PARTITIONS WITH DESIGNATED SUMMANDS NOT DIVISIBLE BY 2^ℓ, 2, AND 3^ℓ MODULO 2, 4, AND 3

Daniel Herden
Department of Mathematics, Baylor University, Waco, Texas
Daniel_Herden@baylor.edu

Mark R. Sepanski
Department of Mathematics, Baylor University, Waco, Texas
Mark_Sepanski@baylor.edu

Jonathan Stanfill
Department of Mathematics, The Ohio State University, Columbus, Ohio
stanfill.13@osu.edu

Cordell C. Hammon
Department of Mathematics, Baylor University, Waco, Texas
Cordell_Hammon@baylor.edu

Joel Henningsen
Department of Mathematics, Baylor University, Waco, Texas
Joel_Henningsen1@baylor.edu

Henry Ickes
Department of Mathematics, Baylor University, Waco, Texas
Henry_Ickes1@baylor.edu

Indalecio Ruiz
Department of Mathematics, Baylor University, Waco, Texas
Indalecio_Ruizbolan1@baylor.edu

Received: 8/18/22, Accepted: 5/30/23, Published: 6/12/23

Abstract

Numerous congruences for partitions with designated summands have been proven since first being introduced and studied by Andrews, Lewis, and Lovejoy. This paper explicitly characterizes the number of partitions with designated summands whose parts are not divisible by 2^ℓ, 2, and 3^ℓ working modulo 2, 4, and 3, respectively, greatly extending previous results on the subject. We provide a few applications of our characterizations throughout in the form of congruences and a computationally fast recurrence. Moreover, we illustrate a previously undocumented connection

DOI: 10.5281/zenodo.8029024

between the number of partitions with designated summands and the number of partitions with odd multiplicities.

1. Introduction

Partitions with designated summands have been studied by many authors since first being introduced by Andrews, Lewis, and Lovejoy in [2]. These partitions are constructed by taking the ordinary partitions and marking exactly one part of each part size, often denoted by affixing a prime to the tagged part. For instance, there are 15 partitions of 5 with designated summands:

$$\begin{gathered} 5',\ 4'+1',\ 3'+2',\ 3'+1'+1,\ 3'+1+1',\ 2'+2+1', \\ 2+2'+1',\ 2'+1'+1+1,\ 2'+1+1'+1,\ 2'+1+1+1', 1'+1+1+1+1, \\ 1+1'+1+1+1,\ 1+1+1'+1+1,\ 1+1+1+1'+1,\ 1+1+1+1+1'. \end{gathered}$$

Hence, $\mathrm{PD}(5) = 15$, where $\mathrm{PD}(n)$ denotes the total number of partitions of n with designated summands.

By [2, Theorem 1], the generating function for the number of partitions with designated summands whose parts belong to a set of positive integers, S, is given by

$$\sum_{n\geq 0} \mathrm{PD}_S(n)q^n = \prod_{n\in S} \frac{1-q^{6n}}{(1-q^n)(1-q^{2n})(1-q^{3n})}.$$

Furthermore, by taking S_k to be the set of positive integers not divisible by k, one obtains the generating function for the number of partitions with designated summands whose parts are not divisible by k. It is given in [2, Corollary 3] by

$$\sum_{n\geq 0} \mathrm{PD}_k(n)q^n = \frac{(q^6;q^6)_\infty (q^k;q^k)_\infty (q^{2k};q^{2k})_\infty (q^{3k};q^{3k})_\infty}{(q;q)_\infty (q^2;q^2)_\infty (q^3;q^3)_\infty (q^{6k};q^{6k})_\infty},$$

where $(a;q)_\infty = \prod_{n\geq 0}(1-aq^n)$ represents the *q-series* or *q-Pochhammer symbol*.

The generating function for $\mathrm{PD}_k(n)$, which will be the central object of study for this paper, has been used to prove numerous congruences for partitions with designated summands in [2, 5, 7, 8, 10, 15]. In some cases, the generating function for $\mathrm{PD}_k(n)$ is more explicitly known. For instance, one has the following complete characterization of $\mathrm{PD}_2(n)$ modulo 2.

Theorem 1 ([2, Corollary 10])**.** *For all $n \geq 0$,*

$$\mathrm{PD}_2(n) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = 0 \text{ or } n = k^2 \text{ for } k \geq 1,\ 3 \nmid k \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

Similarly, a complete characterization of $\mathrm{PD}_3(n)$ modulo 3 exists.

Theorem 2 ([7, Theorem 2])**.** *For all $n \geq 0$,*

$$\mathrm{PD}_3(n) \equiv \begin{cases} 1 \pmod 3, & \text{if } n = 0, \\ r(n) \pmod 3, & \text{otherwise}, \end{cases}$$

where $r(n)$ is the number of solutions to $n = k(k+1)+3m(m+1)+1$ in nonnegative integers k, m.

In this paper, we study further characterizations of $\mathrm{PD}_k(n)$ by means of its generating function. In particular, we characterize $\mathrm{PD}_{2\ell}(n) \pmod 2$ in Section 3, $\mathrm{PD}_2(n) \pmod 4$ in Section 6, and $\mathrm{PD}_{3\ell}(n) \pmod 3$ in Section 8.

In Section 3, Theorem 3, we show that $\mathrm{PD}_{2^\ell}(n) \equiv a_{n\ell} \pmod 2$, where

$$a_{n\ell} = \left|\{\text{solutions to } n = \sum_{m=0}^{\ell-1} 2^m k_m^2 \mid k_m \geq 0,\, 3 \nmid k_m \text{ or } k_m = 0\}\right|.$$

For example, along with a few auxiliary results, this allows us to validate numerous congruences in Theorem 4 and Corollary 1. For $n \geq 0$ and $r \in \{5, 7, 10, 13, 14, 15, 20, 21, 23, 26, 28, 29, 30, 31\}$, we have $\mathrm{PD}_4(32n+r) \equiv 0 \pmod 2$ and $\mathrm{PD}_8(32n+24) \equiv 0 \pmod 2$. Additionally, for $\ell \geq 3$ and $r \in \{4, 6\}$, $\mathrm{PD}_{2^\ell}(8n + r) \equiv 0 \pmod 2$. For $\ell \geq 4$ and $r \in \{4, 6, 10, 12, 14\}$, $\mathrm{PD}_{2^\ell}(16n + r) \equiv 0 \pmod 2$. For $\ell \geq 5$ and $r \in \{4, 6, 10, 12, 14, 16, 20, 22, 24, 26, 28, 30\}$, $\mathrm{PD}_{2^\ell}(32n+r) \equiv 0 \pmod 2$. In general, for $\ell \geq j \geq 3$ and $0 \leq s < 2^{j-1}$ with $s \notin \{0, 4^a(8b+1) \mid a, b \geq 0\}$, $\mathrm{PD}_{2^\ell}(2^j n+2s) \equiv 0 \pmod 2$.

Finally, in Theorem 6 we provide the very interesting congruence, $\mathrm{PD}(n) \equiv b_n \pmod 2$, which links the number $\mathrm{PD}(n)$ of partitions of n with designated summands [19, A077285] to the number b_n of partitions of n with odd multiplicities [19, A055922]. This may be of use in the study of the parity of partitions with odd multiplicities as in [12], [13], and [18].

In Sections 4 and 7, we prove the following remarkably explicit alternate characterization of $\mathrm{PD}_4(n) \pmod 2$ in Theorem 7: For all $n \geq 0$,

$$\mathrm{PD}_4(n) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = mk^2 \text{ for } m, k \geq 0,\ m \mid 6, \\ 0 \pmod 2, & \text{otherwise}. \end{cases}$$

In Section 5, motivated by [2, Corollaries 6 and 9], [11], and Section 3, for general $n \geq 0$ and $k \geq 2$, we also provide a computationally fast recurrence for $\mathrm{PD}_k(n) \pmod 2$ in Theorem 8 of the form

$$\mathrm{PD}_k(n) + \sum_{\ell \geq 1,\, 3 \nmid \ell} \mathrm{PD}_k(n - \ell^2) \equiv \begin{cases} 1 \pmod 2, \text{ if } n = 0 \text{ or } n = km^2 \text{ for some } 3 \nmid m, \\ 0 \pmod 2, \text{ otherwise}. \end{cases}$$

In Section 6, we consider $\mathrm{PD}_2(n) \pmod 4$. Theorem 10 gives the explicit characterization $\mathrm{PD}_2(2n+1) \equiv d_n \pmod 4$, where

$$d_n = \left|\left\{\text{solutions to } n = 3j(3j-1) + 3k(3k-1) = 2\left[\binom{3j}{2} + \binom{3k}{2}\right] \;\middle|\; k, j \in \mathbb{Z}\right\}\right|.$$

Theorem 11 provides an explicit characterization of $\mathrm{PD}_2(2n) \pmod 4$ as

$$\sum_{n\geq 0} \mathrm{PD}_2(2n)q^n \equiv 1 + 2\sum_{k\geq 1,\, 3\nmid k} q^{k^2} + \sum_{k,\ell\geq 1,\, 3\nmid k,\ell} q^{k^2+\ell^2} \pmod 4.$$

Theorem 13 gives the following combined expression,

$$\sum_{n\geq 0} \mathrm{PD}_2(n)q^n \equiv 1 + \Big(\sum_{k\geq 1,\, 3\nmid k} q^{k^2}\Big)\Big(\sum_{k\in\mathbb{Z}} q^{k^2}\Big) \pmod 4.$$

Finally, in Theorem 12, we show that for $n \geq 1$, $\mathrm{PD}_2(3n) \equiv 0 \pmod 4$. Moreover, for all $n \geq 1$ with $6 \nmid n$, $\mathrm{PD}_2(2n+1) \equiv 0 \pmod 4$.

In Section 8, Theorem 15, we show that $\mathrm{PD}_3(n) \equiv e_{n1} \pmod 3$, where

$$e_{n1} = \big|\{\text{solutions to } n = k_0^2 + 3k_1^2 \mid k_m \in \mathbb{Z} \text{ or } \mathbb{N} \text{ when } k_m \text{ is even or odd, respectively}\}\big|.$$

This gives an alternate characterization of Theorem 2. See Remark 2 for alternate forms of e_{n1} and Theorem 16 for a general description of $\mathrm{PD}_{3^\ell}(n) \pmod 3$.

In Theorem 19, for all $n \geq 0$ and $\ell \geq 2$, we show that $\mathrm{PD}_{3^\ell}(3n) \equiv e^*_{n\ell} \pmod 3$, where

$$e^*_{n\ell} = \big|\{\text{solutions to } n = k_0^2 + k_0'^2 + 3^{\ell-1}(k_1^2 + k_1'^2) \mid k_m, k_m' \in \mathbb{Z} \text{ or } \mathbb{N} \text{ when } k_m, k_m' \text{ is even or odd, respectively}\}\big|.$$

As an easy application of our results, Theorem 17, we give very short proofs of the congruences $\mathrm{PD}_3(9n+6) \equiv 0 \pmod 3$ and $\mathrm{PD}_{3^\ell}(3n+2) \equiv 0 \pmod 3$ for which proofs using dissections can be found in [7, Theorem 3] and [8, Theorem 3]. As a further application, Theorem 18, we show that for $n \geq 1$, $\mathrm{PD}_3(2n) \equiv 0 \pmod 3$. Moreover, for all $n \geq 0$ and $\ell \geq 3$, $\mathrm{PD}_{3^\ell}(27n+9) \equiv 0 \pmod 3$ and, for all $n \geq 0$ and $\ell \neq 2$, $\mathrm{PD}_{3^\ell}(27n+18) \equiv 0 \pmod 3$. Finally, in Corollary 2, for all $n \geq 0$, $k \geq 1$ and $\ell \geq 2k+1$, we show that $\mathrm{PD}_{3^\ell}\left(3^{2k}(3n+1)\right) \equiv \mathrm{PD}_{3^\ell}\left(3^{2k}(3n+2)\right) \equiv 0 \pmod 3$.

In Section 9, we conclude with some remarks and conjectured congruences.

2. General Notation and Basic Results

Definition 1. Let

$$f_k = f_k(q) = (q^k; q^k)_\infty = \prod_{n\geq 1}(1 - q^{kn}) = f_1(q^k),$$

$$\text{pd}(q) = \sum_{n\geq 0} \text{PD}(n)q^n,$$

$$\text{pd}_k(q) = \sum_{n\geq 0} \text{PD}_k(n)q^n,$$

where, once again, $\text{PD}(n)$ denotes the number of partitions of n with designated summands (ordinary partitions with exactly one part designated among parts with equal size) and $\text{PD}_k(n)$ denotes the number of partitions of n with designated summands whose parts are not divisible by k.

Using this notation, the generating functions can be written as

$$\text{pd}(q) = \frac{f_6(q)}{f_1(q)f_2(q)f_3(q)}$$

and

$$\text{pd}_k(q) = \frac{f_6(q)}{f_1(q)f_2(q)f_3(q)} \frac{f_k(q)f_{2k}(q)f_{3k}(q)}{f_{6k}(q)} = \frac{f_6(q)}{f_1(q)f_2(q)f_3(q)} \frac{f_1(q^k)f_2(q^k)f_3(q^k)}{f_6(q^k)}.$$

Definition 2. Let

$$g(q) = \frac{1}{\text{pd}(q)} = \frac{f_1(q)f_2(q)f_3(q)}{f_6(q)}.$$

With Definition 2, we have

$$\text{pd}_k(q) = \frac{g(q^k)}{g(q)}. \tag{1}$$

Modulo a prime, p, the Frobenius automorphism implies that

$$f_{pk}(q) = f_k(q^p) \equiv f_k(q)^p \pmod{p}$$

so that $g(q^p) \equiv g(q)^p \pmod{p}$. Therefore, Equation (1) gives us

$$\text{pd}_p(q) \equiv g(q)^{p-1} \pmod{p}. \tag{2}$$

Finally, observe that

$$\text{pd}_{p^\ell}(q) = \frac{g(q^{p^\ell})}{g(q)} = \prod_{m=0}^{\ell-1} \frac{g(q^{p^{m+1}})}{g(q^{p^m})} = \prod_{m=0}^{\ell-1} \text{pd}_p(q^{p^m}). \tag{3}$$

3. The Case of $\text{pd}_{2^\ell}(q) \pmod{2}$

The following can be written more succinctly, but is written to synchronize better with the general case which follows directly after.

Lemma 1. *We have*

$$\mathrm{pd}_2(q) \equiv g(q) \equiv \frac{1}{\mathrm{pd}(q)} \pmod 2 \quad \text{and} \quad \mathrm{pd}_2(q) \equiv \sum_{n\geq 0} a_{n1} q^n \pmod 2,$$

where

$$a_{n1} = \left|\{\text{solutions to } n = k_0^2 \,|\, k_0 \geq 0,\, 3 \nmid k_0 \text{ or } k_0 = 0\}\right|.$$

Proof. Using $p = 2$ in Equation (2), we immediately get $\mathrm{pd}_2(q) \equiv g(q) \equiv \frac{1}{\mathrm{pd}(q)} \pmod 2$.

The second statement is a rephrasing of Theorem 1,

$$\mathrm{pd}_2(q) \equiv 1 + \sum_{k\geq 1,\, 3\nmid k} q^{k^2} \pmod 2. \qquad \square \tag{4}$$

The general case is given by the following.

Theorem 3. *For all $n \geq 0$ and $\ell \geq 1$, $\mathrm{PD}_{2^\ell}(n) \equiv a_{n\ell} \pmod 2$, where*

$$a_{n\ell} = \left|\{\text{solutions to } n = \sum_{m=0}^{\ell-1} 2^m k_m^2 \,|\, k_m \geq 0,\, 3 \nmid k_m \text{ or } k_m = 0\}\right|.$$

Proof. Using Equations (3) and (4), we see that

$$\mathrm{pd}_{2^\ell}(q) = \prod_{m=0}^{\ell-1} \mathrm{pd}_2(q^{2^m}) \equiv \prod_{m=0}^{\ell-1} \Big(1 + \sum_{k\geq 1,\, 3\nmid k} q^{2^m k^2}\Big) \pmod 2.$$

The result follows. $\square$

As a first easy application of Theorem 3, we prove the following newly observed congruences of the form $\mathrm{PD}_{2^\ell}(32n + r)$.

Theorem 4. *For $n \geq 0$ and $r \in \{5, 7, 10, 13, 14, 15, 20, 21, 23, 26, 28, 29, 30, 31\}$, $\mathrm{PD}_4(32n + r) \equiv 0 \pmod 2$, and, for $n \geq 0$, $\mathrm{PD}_8(32n + 24) \equiv 0 \pmod 2$.*

Proof. The first congruence is immediate as the only quadratic residues modulo 32 are $0, 1, 4, 9, 16, 17$, and 25.

For the second congruence, we need to pair off solutions (k_0, k_1, k_2) of the equation

$$32n + 24 = k_0^2 + 2k_1^2 + 4k_2^2. \tag{5}$$

Note that we have $k_0^2 \equiv 0 \pmod 2$, so that $2 \mid k_0$, and we can write $k_0 = 2a$ for some integer a. This gives $32n + 24 = 4a^2 + 2k_1^2 + 4k_2^2$. Note that with $(2a, k_1, k_2)$ also $(2k_2, k_1, a)$ is a solution of (5), and for $k_2 \neq a$ we will pair these two solutions off. This leaves us with considering solutions of the form $(2a, k_1, a)$ and the equation

$32n + 24 = 4a^2 + 2k_1^2 + 4a^2 = 8a^2 + 2k_1^2$. Note that we have $2k_1^2 \equiv 0 \pmod 4$, so that $2 \mid k_1$, and we can write $k_1 = 2b$ for some integer b. This gives the equation $4n + 3 = a^2 + b^2$ which is not satisfiable as the only quadratic residues modulo 4 are 0 and 1. Thus, solutions of the form $(2a, k_1, a)$ do not exist and the pairing is complete. □

Our next result expands on and iterates the idea of pairing off solutions.

Theorem 5. *For $n \geq 0$, $\ell \geq 3$, and $0 \leq s < 2^{\ell-1}$, if $\mathrm{PD}_{2^\ell}(2^\ell n + 2s) \equiv 1 \pmod 2$, then s is a quadratic residue modulo $2^{\ell-1}$. Equivalently, if $0 \leq s < 2^{\ell-1}$ and s is a quadratic nonresidue modulo $2^{\ell-1}$, then $PD_{2^\ell}(2^\ell n + 2s) \equiv 0 \pmod 2$.*

Moreover, for all $n \geq 0$, $\ell \geq 2$, we have $\mathrm{PD}_{2^\ell}(2n) \equiv a^*_{n\ell} \pmod 2$, where

$$a^*_{n\ell} = \left|\{\textit{solutions to } n = k_0^2 + 2^{\ell-1}k_1^2 \mid k_m \geq 0,\ 3 \nmid k_m \textit{ or } k_m = 0\}\right|.$$

Proof. We will be pairing off the solutions $(k_0, k_1, k_2, k_3, \ldots, k_{\ell-1})$ of the equation

$$2^\ell n + 2s = k_0^2 + 2k_1^2 + 4k_2^2 + 8k_3^2 + 16k_4^2 + \ldots + 2^{\ell-1}k_{\ell-1}^2. \tag{6}$$

Note that we have $k_0^2 \equiv 0 \pmod 2$, so that $2 \mid k_0$, and we can write $k_0 = 2a$ for some integer a. This gives

$$2^\ell n + 2s = 4a^2 + 2k_1^2 + 4k_2^2 + 8k_3^2 + 16k_4^2 + \ldots + 2^{\ell-1}k_{\ell-1}^2.$$

Now with $(2a, k_1, k_2, k_3, \ldots, k_{\ell-1})$, we also see that $(2k_2, k_1, a, k_3, \ldots, k_{\ell-1})$ is a solution of (6), and for $k_2 \neq a$ we will pair these two solutions off. This leaves us with considering solutions of the form $(2a, k_1, a, k_3, \ldots, k_{\ell-1})$ and the equation

$$2^\ell n + 2s = 4a^2 + 2k_1^2 + 4a^2 + 8k_3^2 + 16k_4^2 + \ldots + 2^{\ell-1}k_{\ell-1}^2.$$

Note that with $(2a, k_1, a, k_3, k_4 \ldots, k_{\ell-1})$, also $(2k_3, k_1, k_3, a, k_4 \ldots, k_{\ell-1})$ is a solution of (6), and for $k_3 \neq a$ we will pair these two solutions off. This leaves us with considering solutions of the form $(2a, k_1, a, a, k_4 \ldots, k_{\ell-1})$ and the equation

$$2^\ell n + 2s = 4a^2 + 2k_1^2 + 4a^2 + 8a^2 + 16k_4^2 + \ldots + 2^{\ell-1}k_{\ell-1}^2.$$

This process can be iterated until we are left to consider solutions of the form $(2a, k_1, a, a, \ldots, a)$ and the equation

$$2^\ell n + 2s = 4a^2 + 2k_1^2 + 4a^2 + 8a^2 + 16a^2 + \ldots + 2^{\ell-1}a^2 = 2k_1^2 + 2^\ell a^2.$$

In particular, $2^{\ell-1}n + s = k_1^2 + 2^{\ell-1}a^2$. Thus, $s \equiv k_1^2 \pmod{2^{\ell-1}}$ and the first part of the theorem follows. The second part of the theorem is now easy. □

We continue with a general auxiliary result.

Lemma 2. *If there exist j and r such that the congruence* $\mathrm{PD}_{2^j}(2^j n + r) \equiv 0 \pmod 2$ *holds for all* $n \geq 0$*, then* $\mathrm{PD}_{2^\ell}(2^j n + r) \equiv 0 \pmod 2$ *for all* $n \geq 0$ *and* $\ell \geq j$*.*

Proof. Let j, ℓ, n, and r be fixed. With

$$L = \Big\{\text{solutions to } 2^j n + r = s + \sum_{m=j}^{\ell-1} 2^m k_m^2 \,|\, s, k_m \geq 0,\, 3 \nmid k_m \text{ or } k_m = 0\Big\}$$

we have $a_{2^j n + r, \ell} = \sum \big\{a_{sj} \,|\, (s, k_j, k_{j+1}, \ldots, k_{\ell-1}) \in L\big\}$, where $s \equiv r \pmod{2^j}$ for all $(s, k_j, k_{j+1}, \ldots, k_{\ell-1}) \in L$. Hence, $a_{sj} \equiv \mathrm{PD}_{2^j}(s) \equiv 0 \pmod 2$ for all $(s, k_j, k_{j+1}, \ldots, k_{\ell-1}) \in L$, so that $\mathrm{PD}_{2^\ell}(2^j n + r) \equiv a_{2^j n + r, \ell} \equiv 0 \pmod 2$. □

As a demonstration of how Theorem 5 and Lemma 2 combine to efficiently prove congruences, we provide the following corollary. Some of these results were proven with dissections in [8, Equations (3), (5), and (8)].

Corollary 1. *Let* $n \geq 0$*. For* $\ell \geq 3$ *and* $r \in \{4, 6\}$ *holds* $\mathrm{PD}_{2^\ell}(8n + r) \equiv 0 \pmod 2$*. For* $\ell \geq 4$ *and* $r \in \{4, 6, 10, 12, 14\}$*,* $\mathrm{PD}_{2^\ell}(16n + r) \equiv 0 \pmod 2$*. For all* $\ell \geq 5$ *and* $r \in \{4, 6, 10, 12, 14, 16, 20, 22, 24, 26, 28, 30\}$*,* $\mathrm{PD}_{2^\ell}(32n + r) \equiv 0 \pmod 2$*. In general, for* $\ell \geq j \geq 3$ *and* $0 \leq s < 2^{j-1}$ *with s not of the form* 0 *or* $4^a(8b + 1)$ *for some* $a, b \geq 0$*,* $\mathrm{PD}_{2^\ell}(2^j n + 2s) \equiv 0 \pmod 2$*.*

Proof. We apply Theorem 5 first with $\ell = 3$. The quadratic nonresidues modulo 4 are 2 and 3. Thus for $n \geq 0$ and $r \in \{4, 6\}$, $\mathrm{PD}_8(8n + r) \equiv 0 \pmod 2$.

For $\ell = 4$, the quadratic nonresidues modulo 8 are $2, 3, 5, 6$, and 7. So, for $r \in \{4, 6, 10, 12, 14\}$, $\mathrm{PD}_{16}(16n + r) \equiv 0 \pmod 2$.

For $\ell = 5$, the quadratic nonresidues modulo 16 are $2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14$, and 15. So, for $r \in \{4, 6, 10, 12, 14, 16, 20, 22, 24, 26, 28, 30\}$, $\mathrm{PD}_{32}(32n + r) \equiv 0 \pmod 2$.

In general, for $j \geq 3$, the quadratic residues modulo 2^{j-1} are 0 and all numbers of the form $4^a(8b + 1)$ for $a, b \geq 0$. So if $0 \leq s < 2^{j-1}$ is not of this form, then $\mathrm{PD}_{2^j}(2^j n + 2s) \equiv 0 \pmod 2$.

The rest follows from Lemma 2. □

The function $\mathrm{pd}_2(q) \equiv \frac{1}{\mathrm{pd}(q)} \pmod 2$ is well known. We end this section on an interesting new interpretation of its reciprocal.

Theorem 6. *We have* $\mathrm{pd}(q) \equiv \frac{1}{\mathrm{pd}_2(q)} \equiv 1 + \sum_{n \geq 1} b_n q^n \pmod 2$*, where*

$$b_n = \big|\{\textit{partitions of } n \,|\, \textit{the multiplicity of each part is odd}\}\big|.$$

Proof. Observe that

$$\begin{aligned}\frac{1-q^6}{(1-q)(1-q^2)(1-q^3)} &\equiv \frac{(1-q^3)^2}{(1-q)(1-q^2)(1-q^3)}\\ &= \frac{1+q+q^2}{1-q^2} \equiv 1+\sum_{k\geq 0} q^{2k+1} \pmod 2.\end{aligned}$$

Using the above equation in conjunction with the definition of $\mathrm{pd}(q)$ in terms of the f_k, we get $\mathrm{pd}(q) \equiv \prod_{n\geq 1}(1+\sum_{k\geq 0} q^{(2k+1)n}) \pmod 2$ and the result follows. ☐

Theorem 6 provides the hitherto undocumented congruence $\mathrm{PD}(n) \equiv b_n \pmod 2$, which links the number $\mathrm{PD}(n)$ of partitions of n with designated summands [19, A077285] to the number b_n of partitions of n with odd multiplicities [19, A055922]. This may be of use in understanding the parity of partitions with odd multiplicities as studied in [12], [13], and [18].

4. An Alternate Characterization of $\mathrm{pd}_4(q) \pmod 2$

In the case of $\ell = 2$, Theorem 3 says $\mathrm{PD}_4(n) \equiv a_{n2} \pmod 2$, where

$$a_{n2} = \left|\{\text{solutions to } n = k_0^2 + 2k_1^2 \mid k_m \geq 0,\, 3 \nmid k_m \text{ or } k_m = 0\}\right|.$$

In this section we give a remarkably explicit formula for $a_{n2} \pmod 2$ together with a combinatorial proof. A dissection proof for the same result is provided in Section 7.

Theorem 7. *For all $n \geq 0$,*

$$a_{n2} \equiv \begin{cases} 1 \pmod 2, & \textit{if } n = mk^2 \textit{ for } m,k \in \mathbb{Z}_{\geq 0} \textit{ with } m \mid 6,\\ 0 \pmod 2, & \textit{otherwise.}\end{cases}$$

Proof. Let $L(n) = \{(k_0,k_1) \mid n = k_0^2 + 2k_1^2,\, k_m \geq 0,\, 3 \nmid k_m \text{ or } k_m = 0\}$ so that $a_{n2} = |L(n)|$. Recall that $0, 1$ are the only quadratic residues modulo 3.

Case 1: $n \equiv 1 \pmod 3$

For $(a,b) \in L(n)$, $n = a^2 + 2b^2 \equiv 1 \pmod 3$ is only possible for $b = 0$. Thus,

$$|L(n)| = \begin{cases} 1, & \text{if } n = k^2 \text{ for some } k \in \mathbb{Z}_{\geq 0},\\ 0, & \text{otherwise,}\end{cases}$$

confirming the theorem for $n \equiv 1 \pmod 3$.

Case 2: $n \equiv 2 \pmod 3$

For $(a,b) \in L(n)$, $n = a^2 + 2b^2 \equiv 2 \pmod 3$ is only possible for $a = 0$. Thus,

$$|L(n)| = \begin{cases} 1, & \text{if } n = 2k^2 \text{ for some } k \in \mathbb{Z}_{\geq 0},\\ 0, & \text{otherwise,}\end{cases}$$

confirming the theorem for $n \equiv 2 \pmod 3$.

Case 3: $n \equiv 0 \pmod 3$

The case $n \equiv 0 \pmod 3$ is more involved. For $n = 0$, we have $L(0) = \{(0,0)\}$ and $a_{02} = |L(0)| = 1$, confirming the theorem. Thus we may assume $n > 0$. Let $S(n) = \{(k_0, k_1) \mid n = k_0^2 + 2k_1^2,\ k_m \in \mathbb{Z}\}$. We define an equivalence relation on $S(n)$ by

$$(a,b) \sim (c,d) \text{ if and only if } |a| = |c| \text{ and } |b| = |d|, \text{ for } (a,b),(c,d) \in S(n).$$

Next, we investigate the equivalence classes of $S(n)/\sim$.

For $(a,b) \in S(n),\ n = a^2 + 2b^2 \equiv 0 \pmod 3$ is only possible for $a, b \equiv 0 \pmod 3$ or $a, b \not\equiv 0 \pmod 3$. In particular, in this case,

$$L(n) = \{(k_0, k_1) \mid n = k_0^2 + 2k_1^2,\ k_m > 0,\ 3 \nmid k_m\},$$

and $L(n)$ contains exactly one representative for each equivalence class $[(a,b)] \in S(n)/\sim$ with $a, b \not\equiv 0 \pmod 3$.

Noting that

$$a^2 + 2b^2 = n \quad \text{if and only if} \quad \left(\frac{a+4b}{3}\right)^2 + 2\left(\frac{2a-b}{3}\right)^2 = n,$$

we define $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Psi(a,b) = \left(\frac{a+4b}{3}, \frac{2a-b}{3}\right).$$

Observe that Ψ is an involution, i.e., Ψ^2 is the identity map on $\mathbb{R}^2$.

We define a graph whose vertices are the equivalence classes of $S(n)/\sim$ by connecting $[(a,b)]$ with $[(c,d)]$ if $\Psi(a',b') \in [(c,d)]$ for some $(a',b') \in [(a,b)]$. Since Ψ is an involution, $[(a,b)]$ is connected with $[(c,d)]$ if and only if $[(c,d)]$ is connected with $[(a,b)]$, and the resulting graph is undirected. We will further investigate this graph. We begin by discussing the most common occurrences, treating exceptions last.

The equivalence classes $[(a,b)] \in S(n)/\sim$ with $a, b \not\equiv 0 \pmod 3$ are in one-to-one correspondence with $L(n)$. With exceptions to follow below, each such equivalence class is usually connected with exactly one other equivalence class. In particular, for $a \not\equiv b \pmod 3$, $[(a,b)]$ is connected to $[\Psi(a,b)]$; and for $a \equiv b \pmod 3$, we have that $[(a,b)]$ is connected to $[\Psi(a,-b)]$. Similarly, with exceptions to follow, the equivalence classes $[(a,b)] \in S(n)/\sim$ with $a, b \equiv 0 \pmod 3$ are usually connected with exactly two other equivalence classes, $[\Psi(a,b)]$ and $[\Psi(a,-b)]$.

Thus, the resulting graph will usually separate the equivalence classes corresponding to $L(n)$ into connected pairs, hence $|L(n)| \equiv 0 \pmod 2$.

Exceptions to the above happen if either of the following two cases occurs:

(1) One equivalence class is connected to itself, that is, $[\Psi(a,b)] = [(a,b)]$. This case occurs if and only if

$$\left|\frac{2a-b}{3}\right| = |b|, \text{ so that } a = -b \text{ or } a = 2b.$$

Thus, this case occurs if and only if (k,k) or $(2k,k) \in S(n)$, which is equivalent to $n = 3k^2$ or $6k^2$.

(2) One equivalence class connects to the same equivalence class twice, that is, $[\Psi(a,b)] = [\Psi(a,-b)]$. This case occurs if and only if

$$\left|\frac{2a-b}{3}\right| = \left|\frac{2a+b}{3}\right|, \text{ so that } a = 0 \text{ or } b = 0.$$

Thus, this case occurs if and only if $(0,k)$ or $(k,0) \in S(n)$, which is equivalent to $n = 2k^2$ or k^2 with $k \equiv 0 \pmod 3$.

This completes the proof of the theorem. □

Remark 1. The solution set of the Diophantine equation $a^2 + 2b^2 = n$ can be further investigated using unique factorization in the principal ideal domain $\mathbb{Z}\left[\sqrt{-2}\right]$.

5. A Recurrence Relation for $\mathrm{PD}_k(n) \pmod 2$

While Theorem 3 provides an explicit description of $\mathrm{PD}_{2^\ell}(n) \pmod 2$, for general $\mathrm{PD}_k(n) \pmod 2$ we have the following computationally fast recurrence.

Theorem 8. *For $n \geq 0$ and $k \geq 2$,*

$$\mathrm{PD}_k(n) + \sum_{\ell \geq 1,\, 3 \nmid \ell} \mathrm{PD}_k(n - \ell^2) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = 0 \text{ or } n = km^2,\ m \geq 1, 3 \nmid m, \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

Proof. Combining Equation (1) with Lemma 1 gives

$$\mathrm{pd}_k(q) = \frac{g(q^k)}{g(q)} \equiv \frac{\mathrm{pd}_2(q^k)}{\mathrm{pd}_2(q)} \pmod 2.$$

Substituting Equation (4), we have

$$\left(\sum_{n \geq 0} \mathrm{PD}_k(n) q^n\right)\left(1 + \sum_{\ell \geq 1,\, 3 \nmid \ell} q^{\ell^2}\right) \equiv 1 + \sum_{m \geq 1,\, 3 \nmid m} q^{km^2} \pmod 2,$$

and the result follows. □

6. The Case of $\mathrm{pd}_2(q) \pmod 4$

Next we turn to $\mathrm{pd}_2(q) \pmod 4$ beginning with a helpful result.

Lemma 3. *We have* $f_1^3 \equiv \sum_{n\geq 0} q^{\frac{n(n+1)}{2}} \pmod 4$ *and*

$$qf_8^3 \equiv \sum_{n\geq 1,\, 2\nmid n} q^{n^2} \pmod 4. \tag{7}$$

Proof. For any positive integers k and m, we have the identity, [7, Lemma 5],

$$f_{2k}^{2m} \equiv f_k^{4m} \pmod 4. \tag{8}$$

Therefore, the theta identity [1, Equation (2.2.13)] provides

$$\sum_{n\geq 0} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1} \equiv f_1^3 \pmod 4,$$

and the second identity is an immediate consequence of

$$qf_8^3 = q[f_1(q^8)]^3 \equiv q\sum_{n\geq 0} q^{4n(n+1)} = \sum_{n\geq 0} q^{(2n+1)^2} \pmod 4. \qquad \square$$

This result allows one to prove the following characterization.

Theorem 9. *For all* $n \geq 0$, $\mathrm{PD}_2(n) \equiv c_n \pmod 4$ *where*

$$c_n = \Big| \{ \textit{solutions to } n = \frac{3k_0(k_0+1)}{2} + \sum_{m\geq 1} k_m(12a_m + b_m) \mid$$
$$k_m, a_m \geq 0,\, b_m \in \{0,1,2,3\},\ 1 \leq k_1 < k_2 < k_3 < \ldots \} \Big|.$$

Proof. With Equation (8), we have the generating function

$$\mathrm{pd}_2(q) = \frac{f_4 f_6^2}{f_1 f_3 f_{12}} \equiv \frac{f_3^3 f_4}{f_1 f_{12}} \pmod 4.$$

The result follows now easily from $f_3^3 = [f_1(q^3)]^3 \equiv \sum_{k\geq 0} q^{\frac{3k(k+1)}{2}} \pmod 4$ and

$$\frac{1-x^4}{(1-x)(1-x^{12})} = \frac{1}{1+x^4+x^8}\sum_{n\geq 0} x^n = \sum_{\substack{n=12a+b,\\ a\geq 0,\, b\in\{0,1,2,3\}}} x^n. \qquad \square$$

We further prove an explicit characterization of $\mathrm{PD}_2(2n+1)$ modulo 4 related to generalized pentagonal numbers by means of the following dissection of $\mathrm{pd}_2(q)$ into even and odd powers.

Lemma 4. *We have*

$$\mathrm{pd}_2(q) \equiv \left[\frac{f_2^3}{f_6}\right]^2 + qf_{12}^2 \pmod 4, \tag{9}$$

splitting $\mathrm{pd}_2(q)$ *into even and odd powers of* q*, respectively.*

Proof. With Equation (8), we have

$$\mathrm{pd}_2(q) = \frac{f_4 f_6^2}{f_1 f_3 f_{12}} \equiv \frac{f_3^3 f_4}{f_1 f_{12}} \pmod 4.$$

Using the identity (cf., [21, Equation (3.75)])

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4},$$

and repeatedly using Equation (8), this yields

$$\mathrm{pd}_2(q) \equiv \frac{f_3^3 f_4}{f_1 f_{12}} \equiv \frac{f_4^4 f_6^2}{f_2^2 f_{12}^2} + qf_{12}^2 \equiv \left[\frac{f_2^3}{f_6}\right]^2 + qf_{12}^2 \pmod 4.$$

Noting that f_2^3/f_6 and f_{12} are power series in q^2, the first term on the right-hand side gives all even power contributions while the second term gives all odd power contributions due to the multiplication by the extra factor of q. □

An application of Lemma 4 provides the following explicit characterization of $\mathrm{PD}_2(2n+1) \pmod 4$.

Theorem 10. *For all* $n \geq 0$, $\mathrm{PD}_2(2n+1) \equiv d_n \pmod 4$, *where*

$$d_n = \left| \left\{ \textit{solutions to } n = 3j(3j-1) + 3k(3k-1) = 2\left[\binom{3j}{2} + \binom{3k}{2}\right] \;\middle|\; k, j \in \mathbb{Z} \right\} \right|.$$

Proof. One has from (9),

$$\sum_{n\geq 0} \mathrm{PD}_2(2n+1)q^n \equiv f_6^2 = [f_1(q^6)]^2 = \sum_{j,k\in\mathbb{Z}} (-1)^{j+k} q^{3[j(3j-1)+k(3k-1)]} \pmod 4,$$

where we have used Euler's pentagonal number theorem [1, Corollary 1.7]. The result follows from noting that on the right-hand side the only contributions with coefficient -1 occur with different parities of j and k, hence will come in pairs of solutions (j,k), (k,j) each time and thus contribute $2(-1)^{j+k} \equiv 2 \pmod 4$ to the count. □

Similarly, Lemma 4 also provides an explicit characterization of $\mathrm{PD}_2(2n) \pmod 4$.

Theorem 11.

$$\sum_{n\geq 0}\mathrm{PD}_2(2n)q^n \equiv \left[\frac{f_1^3}{f_3}\right]^2 \equiv \Big(1+\sum_{k\geq 1,\,3\nmid k} q^{k^2}\Big)^2$$
$$\equiv 1+2\sum_{k\geq 1,\,3\nmid k} q^{k^2}+\sum_{k,\ell\geq 1,\,3\nmid k,\ell} q^{k^2+\ell^2} \pmod 4.$$

Proof. Note that by [18, Equations (2) and (4)], we have

$$\frac{f_1^3}{f_3}\equiv 1+q\frac{f_9^3}{f_3}\equiv 1+q\sum_{j\in\mathbb{Z}} q^{3j(3j-2)} = 1+\sum_{j\in\mathbb{Z}} q^{(3j-1)^2} = 1+\sum_{k\geq 1,\,3\nmid k} q^{k^2} \pmod 2,$$

which implies

$$\left[\frac{f_1^3}{f_3}\right]^2 \equiv \Big(1+\sum_{k\geq 1,\,3\nmid k} q^{k^2}\Big)^2 \pmod 4. \tag{10}$$

Combining Equation (9) with Equation (10) gives

$$\sum_{n\geq 0}\mathrm{PD}_2(2n)q^n \equiv \left[\frac{f_1^3}{f_3}\right]^2 \equiv \Big(1+\sum_{k\geq 1,\,3\nmid k} q^{k^2}\Big)^2 \pmod 4. \qquad \square$$

As an easy application of these results, we prove a few observed congruences. Alternative proofs using dissections can be found in [3, Corollary 1.4, Theorem 1.5].

Theorem 12. *For* $n\geq 1$, *we have* $\mathrm{PD}_2(3n)\equiv 0 \pmod 4$. *For all* $n\geq 1$ *with* $6\nmid n$, *we have* $\mathrm{PD}_2(2n+1)\equiv 0 \pmod 4$.

Proof. The first congruence is easily seen by noting from [2, Theorem 22] and Equation (8) that

$$\sum_{n\geq 0}\mathrm{PD}_2(3n)q^n = \frac{f_2^2 f_6^4}{f_1^4 f_{12}^2} \equiv \frac{f_1^4 f_6^4}{f_1^4 f_6^4} \equiv 1 \pmod 4.$$

The second congruence follows immediately from Theorem 10 as $n=3j(3j-1)+3k(3k-1)=2\big[\binom{3j}{2}+\binom{3k}{2}\big]$ implies that $n\equiv 0 \pmod 6$. $\square$

We end this section with yet another nice characterization of $\mathrm{PD}_2(n) \pmod 4$.

Theorem 13.

$$\mathrm{pd}_2(q)\equiv 1+\Big(\sum_{k\geq 1,\,3\nmid k} q^{k^2}\Big)\Big(1+2\sum_{k\geq 1} q^{k^2}\Big) = 1+\Big(\sum_{k\geq 1,\,3\nmid k} q^{k^2}\Big)\Big(\sum_{k\in\mathbb{Z}} q^{k^2}\Big) \pmod 4.$$

Proof. Starting from [2, Equation (3.13)], we have

$$\begin{aligned}
\mathrm{pd}_2(q) &= \frac{\sum_{j\in\mathbb{Z}} q^{(3j)^2} - \sum_{j\in\mathbb{Z}} q^{(3j+1)^2}}{1+2\sum_{k\geq 1}(-1)^k q^{k^2}} = \frac{1+2\sum_{k\geq 1,\, 3\mid k} q^{k^2} - \sum_{k\geq 1,\, 3\nmid k} q^{k^2}}{1+2\sum_{k\geq 1}(-1)^k q^{k^2}} \\
&\equiv \frac{1+2\sum_{k\geq 1,\, 3\mid k} q^{k^2} + 3\sum_{k\geq 1,\, 3\nmid k} q^{k^2}}{1+2\sum_{k\geq 1} q^{k^2}} = \frac{1+2\sum_{k\geq 1} q^{k^2} + \sum_{k\geq 1,\, 3\nmid k} q^{k^2}}{1+2\sum_{k\geq 1} q^{k^2}} \\
&= 1 + \frac{\sum_{k\geq 1,\, 3\nmid k} q^{k^2}}{1+2\sum_{k\geq 1} q^{k^2}} \equiv 1 + \Bigg(\sum_{k\geq 1,\, 3\nmid k} q^{k^2}\Bigg)\Bigg(1+2\sum_{k\geq 1} q^{k^2}\Bigg) \pmod 4. \qquad \square
\end{aligned}$$

7. A Dissection Proof for Theorem 7

In this section, we will prove Theorem 7 with the help of dissections. This proof will reuse some of the identities from the last section. In addition, we will also need the following auxiliary result.

Lemma 5. *We have*

$$q^2 f_2^6 f_6^6 \equiv q^2 f_{16}^3 + q^6 f_{48}^3 \pmod 2. \tag{11}$$

Proof. Using the identity (cf., [21, Equation (3.12)])

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}},$$

we have

$$f_1^3 f_3^3 = \frac{f_1^4 f_3^4 f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_1^4 f_3^4 f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}} \equiv f_4^3 + q f_{12}^3 \pmod 2,$$

where we make repeated use of the identity $f_{2k} \equiv f_k^2 \pmod 2$. Raising the last equation to the fourth power gives $f_2^6 f_6^6 \equiv f_{16}^3 + q^4 f_{48}^3 \pmod 2$, and the result follows. $\square$

Alternate proof of Theorem 7. For a proof of Theorem 7, we start with the generating function

$$\mathrm{pd}_4(q) = \frac{f_4 f_6 f_8 f_{12}}{f_1 f_2 f_3 f_{24}} \equiv \left[\frac{f_1^3}{f_3}\right]^3 \pmod 2.$$

Applying Equation (3) and Lemma 4 one can provide the even and odd power dissection of $\mathrm{pd}_4(q)$,

$$\begin{aligned}
\mathrm{pd}_4(q) &= \mathrm{pd}_2(q)\,\mathrm{pd}_2(q^2)\\
&\equiv \left(\left[\frac{f_2^3}{f_6}\right]^2 + qf_{12}^2\right)\left(\left[\frac{f_4^3}{f_{12}}\right]^2 + q^2 f_{24}^2\right) \equiv \left(\left[\frac{f_2^3}{f_6}\right]^2 + qf_6^4\right)\left(\left[\frac{f_2^3}{f_6}\right]^4 + q^2 f_6^8\right)\\
&\equiv \left[\frac{f_2^3}{f_6}\right]^6 + q^2 f_2^6 f_6^6 + qf_2^{12} + q^3 f_6^{12}\\
&\equiv \left[\frac{f_2^3}{f_6}\right]^6 + q^2 f_{16}^3 + q^6 f_{48}^3 + qf_8^3 + q^3 f_{24}^3 \pmod 2, \qquad (12)
\end{aligned}$$

where the last line is a result of Equation (11) and once again applying the identity $f_{2k} \equiv f_k^2 \pmod 2$. In Equation (12), noting that

$$\left[\frac{f_2^3}{f_6}\right]^6 \equiv \left[\frac{f_1^3}{f_3}\right]^{12} \equiv [\mathrm{pd}_4(q)]^4 \equiv \mathrm{pd}_4(q^4) \pmod 2$$

and applying Equation (7) to $q^r f_{8r}^3 = q^r [f_8(q^r)]^3$, $r \in \{1, 2, 3, 6\}$, yields

$$\mathrm{pd}_4(q) \equiv \mathrm{pd}_4(q^4) + \sum_{n\geq 1,\, 2\nmid n} \left[q^{n^2} + q^{2n^2} + q^{3n^2} + q^{6n^2}\right] \pmod 2.$$

Raising this equation to the fourth power yields

$$\mathrm{pd}_4(q^4) \equiv \mathrm{pd}_4(q^{16}) + \sum_{n\geq 1,\, 2\nmid n} \left[q^{(2n)^2} + q^{2(2n)^2} + q^{3(2n)^2} + q^{6(2n)^2}\right] \pmod 2,$$

and combining the last two equations results in

$$\mathrm{pd}_4(q) \equiv \mathrm{pd}_4(q^{4^2}) + \sum_{\substack{n\geq 1,\, 2\nmid n,\\ i\in\{0,1\}}} \left[q^{(2^i n)^2} + q^{2(2^i n)^2} + q^{3(2^i n)^2} + q^{6(2^i n)^2}\right] \pmod 2.$$

Therefore, iterating one obtains, for $\ell \geq 1$,

$$\mathrm{pd}_4(q) \equiv \mathrm{pd}_4(q^{4^\ell}) + \sum_{\substack{n\geq 1,\, 2\nmid n,\\ 0\leq i<\ell}} \left[q^{(2^i n)^2} + q^{2(2^i n)^2} + q^{3(2^i n)^2} + q^{6(2^i n)^2}\right] \pmod 2,$$

where the first summand on the right-hand side contains only powers of q^{4^ℓ}. For $\ell \to \infty$, this results in the identity

$$\begin{aligned}
\mathrm{pd}_4(q) &\equiv 1 + \sum_{n\geq 1,\, 2\nmid n,\, i\geq 0} \left[q^{(2^i n)^2} + q^{2(2^i n)^2} + q^{3(2^i n)^2} + q^{6(2^i n)^2}\right]\\
&\equiv 1 + \sum_{n\geq 1} \left[q^{n^2} + q^{2n^2} + q^{3n^2} + q^{6n^2}\right] \pmod 2,
\end{aligned}$$

completing the proof of Theorem 7. □

We will end this section with a few Rogers-Ramanujan-type identities for $\mathrm{pd}_4(q)$.

Denote by $\varphi(q)$, $\psi(q)$, and $f(a,b)$ Ramanujan's first, second, and general theta functions, respectively, defined by, cf. [4, Entry 22(i), Entry 22(ii), Equation (18.1) and Entry 19],

$$\varphi(q) = f(q,q) = \sum_{n\in\mathbb{Z}} q^{n^2} = \frac{(q^2;q^2)_\infty(-q;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty} = \frac{f_2^2(-q;q)_\infty}{f_1(-q^2;q^2)_\infty^2} = \frac{f_2^5}{f_1^2 f_4^2},$$
$$\psi(q) = f(q,q^3) = \sum_{n\geq 0} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1},$$
$$f(a,b) = \sum_{n\in\mathbb{Z}} a^{n(n+1)/2} b^{n(n-1)/2} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Then, one can show the following identities by means of Theorem 7.

Theorem 14. *Working modulo* 2*, the following identities hold:*

$$\begin{aligned}\mathrm{pd}_4(q) \equiv \left[\frac{f_1^3}{f_3}\right]^3 &\equiv \frac{f(q^6,q^{10})}{\psi(q)} + \frac{f(q^{18},q^{30})}{\psi(q^3)} - 1\\ &\equiv f_1^{13}(q^6;q^{16})_\infty(q^{10};q^{16})_\infty + f_3^{13}(q^{18};q^{48})_\infty(q^{30};q^{48})_\infty - 1\\ &\equiv f_1^{13}(q^6;q^{16})_\infty(q^{10};q^{16})_\infty + q^3 f_3^{13}(q^6;q^{48})_\infty(q^{42};q^{48})_\infty\\ &\equiv f_3^{13}(q^{18};q^{48})_\infty(q^{30};q^{48})_\infty + q f_1^{13}(q^2;q^{16})_\infty(q^{14};q^{16})_\infty \pmod 2.\end{aligned}$$

Proof. Theorem 7 claims

$$\mathrm{pd}_4(q) = \frac{f_4f_6f_8f_{12}}{f_1f_2f_3f_{24}} \equiv \left[\frac{f_1^3}{f_3}\right]^3 \equiv \frac{1}{2}\left[\varphi(q)+\varphi(q^2)+\varphi(q^3)+\varphi(q^6)-2\right] \pmod 2, \quad (13)$$

where we make repeated use of the identity $f_{2k} \equiv f_k^2 \pmod 2$. The identities [4, Example (iv), p. 51] and [4, Corollary (ii), p. 49] imply, working modulo 4 and 2, respectively,

$$\varphi(q)+\varphi(q^2) \equiv \varphi(-q)+\varphi(q^2) = 2\frac{f^2(q^3,q^5)}{\psi(q)} \pmod 4, \quad (14)$$
$$f^2(q^3,q^5) \equiv f(q^6,q^{10}) \equiv \psi(q) + qf(q^2,q^{14}) \pmod 2, \quad (15)$$

while for $\psi(q)$ we have the identity

$$\psi(q) = \frac{f_2^2}{f_1} \equiv f_1^3 \pmod 2. \quad (16)$$

Substituting Equations (14) and (16) into the right-hand side of Equation (13) and applying the definition of $f(a,b)$, yields

$$\begin{aligned}\mathrm{pd}_4(q) \equiv \left[\frac{f_1^3}{f_3}\right]^3 &\equiv \frac{1}{2}\big[\varphi(q)+\varphi(q^2)\big]+\frac{1}{2}\big[\varphi(q^3)+\varphi(q^6)\big]-1\\ &\equiv \frac{f^2(q^3,q^5)}{\psi(q)}+\frac{f^2(q^9,q^{15})}{\psi(q^3)}-1 \equiv \frac{f(q^6,q^{10})}{f_1^3}+\frac{f(q^{18},q^{30})}{f_3^3}-1\\ &\equiv f_1^{13}(q^6;q^{16})_\infty(q^{10};q^{16})_\infty+f_3^{13}(q^{18};q^{48})_\infty(q^{30};q^{48})_\infty-1 \pmod 2.\end{aligned}$$

Now applying Equations (15) and (16) and the definition of $f(a,b)$, one obtains

$$\begin{aligned}\mathrm{pd}_4(q) \equiv \left[\frac{f_1^3}{f_3}\right]^3 &\equiv \frac{f^2(q^3,q^5)}{\psi(q)}+\frac{f^2(q^9,q^{15})}{\psi(q^3)}-1 \equiv \frac{f(q^6,q^{10})}{\psi(q)}+q^3\frac{f(q^6,q^{42})}{\psi(q^3)}\\ &\equiv f_1^{13}(q^6;q^{16})_\infty(q^{10};q^{16})_\infty+q^3f_3^{13}(q^6;q^{48})_\infty(q^{42};q^{48})_\infty \pmod 2\end{aligned}$$

and

$$\begin{aligned}\mathrm{pd}_4(q) \equiv \left[\frac{f_1^3}{f_3}\right]^3 &\equiv \frac{f^2(q^3,q^5)}{\psi(q)}+\frac{f^2(q^9,q^{15})}{\psi(q^3)}-1 \equiv \frac{f(q^{18},q^{30})}{\psi(q^3)}+q\frac{f(q^2,q^{14})}{\psi(q)}\\ &\equiv f_3^{13}(q^{18};q^{48})_\infty(q^{30};q^{48})_\infty+qf_1^{13}(q^2;q^{16})_\infty(q^{14};q^{16})_\infty \pmod 2. \ \square\end{aligned}$$

8. The Case of $\mathrm{pd}_{3^\ell}(q) \pmod 3$

Definition 3. Let

$$h(q)=\frac{f_1(q)^2}{f_2(q)},$$

noting by Gauss's square power identity [1, Equation (2.2.12)],

$$h(q)=\frac{(q;q)_\infty}{(-q;q)_\infty}=1+2\sum_{m\geq 1}(-1)^m q^{m^2}. \tag{17}$$

Using the previous definition, we prove the following.

Theorem 15. *We have* $\mathrm{pd}_3(q)\equiv\sum_{n\geq 0}e_{n1}q^n \pmod 3$ *where*

$$e_{n1}=\big|\{\textit{solutions to } n=k_0^2+3k_1^2 \mid k_m\in\mathbb{Z} \textit{ or } \mathbb{N} \textit{ when } k_m \textit{ is even or odd, respectively}\}\big|.$$

Proof. Using $p=3$ in Equation (2), we get $\mathrm{pd}_3(q)\equiv g(q)^2 \pmod 3$. As $g(q)=\frac{f_1(q)f_2(q)f_3(q)}{f_6(q)}\equiv h(q)^2 \pmod 3$, it follows, in a similar vein to Equation (2), that

$$\mathrm{pd}_3(q)\equiv h(q)^4=h(q)h(q)^3\equiv h(q)h(q^3) \pmod 3. \tag{18}$$

By Equation (17), we may write $h(q)\equiv\sum_{m\in\mathbb{Z},\,2|m}q^{m^2}+\sum_{m\geq 1,\,2\nmid m}q^{m^2} \pmod 3$. The result follows. $\square$

Remark 2. Note that e_{n1} gives an interesting alternate characterization of $\mathrm{PD}_3(n) \pmod 3$ to the one given in [7, Theorem 2] where it was shown that, for $n \geq 1$,

$$\mathrm{PD}_3(n) \equiv \left|\{\text{solutions to } n = k(k+1) + 3m(m+1) + 1 \,|\, k, m \geq 0\}\right| \pmod 3.$$

Note also that we have

$$n = k(k+1) + 3m(m+1) + 1 \quad \text{if and only if} \quad 4n = (2k+1)^2 + 3(2m+1)^2.$$

Thus, for $n \geq 1$, we can also write

$$\begin{aligned} \mathrm{PD}_3(n) &\equiv \left|\{\text{solutions to } 4n = k_0^2 + 3k_1^2 \,|\, k_m \geq 0, k_m \text{ odd}\}\right| \\ &\equiv \left|\{\text{solutions to } 4n = k_0^2 + 3k_1^2 \,|\, k_m \in \mathbb{Z}, k_m \text{ odd}\}\right| \pmod 3. \end{aligned}$$

Theorem 15 extends quite naturally as seen by the next result.

Theorem 16. *For all $n \geq 0$, $\mathrm{PD}_{3^\ell}(n) \equiv e_{n\ell} \pmod 3$ where*

$$\begin{aligned} e_{n\ell} = \Big|\{&\textit{solutions to } n = k_0^2 + \sum_{m=1}^{\ell-1} 3^m(k_m^2 + k_m'^2) + 3^\ell k_\ell^2 \,| \\ &k_m, k_m' \in \mathbb{Z} \textit{ or } \mathbb{N} \textit{ when } k_m, k_m' \textit{ is even or odd, respectively}\}\Big|. \end{aligned}$$

Proof. Using Equations (3) and (18),

$$\mathrm{pd}_{3^\ell}(q) = \prod_{m=0}^{\ell-1} \mathrm{pd}_3(q^{3^m}) \equiv h(q) \left[\prod_{m=1}^{\ell-1} h(q^{3^m})^2\right] h(q^{3^\ell}) \pmod 3.$$

The result follows. □

As an easy application of Theorem 16, we provide very short proofs of the following congruences for which proofs using dissections can be found in [7, Theorem 3] and [8, Theorem 3] for $\mathrm{PD}_3(9n+6)$ and $\mathrm{PD}_{3k}(3n+2)$, respectively.

Theorem 17. *For $n \geq 0$, $\mathrm{PD}_3(9n+6) \equiv 0 \pmod 3$. For all $n \geq 0$ and $\ell \geq 1$, $\mathrm{PD}_{3^\ell}(3n+2) \equiv 0 \pmod 3$.*

Proof. For the first congruence, if $e_{n1} \neq 0$ in Theorem 16, then we can write $n = k_0^2 + 3k_1^2$ for suitable integers k_0, k_1. A straightforward calculation shows that $n \not\equiv 6 \pmod 9$ as $0, 1, 4, 7$ are the only quadratic residues modulo 9. Hence, $e_{n1} = 0$ for all $n \equiv 6 \pmod 9$.

The second congruence follows similarly. If $e_{n\ell} \neq 0$ in Theorem 16, then, since $0, 1$ are the only quadratic residues modulo 3, we can write

$$n = k_0^2 + \sum_{m=1}^{\ell-1} 3^m(k_m^2 + k_m'^2) + 3^\ell k_\ell^2 \equiv k_0^2 \not\equiv 2 \pmod 3.$$

Thus, $e_{n\ell} = 0$ for all $n \equiv 2 \pmod 3$. □

As a further application of Theorem 16, we prove the following newly observed congruences. We have not found the first congruence in the literature, but expect it to be known.

Theorem 18. *For* $n \geq 1$, $\mathrm{PD}_3(2n) \equiv 0 \pmod 3$. *For all* $n \geq 0$ *and* $\ell \geq 3$, $\mathrm{PD}_{3^\ell}(27n+9) \equiv 0 \pmod 3$. *For all* $n \geq 0$ *and* $\ell \neq 2$, $\mathrm{PD}_{3^\ell}(27n+18) \equiv 0 \pmod 3$.

Proof. The first congruence is an immediate consequence of Theorem 2 as $n = k(k+1) + 3m(m+1) + 1$ is odd for nonnegative integers k, m.

The case $\ell = 1$ of the third congruence follows similarly to that of the first congruence in the previous theorem. In particular, a straightforward calculation shows that $n = k_0^2 + 3k_1^2 \not\equiv 18 \pmod{27}$ as $0, 1, 4, 7, 9, 10, 13, 16, 19, 22$, and 25 are the only quadratic residues modulo 27. Hence, $e_{n1} = 0$ for all $n \equiv 18 \pmod{27}$.

For $\ell \geq 3$ in the third congruence, we argue as follows. We want to count the number of solutions to the Diophantine equation

$$27n + 18 = k_0^2 + \sum_{m=1}^{\ell-1} 3^m (k_m^2 + k_m'^2) + 3^\ell k_\ell^2 \tag{19}$$

modulo 3 with sign constraints as stated in Theorem 15. Note that we have $k_0^2 \equiv 0 \pmod 3$, so that $3 \mid k_0$, and we can write $k_0 = 3a$ for some integer a. This gives

$$27n + 18 = 3(k_1^2 + k_1'^2) + 9(a^2 + k_2^2 + k_2'^2) + \ldots + 3^\ell k_\ell^2. \tag{20}$$

In particular, $3(k_1^2 + k_1'^2) \equiv 0 \pmod 9$. Thus, $3 \mid (k_1^2 + k_1'^2)$, but as 0 and 1 are the only quadratic residues modulo 3, we conclude $3 \mid k_1, k_1'$. Hence $3(k_1^2 + k_1'^2) \equiv 0 \pmod{27}$, and Equation (20) implies $9(a^2 + k_2^2 + k_2'^2) \equiv 18 \pmod{27}$, so that we conclude

$$a^2 + k_2^2 + k_2'^2 \equiv 2 \pmod 3. \tag{21}$$

It is immediate that, together with $(3a, k_1, k_1', k_2, k_2', k_3, k_3', \ldots, k_\ell)$, also $(3k_2, k_1, k_1', k_2', a, k_3, k_3', \ldots, k_\ell)$ and $(3k_2', k_1, k_1', a, k_2, k_3, k_3', \ldots, k_\ell)$ solve Equation (19). As a result, the mapping

$$\Psi((k_0, k_1, k_1', k_2, k_2', k_3, k_3', \ldots, k_\ell)) = (3k_2, k_1, k_1', k_2', k_0/3, k_3, k_3', \ldots, k_\ell)$$

defines a bijection on the solution set of Equation (19) such that Ψ^3 is the identity map. The proof will be finished by showing that each orbit of the solution set under iterates of Ψ has order 3. To see this, by way of contradiction, suppose $(3a, k_1, k_1', k_2, k_2', k_3, k_3', \ldots, k_\ell) = (3k_2, k_1, k_1', k_2', a, k_3, k_3', \ldots, k_\ell)$. Comparing components yields $a = k_2 = k_2'$, hence $a^2 + k_2^2 + k_2'^2 \equiv 0 \pmod 3$, contradicting Equation (21) and completing the proof of the third congruence.

The second congruence follows analogously by systematically replacing 18 by 9 in the proof for $\ell \geq 3$. □

It is noteworthy that we can recreate some of our structural results on $\mathrm{pd}_{2^\ell}(q)$ modulo 2 from Section 3 in the context of $\mathrm{pd}_{3^\ell}(q)$ modulo 3. In particular, the grouping argument of the last theorem once again can be iterated.

Theorem 19. *For all $n \geq 0$, $\ell \geq 2$, we have $\mathrm{PD}_{3^\ell}(3n) \equiv e^*_{n\ell} \pmod 3$, where*

$$e^*_{n\ell} = \big|\{\textit{solutions to } n = k_0^2 + k_0'^2 + 3^{\ell-1}(k_1^2 + k_1'^2) \mid \\ k_m, k_m' \in \mathbb{Z} \textit{ or } \mathbb{N} \textit{ when } k_m, k_m' \textit{ is even or odd, respectively}\}\big|.$$

Proof. We will be grouping the solutions $(k_0, k_1, k_1', k_2, k_2', k_3, k_3', \ldots, k_\ell)$ of the equation

$$3n = k_0^2 + 3(k_1^2 + k_1'^2) + 9(k_2^2 + k_2'^2) + 27(k_3^2 + k_3'^2) + \ldots + 3^\ell k_\ell^2 \tag{22}$$

into triplets. Note that we have $k_0^2 \equiv 0 \pmod 3$, so that $3 \mid k_0$, and we can write $k_0 = 3a$ for some integer a. This gives

$$3n = 9a^2 + 3(k_1^2 + k_1'^2) + 9(k_2^2 + k_2'^2) + 27(k_3^2 + k_3'^2) + \ldots + 3^\ell k_\ell^2.$$

Note that with $(3a, k_1, k_1', k_2, k_2', k_3, k_3', \ldots, k_\ell)$ also $(3k_2, k_1, k_1', k_2', a, k_3, k_3', \ldots, k_\ell)$ and $(3k_2', k_1, k_1', a, k_2, k_3, k_3', \ldots, k_\ell)$ are solutions of Equation (22), which provide us with a triplet of solutions unless $k_2 = k_2' = a$. This leaves us with considering solutions of the form $(3a, k_1, k_1', a, a, k_3, k_3', \ldots, k_\ell)$ and the equation

$$3n = 9a^2 + 3(k_1^2 + k_1'^2) + 9(a^2 + a^2) + 27(k_3^2 + k_3'^2) + \ldots + 3^\ell k_\ell^2.$$

Note that with $(3a, k_1, k_1', a, a, k_3, k_3', \ldots, k_\ell)$ also $(3k_3, k_1, k_1', k_3, k_3, k_3', a, \ldots, k_\ell)$ and $(3k_3', k_1, k_1', k_3',\ k_3', a, k_3, \ldots, k_\ell)$ are solutions of Equation (22), which provide us with a triplet of solutions unless $k_3 = k_3' = a$. This leaves us with considering solutions of the form $(3a, k_1, k_1', a, a, a, a, k_4, k_4', \ldots, k_\ell)$, and this process can be iterated until we are left to consider solutions of the form $(3a, k_1, k_1', a, a, \ldots, a, a, k_\ell)$ and the equation $3n = 9a^2 + 3(k_1^2 + k_1'^2) + 9(a^2 + a^2) + 27(a^2 + a^2) + \ldots + 3^{\ell-1}(a^2 + a^2) + 3^\ell k_\ell^2$ $= 3(k_1^2 + k_1'^2) + 3^\ell(a^2 + k_\ell^2)$ In particular, $n = k_1^2 + k_1'^2 + 3^{\ell-1}(a^2 + k_\ell^2)$, and the result follows. □

The next result can be proven the same way as Lemma 2

Lemma 6. *If there exist j and r such that the congruence $\mathrm{PD}_{3^j}(3^j n + r) \equiv 0 \pmod 3$ holds for all $n \geq 0$, then $\mathrm{PD}_{3^\ell}(3^j n + r) \equiv 0 \pmod 3$ for all $n \geq 0$ and $\ell \geq j$.*

As an application of these results, we generalize Theorem 18.

Corollary 2. *For all $n \geq 0$, $k \geq 1$ and $\ell \geq 2k+1$, $\mathrm{PD}_{3^\ell}\left(3^{2k}(3n+1)\right) \equiv \mathrm{PD}_{3^\ell}\left(3^{2k}(3n+2)\right) \equiv 0 \pmod 3$.*

Proof. We will focus on the congruence $\mathrm{PD}_{3^\ell}\left(3^{2k}(3n+1)\right) \equiv 0 \pmod 3$. With Lemma 6, it will be sufficient to prove $\mathrm{PD}_{3^{2k+1}}\left(3^{2k}(3n+1)\right) \equiv 0 \pmod 3$, and according to Theorem 19 we can focus on the equation

$$3^{2k-1}(3n+1) = k_0^2 + k_0'^2 + 3^{2k}(k_1^2 + k_1'^2).$$

Note that $k_0^2+k_0'^2 \equiv 0 \pmod 3$. We conclude $3 \mid k_0, k_0'$ and can write $k_0 = 3a_1, k_0' = 3a_1'$ for integers a_1, a_1'. This leads to the new equation $3^{2k-3}(3n+1) = a_1^2 + a_1'^2 + 3^{2k-2}(k_1^2 + k_1'^2)$. We can iterate this argument to show that $k_0 = 3^k a_k, k_0' = 3^k a_k'$ with $3^{2k-1}(3n+1) = 3^{2k}(a_k^2 + a_k'^2 + k_1^2 + k_1'^2)$. This is an apparent contradiction as the right-hand side of this equation is divisible by 3^{2k} while the left-hand side is not. This shows $e^*_{3^{2k}(3n+1),2k+1} = 0$. ☐

9. Open Problems

We conclude with some additional conjectured congruences. While these congruences may easily be verifiable by a computer proving system along the lines of [17], it would be interesting to find some elementary proofs.

Conjecture 1. For $n \geq 0$, we have

$$\begin{aligned}
\mathrm{PD}_2(16n+12) &\equiv 0 \pmod 4,\\
\mathrm{PD}_2(24n+20) &\equiv 0 \pmod 4,\\
\mathrm{PD}_2(25n+5) &\equiv 0 \pmod 4,\\
\mathrm{PD}_2(32n+24) &\equiv 0 \pmod 4,\\
\mathrm{PD}_2(48n+26) &\equiv 0 \pmod 4,
\end{aligned}$$

and for $r \in \{5, 11, 15, 17\}$,

$$\mathrm{PD}_9(54n+3r) \equiv 0 \pmod 3.$$

References

[1] G.E. Andrews, *The Theory of Partitions*, Cambridge University Press, Cambridge, 1998.

[2] G.E. Andrews, R.P. Lewis, and J. Lovejoy, Partitions with designated summands, *Acta Arith.* **105** (2002), 51–66.

[3] N.D. Baruah and K.K. Ojah, Partitions with designated summands in which all parts are odd, *Integers* **15** (2015), #A9.

[4] B.C. Berndt, *Ramanujan's Notebooks, Part III*, Springer, New York, 1991.

[5] W.Y.C. Chen, K.Q. Ji, H.-T. Jin, and E.Y.Y. Shen, On the number of partitions with designated summands, *J. Number Theory* **133** (2013), 2929–2938.

[6] S.P. Cui and N.S.S. Gu, Arithmetic properties of ℓ-regular partitions, *Adv. in Appl. Math.* **51** (2013), 507–523.

[7] R. da Silva and J. Sellers, New congruences for 3-regular partitions with designated summands, *Integers* **20A** (2020), #A6.

[8] R. da Silva and J. Sellers, Infinitely many congruences for k-regular partitions with designated summands, *Bull. Braz. Math. Soc.*, New Series **51** (2020), 357–370.

[9] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and Its Applications **35**, Cambridge University Press, Cambridge, 1990.

[10] B. Hemanthkumar, H.S. Sumanth Bharadwaj, and M.S. Mahadeva Naika, Congruences modulo small powers of 2 and 3 for partitions into odd designated summands, *J. Integer Seq.* **20** (2017), Art. 17.4.3.

[11] D. Herden, M.R. Sepanski, J. Stanfill, C.C. Hammon, J. Henningsen, H. Ickes, J.M. Menendez, T. Poe, I. Ruiz, and E.L. Smith, Counting the parts divisible by k in all the partitions of n whose parts have multiplicity less than k, *Integers* **22** (2022), #A49.

[12] M.D. Hirschhorn and J.A. Sellers, Parity results for partitions wherein each part appears an odd number of times, *Bull. Aust. Math. Soc.* **99** (2019), 51–55.

[13] W.J. Keith and F. Zanello, Parity of the coefficients of certain eta-quotients, *J. Number Theory* **235** (2022), 275–304.

[14] B.L.S. Lin, The number of tagged parts over the partitions with designated summands, *J. Number Theory* **184** (2018), 216–234.

[15] M.S. Mahadeva Naika and D.S. Gireesh, Congruences for 3-regular partitions with designated summands, *Integers* **16** (2016), #A25.

[16] M.S. Mahadeva Naika and S. Shivaprasada Nayaka, Arithmetic properties of 3-regular bipartitions with designated summands, *Mat. Vesnik* **69** (2017), 192–206.

[17] C.S. Radu and J.A. Sellers, Congruence properties modulo 5 and 7 for the pod function, *Int. J. Number Theory* **7** (2011), 2249–2259.

[18] J.A. Sellers and F. Zanello, On the parity of the number of partitions with odd multiplicities, *Int. J. Number Theory* **17** (2021), 1717–1728.

[19] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://oeis.org

[20] E.X.W. Xia, Arithmetic properties of partitions with designated summands, *J. Number Theory* **159** (2016), 160–175.

[21] E.X.W. Xia and O.X.M. Yao, Analogues of Ramanujan's partition identities, *Ramanujan J.* **31** (2013), 373–396.

[22] O.X.M. Yao, New congruences modulo powers of 2 and 3 for 9-regular partitions, *J. Number Theory* **142** (2014), 89–101.