} s-gonal number is defined by*

$$F_s(n) := \frac{(s-2)n^2-(s-4)n}{2}, \quad n \in \mathbb{Z}.$$

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Henceforth, we call these numbers s-gonal numbers.

The generating function $G_s(q)$ of $F_s(n)$ is given by

$$G_s(q) := \sum_{n=-\infty}^{\infty} q^{F_s(n)} = f(q, q^{s-3}),$$

where $f(a, b)$ is Ramanujan's theta function defined by [1, p. 34]:

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

Also note the exceptional case that $G_3(q)$ generates each triangular number twice while $G_6(q)$ generates only once.

Definition 2 (Comtet, [2, p. 133]) *The partial Bell polynomials are the polynomials $B_{n,k} \equiv B_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$ in an infinite number of variables defined by the formal double series expansion:*

$$\sum_{n,k\geq 0} B_{n,k} \frac{t^n}{n!} u^k = \exp\left(u \sum_{m\geq 1} x_m \frac{t^m}{m!}\right).$$

For more equivalent definitions, exact expressions and further results involving the Bell polynomials, we refer to [2, Chap. 3.3].

The next section contains some lemmas and the main theorem. In the final section, we present some corollaries, including Jha's results (3) and (8).

2. Lemmas and Main Theorem 2.3

In the following, we present two lemmas that lead us to the main theorem, i.e., Theorem 2.3.

Lemma 2.1 *For any positive integer n, we have*

$$\sum_{d|n} \frac{1}{d} \left((-1)^d \delta_1\left(\frac{n}{d}, s-2\right) + \delta_2\left(\frac{n}{d}, s-2\right)\right)$$
$$= \frac{1}{n!} \sum_{k=1}^{n} (-1)^k (k-1)! B_{n,k}(G_s'(0), G_s''(0), \ldots, G_s^{n-k+1}(0)),$$

where, we define $\delta_1(m, v)$ and $\delta_2(m, v)$ for $v \geq 2$ as follows:

$$\delta_1(m, v) = \begin{cases} 2, & \text{if } m \equiv 1 \pmod 2,\ v = 2, \\ 1, & \text{if } m \equiv 1 \text{ or } (v-1) \pmod v,\ v \geq 3, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\delta_2(m,v)=\begin{cases}1, & \text{if } m\equiv 0 (\text{mod } v),\ v\geq 2,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. By Jacobi triple product identity [1, p. 35, Entry 19], we have

$$\begin{aligned}G_s(q)=f(q,q^{s-3})&=(-q;q^{s-2})_\infty(-q^{s-3};q^{s-2})_\infty(q^{s-2};q^{s-2})_\infty\\ &=\prod_{j=0}^{\infty}\left((1+q^{(s-2)j+1})(1+q^{(s-2)j+s-3})(1-q^{(s-2)(j+1)})\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\log G_s(q)&=\sum_{j=0}^{\infty}\left(\log(1+q^{(s-2)j+1})+\log(1+q^{(s-2)j+s-3})+\log(1-q^{(s-2)(j+1)})\right)\\ &=-\sum_{j=0}^{\infty}\sum_{\ell=1}^{\infty}\left(\frac{(-1)^\ell}{\ell}q^{((s-2)j+1)\ell}+\frac{(-1)^\ell}{\ell}q^{((s-2)j+s-3)\ell}+\frac{1}{\ell}q^{(s-2)(j+1)\ell}\right)\\ &=-\sum_{\substack{j\geq 1\\ j\equiv 1 \pmod{s-2}}}\sum_{\ell\geq 1}\frac{(-1)^\ell}{\ell}q^{j\ell}-\sum_{\substack{j\geq 1\\ j\equiv -1 \pmod{s-2}}}\sum_{l\geq 1}\frac{(-1)^\ell}{\ell}q^{j\ell}\\ &\quad-\sum_{\substack{j\geq 1\\ j\equiv 0 \pmod{s-2}}}\sum_{\ell\geq 1}\frac{1}{\ell}q^{j\ell}\\ &=-\sum_{n\geq 1}q^n\left(\sum_{d|n}\frac{1}{d}\left((-1)^d\delta_1\left(\frac{n}{d},s-2\right)+\delta_2\left(\frac{n}{d},s-2\right)\right)\right),\end{aligned}\tag{1}$$

where the given definitions of $\delta_1(m,v)$ and $\delta_2(m,v)$ follow naturally.

Now, let the Taylor series expansion of $G_s(q)$ be

$$G_s(q)=\sum_{n\geq 0}g_n\frac{q^n}{n!}.$$

Then, from [2, p. 140, (5a) and (5b)], we have the following result:

$$\log G_s(q)=\sum_{n\geq 1}L_n\frac{q^n}{n!},\tag{2}$$

where

$$L_n=L_n(g_1,g_2,\ldots,g_n)=\sum_{k=1}^{n}(-1)^k(k-1)!B_{n,k}(g_1,g_2,\ldots,g_n).$$

Comparing (1) and (2), we arrive at the desired result. □

The following lemma is similar to Lemma 2 in [4]. So we omit the proof.

Lemma 2.2 *Let $t_{s,j}(n)$ denote the number of representations of n as a sum of j s-gonal numbers. Then, we have*

$$B_{n,k}(G_s'(0), G_s''(0), \cdots, G_s^{n-k+1}(0)) = \frac{n!}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}t_{s,j}(n).$$

With the use of the above lemmas, we are now able to prove the following theorem.

Theorem 2.3 *For all positive integers n, s with $s \geq 4$, we have*

$$\sum_{d|n}\frac{1}{d}\left((-1)^d\delta_1\left(\frac{n}{d}, s-2\right) + \delta_2\left(\frac{n}{d}, s-2\right)\right) = \sum_{j=1}^{n}\frac{(-1)^j}{j}\binom{n}{j}t_{s,j}(n).$$

Proof. Our proof of the theorem is similar to the one given in [4, Theorem 1]. From Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\sum_{d|n}\frac{1}{d}\left((-1)^d\delta_1(\frac{n}{d}, s-2) + \delta_2(\frac{n}{d}, s-2)\right)\\
&\qquad = \frac{1}{n!}\sum_{k=1}^{n}(-1)^k(k-1)!\frac{n!}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}t_{s,j}(n)\\
&\qquad = \sum_{k=1}^{n}\sum_{j=1}^{k}\frac{(-1)^j}{k}\binom{k}{j}t_{s,j}(n)\\
&\qquad = \sum_{j=1}^{n}(-1)^j t_{s,j}(n)\sum_{k=j}^{n}\frac{1}{k}\binom{k}{j}\\
&\qquad = \sum_{j=1}^{n}(-1)^j\frac{1}{j}\binom{n}{j}t_{s,j}(n),
\end{aligned}$$

where we have used the result

$$\sum_{k=j}^{n}\frac{1}{k}\binom{k}{j} = \frac{1}{j}\binom{n}{j}$$

that can be derived easily from the identity

$$\binom{k}{j-1} = \binom{k+1}{j} - \binom{k}{j}.$$

□

3. Corollaries

In this section, we present the results in [3, 4] and some other interesting results as corollaries of Theorem 2.3.

Corollary 3.1 (Jha [4, Theorem 1]) *For any positive integer n, we have*

$$\sum_{\substack{d|n \\ d \text{ odd}}} \frac{2(-1)^n}{d} = \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{4,j}(n). \tag{3}$$

Proof. Setting $s = 4$ in Theorem 2.3, we find that

$$\sum_{\substack{d|n \\ \frac{n}{d} \text{ odd}}} \frac{2(-1)^d}{d} + \sum_{\substack{d|n \\ \frac{n}{d} \text{ even}}} \frac{1}{d} = \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{4,j}(n). \tag{4}$$

To prove that (4) is equivalent to (3), it is enough to show that

$$\sum_{\substack{d|n \\ d \text{ odd}}} \frac{2(-1)^n}{d} = \sum_{\substack{d|n \\ \frac{n}{d} \text{ odd}}} \frac{2(-1)^d}{d} + \sum_{\substack{d|n \\ \frac{n}{d} \text{ even}}} \frac{1}{d}. \tag{5}$$

We complete it by considering the following three possible cases of n.

`Case I,` n `is odd`: In this case, it is easily seen that both sides of (5) become

$$-\sum_{\substack{d|n \\ d \text{ odd}}} \frac{2}{d}.$$

`Case II,` $n = 2^k$`, where` $k \geq 1$: In this case, the left-hand side of (5) is equal to 2. Now, the right-hand side of (5) becomes

$$\begin{aligned}
\sum_{\substack{d|2^k \\ \frac{2^k}{d} \text{ odd}}} \frac{2(-1)^d}{d} + \sum_{\substack{d|2^k \\ \frac{2^k}{d} \text{ even}}} \frac{1}{d} &= \frac{1}{2^{k-1}} + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}}\right) \\
&= \frac{1}{2^{k-1}} + \frac{2^k - 1}{2^{k-1}} \\
&= 2.
\end{aligned}$$

Thus, (5) is true for this case.

`Case III,` $n = 2^k m$ `where` $k \geq 1$ `and` m `is odd and greater than 1:` The left-hand side of (5) becomes

$$\sum_{\substack{d|n \\ d \text{ odd}}} \frac{2}{d}. \tag{6}$$

Again, the right-hand side of (5) is

$$\begin{aligned}
&\sum_{\substack{d|2^k m \\ \frac{2^k m}{d} \text{ odd}}} \frac{2(-1)^d}{d} + \sum_{\substack{d|2^k m \\ \frac{2^k m}{d} \text{ even}}} \frac{1}{d} \\
&= \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{2}{2^k d} + \Big(\sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{d} + \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{2d} + \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{2^2 d} + \cdots + \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{2^{k-1} d} \Big) \\
&= \left(\frac{1}{2^{k-1}} + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}} \right) \sum_{\substack{d|n \\ d \text{ odd}}} \frac{1}{d} \\
&= \sum_{\substack{d|n \\ d \text{ odd}}} \frac{2}{d}.
\end{aligned} \tag{7}$$

From (6) and (7), we conclude that (5) is true for this case as well. □

Corollary 3.2 (Jha [3, Theorem 1]) *For any positive integer n, we have*

$$\sum_{d|n} \frac{1 + 2(-1)^d}{d} = \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{6,j}(n). \tag{8}$$

Proof. Setting $s = 6$ in Theorem 2.3, we obtain

$$\sum_{\substack{d|n \\ \frac{n}{d} \text{ odd}}} \frac{(-1)^d}{d} + \sum_{\substack{d|n \\ \frac{n}{d} \equiv 0 \pmod 4}} \frac{1}{d} = \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{6,j}(n). \tag{9}$$

To prove the equivalence of (9) and (8), it is enough to show that

$$\sum_{\substack{d|n \\ \frac{n}{d} \text{ odd}}} \frac{(-1)^d}{d} + \sum_{\substack{d|n \\ \frac{n}{d} \equiv 0 \pmod 4}} \frac{1}{d} = \sum_{d|n} \frac{1 + 2(-1)^d}{d}. \tag{10}$$

We show it by considering three possible cases of n.

`Case I,` n `is odd`: In this case, we notice that both sides of (10) become

$$-\sum_{\substack{d|n \\ d \text{ odd}}} \frac{1}{d}.$$

`Case II,` $n = 2m$ `with` m `odd`: In this case, the left-hand side of (10) is

$$\sum_{\substack{d|2m \\ \frac{2m}{d} \text{ odd}}} \frac{(-1)^d}{d} + \sum_{\substack{d|2m \\ \frac{2m}{d} \equiv 0 \pmod 4}} \frac{1}{d} = \sum_{\substack{2d|2m \\ d \text{ odd}}} \frac{(-1)^{2d}}{2d} = \frac{1}{2} \sum_{\substack{d|2m \\ d \text{ odd}}} \frac{1}{d}.$$

The right-hand side of (10) is

$$\sum_{\substack{d|2m \\ d \text{ odd}}} \frac{1 + 2(-1)^d}{d} + \sum_{\substack{d|2m \\ d \text{ even}}} \frac{1 + 2(-1)^d}{d} = -\sum_{\substack{d|2m \\ d \text{ odd}}} \frac{1}{d} + \frac{3}{2} \sum_{\substack{d|2m \\ d \text{ odd}}} \frac{1}{d} = \frac{1}{2} \sum_{\substack{d|2m \\ d \text{ odd}}} \frac{1}{d}.$$

Thus, (10) is true in this case.

`Case III,` $n = 2^k m$ `with` m `odd and` $k \geq 2$: In this case, the left-hand side of (10) is

$$\begin{aligned} &\sum_{\substack{d|2^k m \\ \frac{2^k m}{d} \text{ odd}}} \frac{(-1)^d}{d} + \sum_{\substack{d|2^k m \\ \frac{2^k m}{d} \equiv 0 \pmod 4}} \frac{1}{d} \\ &= \frac{1}{2^k} \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{d} + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \frac{1}{2^{k-2}}\right) \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{d} \qquad (11) \\ &= \left(2 - \frac{3}{2^k}\right) \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{d}. \end{aligned}$$

Again, the right-hand side of (10) is

$$\begin{aligned}\sum_{d\mid 2^k m}\frac{1+2(-1)^d}{d}\\ &= \sum_{\substack{d\mid 2^k m\\ d \text{ odd}}}\frac{1+2(-1)^d}{d} + \sum_{\substack{d\mid 2^k m\\ d=2d_1,\ d_1 \text{ odd}}}\frac{1+2(-1)^d}{d} + \sum_{\substack{d\mid 2^k m\\ d=2^2d_1,\ d_1 \text{ odd}}}\frac{1+2(-1)^d}{d}\\ &\quad + \sum_{\substack{d\mid 2^k m\\ d=2^3d_1,\ d_1 \text{ odd}}}\frac{1+2(-1)^d}{d} + \cdots + \sum_{\substack{d\mid 2^k m\\ d=2^kd_1,\ d_1 \text{ odd}}}\frac{1+2(-1)^d}{d} \qquad (12)\\ &= -\sum_{\substack{d\mid 2^k m\\ d \text{ odd}}}\frac{1}{d} + \frac{3}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\frac{1}{2^{k-1}}\right)\sum_{\substack{d\mid 2^k m\\ d \text{ odd}}}\frac{1}{d}\\ &= \left(2-\frac{3}{2^k}\right)\sum_{\substack{d\mid 2^k m\\ d \text{ odd}}}\frac{1}{d}.\end{aligned}$$

From (11) and (12), we arrive at (10) for this case. □

Corollary 3.3 *For any positive integer n, we have*

$$\sum_{\substack{d\mid n\\ \frac{n}{d}\equiv 1 \text{ or } 2 \pmod 3}}\frac{(-1)^d}{d} + \sum_{\substack{d\mid n\\ \frac{n}{d}\equiv 0 \pmod 3}}\frac{1}{d} = \sum_{j=1}^{n}\frac{(-1)^j}{j}\binom{n}{j}t_{5,j}(n).$$

Proof. The result follows by setting $s=5$ in Theorem 2.3. □

Corollary 3.4 *Let n be a positive integer and $\sigma(n)$ denote the sum of positive divisors of n. Let p be an odd prime such that $p \mid n$ and $p^2 \nmid n$. If $\frac{n}{p} \equiv 1$ or $p-1 \pmod p$, then*

$$\sum_{j=1}^{n}\frac{(-1)^j}{j}\binom{n}{j}t_{p+2,j}(n) = \frac{\sigma(n)}{n} - \frac{2}{p}.$$

Otherwise,

$$\sum_{j=1}^{n}\frac{(-1)^j}{j}\binom{n}{j}t_{p+2,j}(n) = \frac{\sigma(n)}{n} - \frac{1}{p}.$$

Proof. Let p be as stated in the corollary. Setting $s = p+2$ in Theorem 2.3, it follows that, if $\frac{n}{p} \equiv 1$ or $p-1 \pmod{p}$, then

$$\sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{p+2,j}(n) = -\frac{1}{p} + \sum_{\substack{d|n \\ d \neq p}} \frac{1}{d}.$$

Otherwise,

$$\sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{p+2,j}(n) = \sum_{\substack{d|n \\ d \neq p}} \frac{1}{d}.$$

□

As

$$\sum_{\substack{d|n \\ d \neq p}} \frac{1}{d} = \sum_{d|n} \frac{1}{d} - \frac{1}{p} = \frac{1}{n} \sum_{d|n} \frac{n}{d} - \frac{1}{p} = \frac{1}{n} \sum_{d|n} d - \frac{1}{p} = \frac{\sigma(n)}{n} - \frac{1}{p},$$

we readily arrive at the desired results.

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