A JACOBI SYMBOL CRITERION INVOLVING k-FIBONACCI AND k-LUCAS NUMBERS AND INTEGER POINTS ON ELLIPTIC CURVES

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Received: 11/30/22, Accepted: 5/30/23, Published: 6/12/23

Abstract

In 1989, Ming Luo showed that the Fibonacci number U_n is triangular if and only if $n = \pm 1, 2, 4, 8,$ or 10. Over the course of his demonstration, he established a Jacobi symbol criterion. Moreover, he observed that this problem is equivalent to finding all integer points on two elliptic curves. In this paper, we shall prove a Jacobi symbol criterion for more general families of binary recurrences. In addition, applying the criterion and elementary methods, we shall determine all integer points on the elliptic curves $y^2 = 5x^2(x+3)^2 + 4(-1)^n$.

1. Introduction

Let $U = (U_n)_{n\geq 0}$ be a binary recurrence sequence defined by initial terms $U_0, U_1 \in \mathbb{Z}$ and the recurrence relation

$$U_{n+2} = AU_{n+1} + BU_n \qquad (n \geq 0),$$

where A and B are non-zero integers. Let $\alpha = (A + \sqrt{D})/2$ and $\beta = (A - \sqrt{D})/2$ be the zeros of the characteristic polynomial of U given by $p(x) = x^2 - Ax - B$,

DOI: 10.5281/zenodo.8028979

[1]E.S. was partially supported by CNPq (Grant 409198/2021-8) and FAPEAL (Grant E:60030.0000002581/2022).

where $D = A^2 + 4B$ is the discriminant of U and

$$a = U_1 - \beta U_0, \quad b = U_1 - \alpha U_0, \quad C = U_1^2 - AU_0U_1 - BU_0^2.$$

The sequence U is called non-degenerate if $C \neq 0$ and α/β is not a root of unity. It is well-known that if U is non-degenerate, then we have that

$$U_n = \frac{a\alpha^n - b\beta^n}{\alpha - \beta} \tag{1}$$

for all integers $n \geq 0$. In fact, (1) holds whenever $\alpha - \beta \neq 0$.

From this point on, we assume that $B = 1$, $A \geq 1$ and U is non-degenerate. Therefore, it is also well-known that U has a so-called associate sequence $V = (V_n)_{n\geq 0}$ for which

$$V_n^2 - DU_n^2 = 4C(-1)^n$$

holds for all $n \geq 0$, where $V_0 = 2U_1 - AU_0$, $V_1 = AU_1 + 2BU_0$ and V satisfies the same recurrence relation of U (for more details see [8]).

It follows from our assumption that U and V are the Fibonacci and Lucas sequences, respectively, for $U_0 = 0$, $U_1 = 1$ and $A = 1$, since we assume $B = 1$. These sequences are famous for having several identities and interesting properties associated with them. For these reasons, it is common to find several generalizations of these sequences. Many authors consider the problem of studying the binary recurrence, the so-called k-Fibonacci sequence $F_k = (F_{k,n})_{n\geq 0}$ given by $F_{k,0} = 0$, $F_{k,1} = 1$ and

$$F_{k,n+2} = kF_{k,n+1} + F_{k,n}$$

for $n \geq 0$, and its associated k-Lucas sequence $L_k = (L_{k,n})_{n\geq 0}$, where $L_{k,0} = 2$ and $L_{k,1} = k$ (for more details see [1, 2, 3, 4]).

There are many articles concerning the mixed exponential-polynomial Diophantine equation

$$U_n = P(x),$$

where $P \in \mathbb{Z}[x]$ is a polynomial. In particular, there is a special interest in the case that $P(x)$ has degree 2. Since $Y = \pm L_{k,n}$ and $X = \pm F_{k,n}$ are the complete set of solutions of the Pell equations

$$Y^2 - (k^2 + 4)X^2 = 4(-1)^n,$$

the condition $F_{k,n} = P(x)$ is equivalent to finding all integer solutions of the Diophantine equation $Y^2 - (k^2+4)(P(x))^2 = \pm 4$ (for more details see [5]). In particular, if $P(x)$ has degree 2, this problem is equivalent to finding all integer points on these two elliptic curves.

Thus, given the integers a, b and c with $a \neq 0$, we have that the solutions of the equation $\pm F_{k,n} = a(x + b)(x + c)$ are the X-coordinates of integer points on the following elliptic curves:

$$y^2 = a^2(k^2+4)(x+b)^2(x+c)^2 + 4(-1)^n. \tag{2}$$

For some values of the parameters in Equation (2), we can determine all integer points on the previous curves by estimating linear forms in elliptic logarithms. For example, the `IntegralQuarticPoints()` subroutine of Magma can be used to determine the solutions of some equations of this type. However, in this paper, we follow the ideas in [10, 11, 12] and consider the problem of determining these integer points with elementary number theory methods, especially the Jacobi symbol.

Suppose that

$$\pm F_{k,n} = a(x+b)(x+c) \tag{3}$$

with a, b, c integers with $a \neq 0$. Note that the Equations in (3) have a solution if and only if $(\pm 4aF_{k,n} + \Delta)$ is a perfect square, where $\Delta = a^2(b+c)^2 - 4a^2bc = a^2(b-c)^2 = d^2$. In this case, the Jacobi symbol

$$\left(\frac{\pm 4aF_{k,n} + d^2}{s}\right) = 1$$

is valid for all odd positive integers s. The Jacobi symbol is an important tool in the study of Diophantine equations involving perfect squares (for more examples see [6, 7, 10, 11]). In order to determine an s such that the Jacobi symbol above is -1, which implies that $(\pm 4aF_{k,n} + \Delta)$ is not a perfect square, we shall prove the following Jacobi Symbol Criterion, which we believe to have independent interest.

Theorem 1 (Jacobi Symbol Criterion)**.** *Let a, d, k be positive integers, where d and k are odd with $d^2 > 8a$. If $n \equiv \pm 2 \pmod 6$ and $\gcd(a, L_n) = 1$, then*

$$\left(\frac{\pm 4aF_{k,2n} + d^2}{L_{k,2n}}\right) = -\left(\frac{\pm 8aF_{k,n} + d^2L_{k,n}}{64a^2 + (k^2+4)d^4}\right),$$

whenever the right Jacobi symbol is proper.

We organize this paper as follows. In Section 2, we will prove the Jacobi Symbol Criterion given by Theorem 1. In Section 3, we will use this criterion and some auxiliary lemmas to determine all integer points on the elliptic curves $y^2 = 5x^2(x+3)^2 + 4(-1)^n$.

2. Proof of Jacobi Symbol Criterion

Proof of Theorem 1. Firstly, let $(L_{k,n})_{n\geq 0}$ be the associate sequence given by $L_{k,n+2} = kL_{k,n+1} + L_{k,n}$, for $n \geq 0$ and initial terms $L_{k,0} = 2$ and $L_{k,1} = k$. The proof of Theorem 1 requires the following identities:

$$L_{k,2n} = L_{k,n}^2 - 2(-1)^n; \quad (4)$$
$$F_{k,2n} = F_{k,n}L_{k,n}; \quad (5)$$
$$2L_{k,2n} = (k^2+4)F_{k,n}^2 + L_{k,n}^2. \quad (6)$$

These identities are generalizations of well-known identities associated with the Fibonacci and Lucas sequences; for more details see [3]. Since k is odd and $n \equiv \pm 2 \pmod 6$, we have that $L_{k,n} \equiv 3 \pmod 4$ and $L_{k,2n} \equiv 7 \pmod 8$. So, we can consider the Jacobi symbol

$$\left(\frac{\pm 4aF_{k,2n} + d^2}{L_{k,2n}}\right).$$

Moreover, the Jacobi symbol $(2 \mid L_{k,2n}) = 1$, and thus

$$\left(\frac{\pm 4aF_{k,2n} + d^2}{L_{k,2n}}\right) = \left(\frac{2}{L_{k,2n}}\right)\left(\frac{\pm 4aF_{k,2n} + d^2}{L_{k,2n}}\right) = \left(\frac{\pm 8aF_{k,2n} + 2d^2}{L_{k,2n}}\right).$$

By (4), we have $2 \equiv L_{k,n}^2 \pmod{L_{k,2n}}$, since n is an even integer. Further, using (5) we obtain

$$\left(\frac{\pm 4aF_{k,2n} + d^2}{L_{k,2n}}\right) = \left(\frac{\pm 8aF_{k,n}L_{k,n} + d^2L_{k,n}^2}{L_{k,2n}}\right).$$

Note that $8a < d^2$ by hypothesis, thus $\pm 8aF_{k,n}L_{k,n} + d^2L_{k,n}^2 > 0$. So, by quadratic reciprocity it follows that

$$\left(\frac{\pm 4aF_{k,2n} + d^2}{L_{k,2n}}\right) = \left(\frac{L_{k,2n}}{\pm 8aF_{k,n}L_{k,n} + d^2L_{k,n}^2}\right) = \left(\frac{L_{k,2n}}{L_{k,n}}\right)\left(\frac{L_{k,2n}}{\pm 8aF_{k,n} + d^2L_{k,n}}\right),$$

since $d^2L_{k,n}^2 \equiv 1 \pmod 4$. Now, by (4) and using $L_{k,n} \equiv 3 \pmod 4$ we obtain $(L_{k,2n} \mid L_{k,n}) = -(2 \mid L_{k,n})$. Furthermore, (6) gives us

$$\left(\frac{\pm 4aF_{k,2n} + d^2}{L_{k,2n}}\right) = -\left(\frac{2}{L_{k,n}}\right)\left(\frac{\frac{1}{2}((k^2+4)F_{k,n}^2 + L_{k,n}^2)}{\pm 8aF_{k,n} + d^2L_{k,n}}\right).$$

In order to exchange the sum $((k^2+4)F_{k,n}^2 + L_{k,n}^2)/2$ for a product, we can multiply it by a Jacobi symbol of a suitable perfect square. For this, we use the following identity

$$\frac{16a^2d^2((k^2+4)F_{k,n}^2 + L_{k,n}^2)}{2} = (\pm 8aF_{k,n} + d^2L_{k,n})Q \mp (64a^3 + (k^2+4)ad^4)F_{k,n}L_{k,n},$$

where we consider $Q = (k^2+4)ad^2F_{k,n} + 8a^2L_{k,n}$ in the case $8aF_{k,n} + d^2L_{k,n}$ and $Q = -(k^2+4)ad^2F_{k,n} + 8a^2L_{k,n}$ in other case. It follows that

$$\left(\frac{\pm 4aF_{k,2n} + d^2}{L_{k,2n}}\right) = -\left(\frac{2}{L_{k,n}}\right)\left(\frac{\mp(64a^3 + (k^2+4)ad^4)}{\pm 8aF_{k,n} + d^2L_{k,n}}\right)\left(\frac{F_{k,n}L_{k,n}}{\pm 8aF_{k,n} + d^2L_{k,n}}\right).$$

Now,

$$\left(\frac{L_{k,n}}{\pm 8aF_{k,n}+d^2L_{k,n}}\right) = -\left(\frac{\pm 8aF_{k,n}+d^2L_{k,n}}{L_{k,n}}\right) = \mp\left(\frac{2a}{L_{k,n}}\right)\left(\frac{F_{k,n}}{L_{k,n}}\right),$$

and $n \equiv \pm 2 \pmod 6$ implies $F_{k,n} \equiv (\pm 1)(-1)^{\frac{k-1}{2}} \pmod 4$, thus

$$\left(\frac{F_{k,n}}{\pm 8aF_{k,n}+d^2L_{k,n}}\right) = (\pm 1)(-1)^{\frac{k-1}{2}}\left(\frac{\pm 8aF_{k,n}+d^2L_{k,n}}{F_{k,n}}\right) = \left(\frac{F_{k,n}}{L_{k,n}}\right).$$

From this,

$$\begin{aligned}\left(\frac{\pm 4aF_{k,2n}+d^2}{L_{k,2n}}\right) &= \pm\left(\frac{2}{L_{k,n}}\right)\left(\frac{\mp(64a^3+(k^2+4)ad^4)}{\pm 8aF_{k,n}+d^2L_{k,n}}\right)\left(\frac{2a}{L_{k,n}}\right)\\ &= -\left(\frac{a}{L_{k,n}}\right)\left(\frac{a}{\pm 8aF_{k,n}+d^2L_{k,n}}\right)\left(\frac{64a^2+(k^2+4)d^4}{\pm 8aF_{k,n}+d^2L_{k,n}}\right)\\ &= -\left(\frac{a}{\pm 8aL_{k,n}F_{k,n}+d^2L_{k,n}^2}\right)\left(\frac{64a^2+(k^2+4)d^4}{\pm 8aF_{k,n}+d^2L_{k,n}}\right).\end{aligned}$$

Writing $a = 2^s b$ with $2 \nmid b$, we obtain

$$\begin{aligned}\left(\frac{a}{\pm 8aL_{k,n}F_{k,n}+d^2L_{k,n}^2}\right) &= \left(\frac{2^s b}{\pm 8aL_{k,n}F_{k,n}+d^2L_{k,n}^2}\right)\\ &= \left(\frac{2^s}{\pm 8aF_{k,n}L_{k,n}+d^2L_{k,n}^2}\right)\left(\frac{d^2L_{k,n}^2}{b}\right)\\ &= 1,\end{aligned}$$

since $\pm 8aF_{k,n}L_{k,n}+d^2L_{k,n}^2 \equiv 1 \pmod 8$ and $b \mid a$. Finally,

$$\left(\frac{\pm 4aF_{k,2n}+d^2}{L_{k,2n}}\right) = -\left(\frac{64a^2+(k^2+4)d^4}{\pm 8aF_{k,n}+d^2L_{k,n}}\right) = -\left(\frac{\pm 8aF_{k,n}+d^2L_{k,n}}{64a^2+(k^2+4)d^4}\right)$$

and this concludes the proof. □

3. The Curves $y^2 = 5x^2(x+3)^2 + 4(-1)^n$

In this section, we consider the elliptic curves $y^2 = 5x^2(x+3)^2+4(-1)^n$ to exemplify the method. It is well-known that $X = \pm F_n$ and $Y = \pm L_n$ are the complete set of solutions of the Diophantine equations

$$Y^2 - 5X^2 = 4(-1)^n.$$

So, we conclude that the curves $y^2 = 5x^2(x+3)^2 + 4(-1)^n$ have integer points if and only if the equation $\pm F_n = x(x+3)$ has a solution. We shall prove the following theorem.

Theorem 2. *If (x, y) is a integer point on the elliptic curves*

$$y^2 = 5x^2(x+3)^2 + 4(-1)^n,$$

then $(x, y, n) \in \{(-3, \pm 2, 0), (-2, \pm 4, 3), (-1, \pm 4, 3), (0, \pm 2, 0)\}$.

Clearly, $-F_n = x(x+3)$ if and only if $x \in \{-3, -2, -1, 0\}$. Further, if $n = 0$ and $x = 0$, then $F_n = x(x+3)$. We shall prove that there are no other solutions. To achieve this, we will use the Jacobi Symbol Criterion and some auxiliary lemmas.

The following lemma will be used to reduce F_n modulo L_{2k}, in order to apply Theorem 1.

Lemma 1. *For all integers k and m, and g odd,*

$$F_{2kg+m} \equiv \begin{cases} F_{2k+m} \pmod{L_{2k}} & if \quad g \equiv 1 \pmod 4 \\ -F_{2k+m} \pmod{L_{2k}} & if \quad g \equiv 3 \pmod 4. \end{cases}$$

The proof of this lemma can be found in [11]. In our proof of Theorem 2 we use the fact that $n = 2 \cdot 2^w \cdot 5^2 \cdot 7t$ with $w \geq 3$ and t odd. In order to guarantee this condition we will prove the following lemma.

Lemma 2. *If $F_n = x(x+3)$ for some $n, x \in \mathbb{N}$, then $n \equiv 0 \pmod{2800}$.*

Proof. If n and x are integers such that $F_n = x(x+3)$, then $4F_n + 9$ is a perfect square. Hence, the Jacobi symbol $(4F_n + 9 \mid Q)$ is equal to 1 for every odd positive integer Q. So the idea of the proof is to show that if n is not congruent to 0 modulo 2800, then there is an odd prime Q such that $(4F_n + 9 \mid Q) = -1$.

The proof will be divided into five steps. In each step, we use some Jacobi symbol properties and that $(F_n)_{n\geq 0}$ is periodic modulo Q.

Step 1. $n \equiv 0 \pmod 4$.

In fact, F_n is periodic modulo 3 and 7 with period 8 and 16, respectively, and

$$\left(\frac{4F_n + 9}{3}\right) = -1, \qquad \text{if } n \equiv 3, 5, \text{or } 6 \pmod 8;$$
$$\left(\frac{4F_n + 9}{7}\right) = -1, \qquad \text{if } n \equiv \pm 1, \pm 2, \pm 3, \pm 6, \text{or} \pm 7 \pmod{16}.$$

Thus, if $(4F_n + 9 \mid Q) = 1$ for all odd positive integers Q, then $n \equiv 0 \pmod 4$.

Step 2. $n \equiv 0 \pmod{20}$.

By step 1, we have that $n \equiv 0, 4, 8, 12$, or $16 \pmod{20}$ and $n \equiv 0, 4, 8, 2$, or $6 \pmod{10}$. Using that F_n is periodic modulo 5 and modulo 11 with periods 20 and 10, respectively, and

$$\left(\frac{4F_n + 9}{Q}\right) = -1, \quad \text{if} \quad (n, Q) \in \{(4, 11), (8, 5), (12, 11), (16, 5)\},$$

we get $n \equiv 0 \pmod{20}$.

Step 3. $n \equiv 0 \pmod{100}$.

Since $n \equiv 0 \pmod{20}$, we have $n \equiv 0, \pm 20, \pm 40, \pm 60, \pm 80$, or $100 \pmod{200}$. Note that F_n is periodic modulo 401 with period 200 and

$$\left(\frac{4F_n + 9}{401}\right) = -1, \quad \text{if} \quad n \equiv \pm 20, 40, 60, \text{or} \pm 80 \pmod{200}.$$

Moreover, if $n \equiv 140 \pmod{200}$, then $n \equiv 40 \pmod{100}$ and if $n \equiv 160 \pmod{200}$, then $n \equiv 10 \pmod{50}$. Using the fact that F_n is periodic modulo 3001 and 101 with periods 100 and 50, respectively, and

$$\left(\frac{4F_n + 9}{Q}\right) = -1, \quad \text{if} \quad (n, Q) \in \{(40, 3001), (10, 101)\},$$

we obtain that $n \equiv 0 \pmod{100}$.

Step 4. $n \equiv 0 \pmod{700}$.

First of all, F_n is periodic modulo 13 and 29 with periods 28 and 14, respectively. Using that $100k \equiv 0, 16, 4, 20, 8, 24$, or $12 \pmod{28}$ and $100k \equiv 0, 2, 4, 8, 10$, or $12 \pmod{14}$ for $k = 0, 1, 2, 3, 4, 5$, or 6, respectively, we get $(4F_n + 9 \mid Q) = -1$ if $(n, Q) \in \{(16, 13), (4, 13), (6, 29), (8, 13), (10, 29)\}$. Thus, either $n \equiv 0 \pmod{700}$ as we claim or $n \equiv 600 \pmod{700}$.

Note that F_n is periodic modulo 281 and 2801 with periods 56 and 1400, respectively, and $n \equiv 40$, or $12 \pmod{56}$ and $n \equiv 600$, or $1300 \pmod{1400}$ in the case $n \equiv 600 \pmod{700}$, but we have that $(4F_n + 9 \mid Q) = -1$ if $(n, Q) \in \{(600, 2801), (12, 281)\}$, so we conclude that $n \equiv 0 \pmod{700}$.

Step 5. $n \equiv 0 \pmod{2800}$.

Finally, as $n \equiv 0 \pmod{700}$ and F_n is periodic modulo 47 and 1601 with periods 32 and 160, respectively, we consider that $700k \equiv 0, 28, 24, 20, 16, 12, 8$, or $4 \pmod{32}$ and $700k \equiv 0, 60, 120, 20, 80, 140, 40$, or $100 \pmod{160}$, where we take $k = 0, 1, 2, 3, 4, 5, 6$, or 7, respectively. Since the Jacobi symbol $(4F_n + 9 \mid Q) = -1$ whenever $(n, Q) \in \{(28, 47), (24, 47), (20, 47), (140, 1601), (8, 47), (100, 1601)\}$, so we get $k = 0$ or $k = 4$. Hence, we conclude $n \equiv 0 \pmod{2800}$ and this completes the proof. $\square$

For the purpose of simplifying the Jacobi Symbol Criterion proved in Theorem 1, we shall prove the following lemma. Before that, let $\nu_p(r)$ be the p-adic valuation of a integer r, i.e., the exponent of the highest power of a prime p which divides r. The p-adic valuation of a Fibonacci number was completely characterized, see [9]. For instance, if $m \equiv 0 \pmod 8$ we have $\nu_7(F_m) = \nu_7(m) + 1$, and now we have conditions to prove the following lemma.

Lemma 3. *If $m \equiv 0 \pmod{16}$, then $(\pm 8F_m + 9L_m \mid 7) = 1$.*

Proof. Note that $7 \mid F_{m/2}$, since $m/2 \equiv 0 \pmod 8$ and $\nu_7(F_{m/2}) = \nu_7(m/2)+1 \geq 1$. Thus, by (6),

$$\left(\frac{L_m}{7}\right) = \left(\frac{\frac{1}{2}(5F_{m/2}^2 + L_{m/2}^2)}{7}\right) = \left(\frac{10F_{m/2}^2 + 2L_{m/2}^2}{7}\right) = \left(\frac{2L_{m/2}^2}{7}\right) = 1.$$

Therefore,

$$\left(\frac{\pm 8F_m + 9L_m}{7}\right) = \left(\frac{9L_m}{7}\right) = \left(\frac{L_m}{7}\right) = 1$$

and we conclude the proof of the lemma. □

Now, considering $a = 1$, $k = 1$, $d = 3$ and $n \equiv \pm 16 \pmod{48}$ in the Jacobi Symbol Criterion and applying the lemma above, we obtain

$$\begin{aligned}\left(\frac{\pm 4F_{2n} + 9}{L_{2n}}\right) &= -\left(\frac{\pm 8F_n + 9L_n}{64 + 5 \cdot 81}\right) = -\left(\frac{\pm 8F_n + 9L_n}{469}\right) \\ &= -\left(\frac{\pm 8F_n + 9L_n}{7}\right)\left(\frac{\pm 8F_n + 9L_n}{67}\right) \\ &= -\left(\frac{\pm 8F_n + 9L_n}{67}\right),\end{aligned}$$

where we use the fact that $n \equiv \pm 16 \pmod{48}$ implies $n \equiv \pm 2 \pmod 6$ and $n \equiv 0 \pmod{16}$.

The following lemma will be useful to calculate $(\pm 8F_n + 9L_n \mid 67)$ and we use the Sage Mathematics Software System [13] to simplify the calculations of its proof.

Lemma 4. *Let $w \geq 3$ be a positive integer. If $w \equiv 0, 1, 2, 3, 4, 5, 6,$ or $7 \pmod 8$, then we have*

$$\begin{array}{lcllllllll l} F_{2^w} & \equiv & 18, & 62, & 64, & 21, & 49, & 5, & 3, & \textit{or } 46 & \pmod{67}, \\ L_{2^w} & \equiv & 63, & 14, & 60, & 47, & 63, & 14, & 60, & \textit{or } 47 & \pmod{67}, \\ F_{2^w \cdot 7} & \equiv & 4, & 65, & 37, & 10, & 63, & 2, & 30, & \textit{or } 57 & \pmod{67}, \\ L_{2^w \cdot 7} & \equiv & 33, & 15, & 22, & 13, & 33, & 15, & 22, & \textit{or } 13 & \pmod{67}, \\ F_{2^w \cdot 5^2} & \equiv & 21, & 49, & 5, & 3, & 46, & 18, & 62, & \textit{or } 64 & \pmod{67}, \\ L_{2^w \cdot 5^2} & \equiv & 47, & 63, & 14, & 60, & 47, & 63, & 14, & \textit{or } 60 & \pmod{67}, \\ F_{2^w \cdot 5^2 \cdot 7} & \equiv & 10, & 63, & 2, & 30, & 57, & 4, & 65, & \textit{or } 37 & \pmod{67}, \\ L_{2^w \cdot 5^2 \cdot 7} & \equiv & 13, & 33, & 15, & 22, & 13, & 33, & 15, & \textit{or } 22 & \pmod{67}, \end{array}$$

respectively.

Proof. Note that F_n and L_n are periodic modulo 67 with period 136. Let $n_w = 2^w t$ be a positive integer, where $w \geq 3$ and t is a fixed integer with $\gcd(t, 17) = 1$. Since $(n_w/2^3)_{w\geq 3}$ is a sequence of integers periodic modulo 17 with period 8, we have that F_{n_w} and L_{n_w} are periodic modulo 67 with period 8. After some calculations, we obtain the values in the lemma and this completes the proof. □

Proof of Theorem 2. Now, we are able to study the equation $y^2 = 5x^2(x+3)^2 \pm 4$. We shall prove that the only integer points on these elliptic curves are $(x, y) = (0, \pm 2)$. To obtain a contradiction, suppose that $F_n = x(x+3)$ has a solution with $(n, x) \neq (0, 0)$. By Lemma 2, we have that $n = 2 \cdot 2^w \cdot 5^2 \cdot 7t$ with $w \geq 3$ and t odd. Moreover, we can write $n = 2gk$ such that $3 \nmid k$ and g is odd. Hence, by Lemma 1, we have

$$\left(\frac{4F_n+9}{L_{2k}}\right) = \begin{cases} \left(\frac{4F_{2k}+9}{L_{2k}}\right), & \text{if } g \equiv 1 \pmod 4, \\ \left(\frac{-4F_{2k}+9}{L_{2k}}\right), & \text{if } g \equiv 3 \pmod 4. \end{cases}$$

`Case 1:` $t \equiv 1 \pmod 4$.

- If $w \equiv 0, 3, 5, 6, \text{or } 7 \pmod 8$, then we consider $k = 2^w$ and $g = 5^2 \cdot 7t$. Note that $g \equiv 3 \pmod 4$, thus by Lemma 1, the Jacobi Symbol Criterion and Lemma 4, we obtain

$$\left(\frac{4F_n+9}{L_{2k}}\right) = \left(\frac{-4F_{2k}+9}{L_{2k}}\right) = -\left(\frac{-8F_k+9L_k}{67}\right) = -1.$$

- If $w \equiv 1, 2, \text{or } 4 \pmod 8$, then we take $k = 2^w \cdot 5^2 \cdot 7$ and $g = t$. Using again Lemma 1, the Jacobi Symbol Criterion and Lemma 4, we have

$$\left(\frac{4F_n+9}{L_{2k}}\right) = \left(\frac{4F_{2k}+9}{L_{2k}}\right) = -\left(\frac{8F_k+9L_k}{67}\right) = -1.$$

`Case 2:` $t \equiv 3 \pmod 4$.

- If $w \equiv 1, 2, 3, 4, \text{or } 7 \pmod 8$, then we take $k = 2^w$ and $g = 5^3 \cdot 7t$. Note that $g \equiv 1 \pmod 4$, so in this case we have

$$\left(\frac{4F_n+9}{L_{2k}}\right) = \left(\frac{4F_{2k}+9}{L_{2k}}\right) = -\left(\frac{8F_k+9L_k}{67}\right) = -1.$$

- Finally, if $w \equiv 0, 5, \text{or } 6 \pmod 8$, then putting $k = 2^w \cdot 5^2 \cdot 7$ and $g = t$, we conclude that

$$\left(\frac{4F_n+9}{L_{2k}}\right) = \left(\frac{-4F_{2k}+9}{L_{2k}}\right) = -\left(\frac{-8F_k+9L_k}{67}\right) = -1.$$

Therefore, we obtain a contradiction in all cases. These contradictions occur by supposing that $F_n = x(x+3)$ has a solution other than $(n, x) = (0, 0)$. Accordingly, if (x, y) is an integer point on the elliptic curves

$$y^2 = 5x^2(x+3)^2 + 4(-1)^n,$$

then $(x, y, n) \in \{(-3, \pm 2, 0), (-2, \pm 4, 3), (-1, \pm 4, 3), (0, \pm 2, 0)\}$, and this completes the proof of Theorem 2. $\square$

Acknowledgements. The authors are grateful to the referee and the editors for their valuable comments about this work. They also thank D. Marques and P. Trojovský for initial discussions on the subject.

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