RAINBOW NUMBERS OF $[m] \times [n]$ FOR $x_1 + x_2 = x_3$

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Abstract

Consider the set $[m] \times [n] = \{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\}$ and the equation *eq*: $x_1 + x_2 = x_3$. The *rainbow number* of $[m] \times [n]$ for *eq*, denoted $\mathrm{rb}([m] \times [n], eq)$, is the smallest number of colors such that for every surjective $\mathrm{rb}([m] \times [n], eq)$-coloring of $[m] \times [n]$ there must exist a solution to *eq*, with component-wise addition, where every element of the solution set is assigned a distinct color. This paper determines that $\mathrm{rb}([m] \times [n], eq) = m + n + 1$ for all values of m and n that are greater than or equal to two.

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1. Introduction

Given a set S, a coloring of S assigns each element a color. Ramsey theory is the study of guaranteeing monochromatic structures in S. On the other hand, anti-Ramsey theory is the study of guaranteeing polychromatic (or rainbow) structures in S and has gained the interest of many authors when S is $[n] = \{1, 2, \ldots, n\}$ or $\mathbb{Z}_n$ [1, 2, 5, 8, 9, 10]. As an example, the *anti-van der Waerden number on* $[n]$, denoted $\mathrm{aw}([n], k)$, is the smallest number of colors such that every exact (onto) $\mathrm{aw}([n], k)$-coloring of $[n]$ is guaranteed to have an arithmetic progression of length k, where each element of the progression is colored distinctly, [4, 6, 13]. The anti-van der Waerden number has also been studied in graphs, [3, 11, 12], on finite abelian groups, [14], and were generalized further on $\mathbb{Z}_n$ for linear equations, [1, 5, 8, 10]. This inspired the authors in [7] to look into rainbow numbers of $[n]$ for linear equations.

The rainbow number of $[n]$ for eq, denoted $\mathrm{rb}([n], eq)$, is the smallest number of colors such that for every exact $\mathrm{rb}([n], eq)$-coloring of $[n]$, there exists a solution to eq with every member of the solution set assigned a distinct color. In this paper, the rainbow number of $[m]\times[n]$ for equation $x_1+x_2 = x_3$ is solved completely. Section 2 establishes vocabulary, contains preliminary results, and provides examples and figures that are referenced throughout the paper. Each succeeding section contains a case analysis that ultimately leads to a complete solution. Section 3 analyzes $(m + n + 1)$-colorings of $[m] \times [n]$ when the main diagonal has three colors. The analysis continues into Section 4 when the main diagonal has four or more colors. Finally, Section 5 discusses some forbidden structures in rainbow-free $(m + n + 1)$-colorings of $[m] \times [n]$ and completes the analysis.

2. Preliminaries

An *r-coloring* of a set S is a function $c : S \to [r]$, where $[r] = \{1, 2, \ldots, r\}$, and an r-coloring is *exact* if it is surjective. If $X \subseteq S$, then $c(X) = \{c(x) : x \in X\}$. Given an equation eq, a solution in S to eq is a subset of elements of S that satisfy the equation. A solution s is a *rainbow solution* to eq with respect to coloring c if the colors of the elements of s are pairwise distinct. If the context is clear the reference to the coloring c will be dropped and 's is a rainbow solution' will often be used. The *rainbow number of S for eq*, denoted $\mathrm{rb}(S, eq)$, is the smallest number such that every exact $\mathrm{rb}(S, eq)$-coloring of S contains a rainbow solution to eq. A coloring of S that has no rainbow solutions to eq is called *rainbow-free*. Although solutions to eq are sets, they will often be thought of as lists. That is, if it is claimed that $\{\alpha, \beta, \gamma\}$ is a solution then, in most cases, $\alpha + \beta = \gamma$. Solutions with repeated elements cannot be rainbow and will be referred to as *degenerate*. If a set S has no solutions to eq then the convention is that $\mathrm{rb}(S, eq) = |S| + 1$.

Throughout the paper, the set S that will be discussed is

$$[m] \times [n] = \{(i, j) : i, j \in \mathbb{Z}, 1 \leq i \leq m \text{ and } 1 \leq j \leq n\},$$

with $m \leq n$, and the equation, eq, is $x_1 + x_2 = x_3$. Note that addition is componentwise and that if $m = 1$, there are no solutions. Following convention, $\mathrm{rb}([1] \times [n], eq) = n + 1$.

Define the k^{th} *diagonal* D_k as the set

$$D_k = \big\{(i, j) \in [m] \times [n] \,:\, m - k = i - j\big\}.$$

For reference, the authors view the array with m rows and n columns and the upper left element as $(1, 1)$. Thus, $D_1 = \{(m, 1)\}$ and is in the bottom left corner. The diagonal D_m is called the *main diagonal*. A diagonal that is not the main diagonal will be referred to as an *off-diagonal*. Observe that $[m] \times [n]$ has $m + n - 1$ diagonals.

Definition 1 was established by the authors in [7].

Definition 1. Let c be an exact r-coloring of $[n]$. Define

$$\mathcal{C}_i = \{a \in [n] \,:\, c(a) = i\}$$

and $s_i = \min_{s \in \mathcal{C}_i} s$.

Note that for any exact r-coloring c, it is always possible to have $s_i < s_j$ for $i < j$. A modified definition is introduced for the purposes of this paper. Let c be a coloring of $[m] \times [n]$. Without loss of generality, it can and will be assumed that if $|c(D_m)| = \ell$, then $c(D_m) = \{1, 2, \ldots, \ell\}$. Now, define $s_1 = 1$ and for $k \geq 2$, define

$$s_k = \min_{1 \leq x \leq m} \big\{x : c((x, x)) \neq c((s_j, s_j)) \text{ for all } j < k\big\}.$$

For any exact r-coloring c, it is always possible to have $s_i < s_j$ for $i < j$. If that is not the case, say $s_i > s_j$ and $i < j$, an isomorphic coloring can be created by swapping the color of any element with color i to have color j and vice versa. Thus, it will also be assumed that $s_i < s_j$ when $i < j$.

In [7], rainbow numbers for the set $[m]$ for equation $x_1 + x_2 = x_3$ are investigated. Note that that main diagonal of $[m] \times [n]$ behaves similarly to $[m]$. Thus, results from [7] can be applied to the main diagonal. Further, s_i in Lemma 1 is defined similarly to the s_i established in this paper.

The following four results are from [7] or extensions of results from [7] to $[m] \times [n]$. In particular, Lemma 2 will limit the number of colors that can appear in D_m. The wording has been modified to match the language used in this paper.

Lemma 1 ([7]). *Let c be an exact, rainbow-free r-coloring of $[m]$ for $x_1 + x_2 = x_3$, with color set $\{0, 1, \ldots, r-1\}$, then*

(a) *if $s_i = \ell$, then $2\ell \le s_{i+1}$ for $0 \le i \le r-2$,*

(b) *$2^i \le s_i$ for $0 \le i \le r-1$.*

Theorem 1. *If c is a rainbow-free coloring of $[m]$ and $\ell = |c([m])|$, then $2^{i-2}s_2 \le s_i$ for $2 \le i \le \ell$. In particular, $2^{\ell-2}s_2 \le m$.*

Proof. The proof will proceed by induction on i. The base case, when $i = 2$, is clear. As the inductive hypothesis, assume for some $2 \le j \le \ell - 1$ that $2^{j-2}s_2 \le s_j$ which gives $2^{j-1}s_2 \le 2s_j$. Further, by Lemma 1 (a), $2s_j \le s_{j+1}$. Therefore, $2^{j-1}s_2 \le s_{j+1}$ which completes the induction. □

Theorem 2 ([7]). *For $m \ge 3$, $\mathrm{rb}([m], x_1 + x_2 = x_3) = \lfloor \log_2(m) + 2 \rfloor$.*

Lemma 2. *If c is a rainbow-free coloring of $[m] \times [n]$ for eq with $m \le n$, then $|c(D_m)| \le \lfloor \log_2(m) + 1 \rfloor$ and $|c(D_m)| \le \log_2(m/s_2) + 2$.*

Proof. Note that solutions to $x + y = z$ in D_m correspond bijectively to solutions to $x + y = z$ in $[m]$, so Theorem 2 can be applied to D_m. This gives $|c(D_m)| \le \lfloor \log_2(m) + 1 \rfloor$ immediately. Further, define $\ell = |c(D_m)|$, and note that Theorem 1 implies that $2^{\ell-2}s_2 \le m$. Thus, $\ell \le \log_2(m/s_2) + 2$, as desired. □

Lemma 3 will provide an exact $(m+n)$-coloring that avoids rainbow solutions, which establishes a lower bound on $\mathrm{rb}([m] \times [n], eq)$. The remaining results analyze the appearance of solutions to eq in $[m] \times [n]$ and the structure of a rainbow-free coloring on $[m] \times [n]$ to help establish an upper bound on $\mathrm{rb}([m] \times [n], eq)$.

Lemma 3. *If $2 \le m \le n$, then $m + n + 1 \le \mathrm{rb}([m] \times [n], eq)$.*

Proof. Let $c : [m] \times [n] \to [m+n]$ be defined by

$$c((i,j)) = \begin{cases} 1 & \text{if } i < m \text{ and } j < n, \\ i+1 & \text{if } i < m \text{ and } j = n, \\ j+m & \text{if } i = m. \end{cases}$$

Note that $(1, n)$ and $(m, 1)$ are not in any solution to eq. If $\alpha, \beta, \gamma \in [m] \times [n]$ such that $\alpha + \beta = \gamma$, then $c(\alpha) = c(\beta) = 1$. Therefore, c is rainbow-free for eq and $m + n + 1 \le \mathrm{rb}([m] \times [n], eq)$. □

Lemma 4. *If c is a rainbow-free coloring of $[m] \times [n]$ for eq with $m \le n$, then for all D_x with $x \ne m$, $\big|c(D_x) \setminus c\big(D_m\big)\big| \le 1$.*

Proof. If $m = 1$ each diagonal has one element and the result follows. Thus, assume $2 \le m$. Since $|D_1| = |D_{m+n-1}| = 1$, it is certainly true that $|c(D_1) \setminus c(D_m)| \le 1$ and $|c(D_{m+n-1}) \setminus c(D_m)| \le 1$.

For the purpose of contradiction, assume $1 < x < m+n-1$ and $|c(D_x) \setminus c(D_m)| \ge 2$. Then there exists $\beta,\ \gamma \in D_x$ such that $c(\beta), c(\gamma) \in c(D_x) \setminus c(D_m)$. However, there exists $\alpha \in D_m$ such that $\{\alpha, \beta, \gamma\}$ is a rainbow solution, a contradiction. Thus, $\big|c(D_x) \setminus c\big(D_m\big)\big| \le 1$. □

Corollary 1 follows quickly from Lemmas 3 and 4. This allows most of the rest of the paper to focus on the situation when $3 \le m \le n$.

Corollary 1. *If $m = 2$ and $m \le n$, then* $\mathrm{rb}([m] \times [n]) = m + n + 1$*.*

Corollary 2. *If c is an exact, rainbow-free $(m + n + 1)$-coloring of $[m] \times [n]$ for eq with $3 \le m \le n$, then $|c(D_m)| \ge 3$.*

Proof. Lemma 4 implies that $\big|c(D_x) \setminus c(D_m)\big| \le 1$ for all $x \ne m$. So, $\big|c\big([m] \times [n]\big) \setminus c(D_m)\big| \le m + n - 2$. Thus, $|c(D_m)| \ge 3$. □

Let c be a coloring of $[m] \times [n]$. Diagonal D_j *contributes color* x if $x \in c(D_j) \setminus c(D_m)$ and $x \notin c(D_i)$ for all $i < j$. Otherwise, D_j does not contribute color x. In general, if D_j contributes any color it is called a *contributing diagonal*. If a diagonal does not contribute any color, it is called a *non-contributing diagonal*.

Lemma 5 and the proceeding corollary will establish results related to contributing off-diagonals. In particular, they provide an upper bound on the number of non-contributing off-diagonals and subsequently a lower bound on the number of contributing off-diagonals. This furthers the restrictions that applying an exact, rainbow-free $(m + n + 1)$-coloring has on $[m] \times [n]$ when the main diagonal has exactly three colors, as will be shown in Lemma 10, when the main diagonal has exactly four or more colors, as will be shown in Theorem 4 and more generally in Lemma 18.

Lemma 5. *If c is an exact, rainbow-free $(m + n + 1)$-coloring of $[m] \times [n]$ with $3 \le m \le n$, then there are at most $|c(D_m)| - 3$ off-diagonals that do not contribute a color.*

Proof. Define $\ell = |c(D_m)|$, and note that there are $m + n - 2$ off-diagonals and $m+n+1-\ell$ colors that appear in the off-diagonals that do not appear in the main diagonal. Let k be the number of off-diagonals that do not contribute a color. For the sake of contradiction, assume $k \ge \ell-2$. This implies that $m+n-2-k \le m+n-\ell$ off-diagonals must contribute a color. Thus, by the pigeon hole principle, some off-diagonal contains two colors that do not appear in the main diagonal. Since c was assumed to be rainbow-free, Lemma 4 is contradicted. Therefore, $k \le \ell - 3$. □

Corollary 3. *If c is an exact, rainbow-free $(m+n+1)$-coloring of $[m] \times [n]$ with $3 \leq m \leq n$, then there are at least $m+n-1-\log_2(m/s_2)$ contributing off-diagonals.*

Proof. Define $\ell = |c(D_m)|$, and observe that Lemma 5 indicates that there are at most $\ell - 3$ non-contributing off-diagonals. Since there are $m+n-2$ off-diagonals in total, and off-diagonals are either contributing or non-contributing, there are at least $(m+n-2)-(\ell-3) = m+n-\ell+1$ contributing off-diagonals. Since Lemma 2 implies that $\ell \leq \log_2(m/s_2)+2$, there are at least $m+n-\log_2(m/s_2)-1$ contributing off-diagonals. □

Let $\alpha = (a_1, a_2) \in D_a$ and $\beta = (b_1, b_2) \in D_b$ with $a \neq b$. If $a_1 < b_1$ and $a_2 < b_2$, there is a *jump from α to β*, and the *jump distance* is defined to be $(b_1 - a_1) + (b_2 - a_2)$. Alternatively, there is a jump from α to β when there exists some $\delta \in [m] \times [n]$ such that $\alpha + \delta = \beta$, where $\delta = (d_1, d_2)$ is called *the jump* from α to β and has jump distance $d_1 + d_2$.

Example 1. In Figure 1, $\alpha = (2,7)$, $\beta = (4,11)$, $\gamma = (5,2)$ and $\tau = (7,3)$. The jump from α to β is $\delta = (2,4)$ and has jump distance six. The jump from γ to τ is $(2,1)$ and has jump distance three.

When there is a jump from $\alpha = (a_1, a_2)$ to $\beta = (b_1, b_2)$ with $\alpha \in D_a$ and $\beta \in D_b$, certain diagonals exhibit the special property that all elements within the diagonals make additional jumps with α or β. This set of diagonals is

$$S := \{D_x \mid m + a_2 - b_1 < x < m + b_2 - a_1 \text{ and } x \notin \{a, b, m\}\}. \quad (1)$$

To visualize this, a rectangle is drawn using α and β as corners. The diagonals with indices between $m + a_2 - b_1$ and $m + b_2 - a_1$ are those that intersect this rectangle not including the lower left and upper right corners. To finalize S, simply remove the diagonals containing α and β and the main diagonal if applicable.

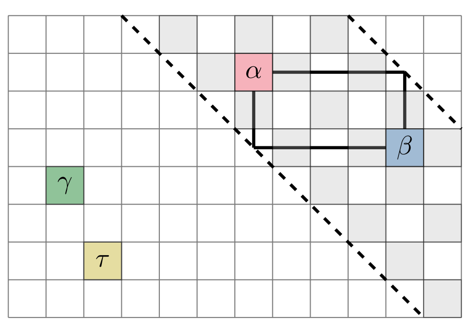

Figure 1: Jumps from α to β and from γ to τ from Example 1 are shown. Also, a visualization of the set of diagonals S (in gray) with respect to α and β is given.

The next two lemmas will show that if γ is in a diagonal of S, then either α makes a jump to γ or γ makes a jump to β. Lemma 6 will show this for all diagonals of S between D_a and D_b and Lemma 7 will consider the remainder of the diagonals in S.

Lemma 6. *Suppose there is a jump from $\alpha = (a_1, a_2) \in D_a$ to $\beta = (b_1, b_2) \in D_b$. If $\gamma = (g_1, g_2) \in D_g$ such that $a < g < b$ or $b < g < a$, then either α makes a jump to γ or γ makes a jump to β.*

Proof. Since there is a jump from α to β, it follows that $a_1 < b_1$ and $a_2 < b_2$. First consider when $a < g < b$. Then $m - a_1 + a_2 < m - g_1 + g_2 < m - b_1 + b_2$, so

$$a_2 - a_1 < g_2 - g_1 < b_2 - b_1.$$

Suppose $g_1 < b_1$. Then $g_2 - g_1 < b_2 - b_1 < b_2 - g_1$ which implies that $g_2 < b_2$. Thus, γ makes a jump to β.

On the other hand, suppose $b_1 \leq g_1$. In this case, $a_1 < b_1 \leq g_1$, so $a_2 - g_1 < a_2 - a_1 < g_2 - g_1$. This yields $a_2 < g_2$, which means α makes a jump to γ.

The case where $b < g < a$ follows a similar argument and leads to the desired conclusion. □

Lemma 7. *Suppose there is a jump from $\alpha = (a_1, a_2) \in D_a$ to $\beta = (b_1, b_2) \in D_b$ and define $\ell = \min\{b_1 - a_1, b_2 - a_2\}$.*

(a) *If $a < b$ and $\gamma \in D_g$ such that $a - \ell < g < a$ or $b < g < b + \ell$, then either there is a jump from α to γ or there is a jump from γ to β.*

(b) *If $b < a$ and $\gamma \in D_g$ such that $b - \ell < g < b$ or $a < g < a + \ell$, then either there is a jump from α to γ or there is a jump from γ to β.*

Proof. Let $a < b$ and $\gamma = (g_1, g_2) \in D_g$ and first consider $a - \ell < g < a$. If $a_2 < g_2$, then $a_2 - g_1 < g_2 - g_1 < a_2 - a_1$ implying that $a_1 < g_1$ and α makes a jump to γ. Alternatively, suppose $g_2 \leq a_2$. Since $a_2 < b_2$, it follows that $g_2 < b_2$. Note that $a - \ell < g < a$ implies that $a_2 - a_1 - \ell < g_2 - g_1 < a_2 - a_1$. This yields $g_1 < a_1 + \ell$. Since $\ell \leq b_1 - a_1$, it follows that $g_1 < b_1$. Thus, γ makes a jump to β.

Second, consider $b < g < b + \ell$. If $g_2 < b_2$, then $g_2 - b_1 < b_2 - b_1 < g_2 - g_1$, implying that $g_1 < b_1$, and there is a jump from γ to β. Alternatively, suppose $b_2 \leq g_2$. Since $a_2 < b_2$, it follows that $a_2 < g_2$. Then $b < g < b + \ell$ implies that $b_2 - b_1 < g_2 - g_1 < b_2 - b_1 + \ell$. This yields $b_1 - \ell < g_1$. Since $\ell \leq b_1 - a_1$, it follows that $a_1 < g_1$. Thus, α makes a jump to γ.

The result when $b < a$ follows from a similar argument. □

Given the diagonals of two elements, Lemma 8 will determine the diagonals that the sum and difference of the elements land in. For this reason, the authors refer to this as the 'Landing Lemma.' This lemma becomes important quickly, as many succeeding proofs require a working knowledge of the locations of contributing diagonals, non-contributing diagonals, or elements of a jump.

Lemma 8 (Landing Lemma)**.** *If $\alpha \in D_a$, $\beta \in D_b$, and $\alpha+\beta \in [m]\times[n]$, then $\alpha+\beta \in D_{a+b-m}$. Similarly, if $\alpha-\beta \in [m]\times[n]$ then $\alpha-\beta \in D_{a-b+m}$.*

Proof. Let $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2)$ and suppose $\alpha+\beta \in D_k$ for some appropriately defined k. Then, $m-a = a_1 - a_2$ and $m - b = b_1 - b_2$. Moreover, $m - k = a_1 + b_1 - (a_2 + b_2)$ and thus

$$k = m - (a_1 + b_1) + a_2 + b_2 = a + b - m.$$

A similar analysis gives that $\alpha - \beta \in D_{a-b+m}$. □

Let c be a coloring of $[m]\times[n]$. Recall that $s_2 = \min\{x \mid c((x,x)) \neq c((1,1))\}$. Define

$$W_1 := \{(x,y) \in [m]\times[n] \mid (x,y) + (s_2,s_2) \in [m]\times[n]\},$$
$$W_2 := \{(x,y) \in [m]\times[n] \mid (x,y) - (s_2,s_2) \in [m]\times[n]\},$$

and $W := W_1 \cup W_2$. See Figure 2 for a visualization of these and the following definitions.

The next definitions will not be used until Section 5 but are included here as they are closely related to W, W_1 and W_2. Define

$$Y_1 := \{(x,y) \in [m]\times[n] : m < x + s_2 \text{ and } y - s_2 < 0\},$$
$$Y_2 := \{(x,y) \in [m]\times[n] : x - s_2 < 0 \text{ and } m < y + s_2\},$$

and $Y := Y_1 \cup Y_2$.

Let c be a coloring of $[m]\times[n]$. If $\alpha \in D_a, \beta \in D_{a+1}$ where D_a and D_{a+1} are contributing, and $c(\alpha), c(\beta) \notin c(D_m)$, then $\{\alpha,\beta\}$ are a *consecutive contributing pair of elements*. If $\{\alpha,\beta\}$ is a consecutive contributing pair of elements with $\alpha = (a_1,a_2)$ and $\beta = (a_1, a_2+1)$, then $\{\alpha,\beta\}$ is a *horizontal pair*. If instead $\beta = (a_1 - 1, a_2)$, then $\{\alpha,\beta\}$ is a *vertical pair*. Let P_v be a vertical pair, P_h be a horizontal pair, and define $P := P_v \cup P_h$. If $P_v \cap P_h = \emptyset$, $P_v \cap W \neq \emptyset$, and $P_h \cap W \neq \emptyset$, P is called a *contributing disjoint corner*.

Lemma 9 will show that a contributing disjoint corner will force a rainbow solution in $[m]\times[n]$. The proof of Theorem 3 will use Lemma 9 to arrive at a contradiction once a contributing disjoint corner is found.

Lemma 9. *If c is a rainbow-free coloring of $[m]\times[n]$ for eq with $m \leq n$, then there are no contributing disjoint corners $P = P_v \cup P_h$ in $[m]\times[n]$.*

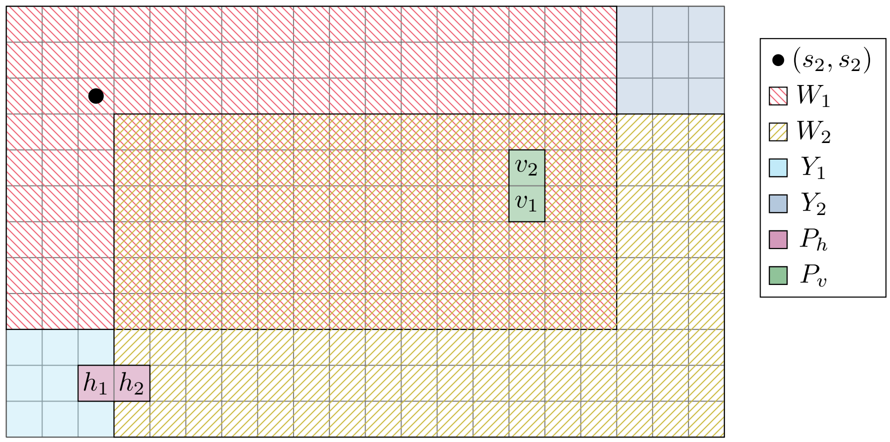

Figure 2: This figure is an example of $[m] \times [n]$ when $(s_2, s_2) = (3, 3)$ with corresponding Y_1, Y_2, W_1 and W_2 highlighted. Further, a contributing disjoint corner, $P_h \cup P_v$, is shown. By definition of contributing disjoint corner, the colors of h_1, h_2, v_1 and v_2 are pairwise distinct.

Proof. If $|c(D_m)| = 1$, then (s_2, s_2) does not exist and W is not defined, so the statement is vacuously true. Therefore, let $2 \leq |c(D_m)|$. For the sake of contradiction, assume $P_v = \{(a_1, b_1), (a_1 - 1, b_1)\}$ and $P_h = \{(a_2, b_2), (a_2, b_2 + 1)\}$ form a contributing disjoint corner. Define $c_3 = c((a_1, b_1))$, $c_4 = c((a_1 - 1, b_1))$, $c_5 = c((a_2, b_2))$, and $c_6 = c((a_2, b_2 + 1))$. Recall that convention implies that $c((1, 1)) = 1$ and $c((s_2, s_2)) = 2$. Note that colors 1, 2, c_3, c_4, c_5 and c_6 are pairwise distinct. It will be shown that every possible color assigned to $(s_2 - 1, s_2)$ gives a rainbow solution, which is a contradiction.

First, note that P_v intersects either W_1 or W_2. Suppose $P_v \cap W_1 \neq \emptyset$. Then $(a_1 - 1, b_1) \in W_1$ which implyies that the elements of

$$(a_1 - 1, b_1) + (s_2, s_2) = (a_1 - 1 + s_2, b_1 + s_2)$$

are in $[m] \times [n]$. Further, this equation implies that $c((a_1 - 1 + s_2, b_1 + s_2)) \in \{2, c_4\}$. So, the equation

$$(a_1, b_1) + (s_2 - 1, s_2) = (a_1 - 1 + s_2, b_1 + s_2)$$

implies that $c((s_2 - 1, s_2)) \in \{2, c_3, c_4\}$. Alternatively, if $P_v \cap W_2 \neq \emptyset$, then $(a_1, b_1) \in W_2$. Similar to the previous case, the equations

$$(a_1 - s_2, b_1 - s_2) + (s_2, s_2) = (a_1, b_1)$$

and

$$(a_1 - s_2, b_1 - s_2) + (s_2 - 1, s_2) = (a_1 - 1, b_1)$$

imply that $c((s_2-1,s_2)) \in \{2,c_3,c_4\}$.

Second, note that P_h either intersects W_1 or W_2. If $P_h \cap W_1 \neq \emptyset$, then $(a_2,b_2) \in W_1$. So $(a_2+s_2,b_2+s_2) \in [m]\times[n]$. Since $a_2 < a_2+s_2-1 < a_2+s_2$, it follows that $(a_2+s_2-1,b_2+s_2) \in [m]\times[n]$. Thus, the equation

$$(a_2,b_2+1)+(s_2-1,s_2-1)=(a_2+s_2-1,b_2+s_2)$$

implies that $c((a_2+s_2-1,b_2+s_2)) \in \{1,c_6\}$ because $c((s_2-1,s_2-1))=1$. Finally, the equation

$$(a_2,b_2)+(s_2-1,s_2)=(a_2+s_2-1,b_2+s_2)$$

implies that $c((s_2-1,s_2)) \in \{1,c_5,c_6\}$. On the other hand, if $P_h \cap W_2 \neq \emptyset$, then $(a_2,b_2+1) \in W_2$ and the equations

$$(a_2-s_2+1,b_2-s_2+1)+(s_2-1,s_2)=(a_2,b_2+1)$$

and

$$(a_2-s_2+1,b_2-s_2+1)+(s_2-1,s_2-1)=(a_2,b_2)$$

imply that $c((s_2-1,s_2)) \in \{1,c_5,c_6\}$.

Therefore, no matter where P_v and P_h are in $[m]\times[n]$, it must be that $c((s_2-1,s_2)) \in \{2,c_3,c_4\} \cap \{1,c_5,c_6\}$, a contradiction. □

3. Main Diagonal Has Three Colors

This section focuses on the situation where $[m]\times[n]$ has an exact $(m+n+1)$-coloring and the main diagonal has exactly three colors. The overall strategy of the section is to analyze the structure of an arbitrary coloring with the previously stated assumptions and show that the structural restrictions must admit a rainbow solution to *eq*.

Lemma 10 is a direct consequence of Lemmas 4 and 5.

Lemma 10. *If c is an exact, rainbow-free $(m+n+1)$-coloring of $[m]\times[n]$ for eq with $3 \le m \le n$ and $|c(D_m)|=3$, then each off-diagonal D_k contributes exactly one color c_k such that $c_k \notin c([m]\times[n]\setminus D_k)$.*

Lemma 11 will prove that there are no jumps between distinctly colored elements when there are three colors in the main diagonal. A brief outline of the argument follows. First, it is shown that if such a jump exists between elements in D_a and D_b, then there is a k such that $a=m+k$, and $b=m+2k$. Then it is shown that $|k|=1$, so either $a=m+1$ and $b=m+2$ or $a=m-1$ and $b=m-2$. This ultimately forces $\alpha=(1,2)$ or $\alpha=(2,1)$, respectively. Finally, two cases pertaining to the size of s_3 lead to contradictions.

Lemma 11. *If c is an exact, rainbow-free $(m+n+1)$-coloring of $[m] \times [n]$ for eq with $3 \leq m \leq n$ and $|c(D_m)| = 3$, then there are no jumps from α to β when $c(\alpha), c(\beta) \notin c(D_m)$ and $c(\alpha) \neq c(\beta)$.*

Proof. Suppose there is a jump from $\alpha \in D_a$ to $\beta \in D_b$ with $c(\alpha), c(\beta) \notin c(D_m)$ and $c(\alpha) \neq c(\beta)$. There must exist some $\delta \in D_t$ with $\alpha + \delta = \beta$, and $c(\delta) \in \{c(\alpha), c(\beta)\}$. If $t \notin \{a, b, m\}$, then Lemma 10 is contradicted. Further, $t \neq m$ because $c(\delta) \notin c(D_m)$, and $t \neq b$ because $\alpha \notin D_m$. Thus, $t = a$, $c(\delta) = c(\alpha)$, and Lemma 8 implies that $b = 2a - m$.

Define $k = a - m$ so that $a = m + k$ and $b = m + 2k$. For the sake of contradiction, assume $1 < |k|$. Then there exists some off-diagonal D_g with g strictly between a and b. By Lemma 10, D_g must contribute a color, say D_g contributes $c(\gamma)$ for some $\gamma \in D_g$. Lemma 6 implies that there must be either a jump from α to γ or a jump from γ to β. In either case, there exists some $\delta' \in D_{t'}$ such that $\alpha + \delta' = \gamma$ or $\gamma + \delta' = \beta$, and Lemma 8 implies that $t' \in \{m+g-a, m+b-g\}$. For the sake of contradiction, assume $t' \in \{a, b, m, g\}$ implying that $t' \in \{a, b, m, g\} \cap \{m+g-a, m+b-g\}$. Since g is strictly between a and b, it follows that $g = m+k+j$ for some j strictly between 0 and k. Converting all elements in the two sets to be in terms of m, k, and j gives

$$t' \in \{m+k, m+2k, m, m+k+j\} \cap \{m+j, m+k-j\}.$$

Note that the restrictions on j, and the fact that $1 < |k|$, imply $m+j$ and $m+k-j$ are strictly between m and $m+k$. Thus, the intersection is empty and $t' \notin \{a, b, m, g\}$. Since c is rainbow-free, $c(\delta') \in \{c(\alpha), c(\beta), c(\gamma)\}$. This contradicts Lemma 10.

This means $|k| = 1$ which implies that either $a = m+1$ and $b = m+2$ or that $a = m-1$ and $b = m-2$. First, consider the case where $a = m+1$ and $b = m+2$. Assume, for the sake of contradiction, that $\alpha = (a_1, a_2), \beta = (b_1, b_2), \delta = (t_1, t_2)$ are defined such that $2 \leq a_1, t_1$. Since $2 \leq a_1, t_1$, it must be the case that $4 \leq b_1$. Additionally, note that D_{m+3} contributes $c(\gamma')$ for some $\gamma' = (g_1, g_2) \in D_{m+3}$ by Lemma 10. Since $\alpha \in D_{m+1}$, $\beta \in D_{m+2}$, $\delta \in D_{m+1}$, and $\gamma' \in D_{m+3}$, each of the equations

$$a_2 = a_1 + 1 \tag{2}$$

$$t_2 = t_1 + 1 \tag{3}$$

$$g_2 = g_1 + 3 \tag{4}$$

hold. Using the above information, it will be shown that irrespective of the location of $\gamma' = (g_1, g_2)$ in the diagonal D_{m+3}, a contradiction can be found.

`Case 1.` Suppose that $a_1 < g_1$.

Equations (2) and (4) give that $a_2 = a_1 + 1 < a_1 + 3 < g_1 + 3 = g_2$. Since $a_1 < g_1$ and $a_2 < g_2$ there is a jump from α to γ'. So, by Lemma 8, there exists a $\delta' \in D_{m+2}$ such that $\alpha + \delta' = \gamma'$. Since c is a rainbow-free coloring,

$c(\delta') \in \{c(\alpha), c(\gamma')\}$. However, D_{m+2} contributes the color $c(\beta)$ so Lemma 4 is contradicted.

Case 2. Suppose that $g_1 \leq a_1$.
Since there is a jump from α to β it follows that $a_1 < b_1$ and $a_2 < b_2$. Thus, $g_1 < b_1$. Equations (2) and (4) give that $g_2 \leq a_2 + 2$. Further, Equation (3) implies that $t_1 \leq 3$, so $\alpha + \delta = \beta$ gives that $a_2 + 3 \leq a_2 + t_2 = b_2$. So $g_2 < b_2$. This implies there is a jump from γ' to β, so by Lemma 8 there exists a $\delta' \in D_{m-1}$ such that $\gamma' + \delta' = \beta$. Lemma 10 implies that D_{m-1} must contribute a color. Thus, $c(\delta') \notin \{c(\gamma'), c(\beta)\}$, contradicting that c is rainbow-free.

In either case, a contradiction is obtained, so $a_1 = 1$ or $t_1 = 1$, that is, $\alpha = (1, 2)$ or $\delta = (1, 2)$, respectively.

Since $c(\alpha) = c(\delta)$, without loss of generality, suppose $\alpha = (1, 2)$. Now, for the sake of contradiction, assume that $\frac{m}{2} + 1 \leq s_3$. Define $\rho = (s_3 - 1, s_3 - 2)$ so that $\alpha + \rho = (s_3, s_3)$ and $c(\rho) \in \{c(\alpha), 3\}$. Since $\rho \in D_{m-1}$, Lemmas 4 and 10 imply that $c(\rho) = 3$. Lemma 10 further implies that D_{m-1} must contribute a color. Say that D_{m-1} contributes $c(\xi)$ for some $\xi = (e_1, e_2) \in D_{m-1}$. If $e_1 < s_3 - 1$, then there exists some $\kappa = (k, k) \in D_m$ such that $k < s_3$ and $\xi + \kappa = \rho$, a rainbow solution. Otherwise, suppose $s_3 - 1 < e_1$. Then there exists some $\kappa = (k, k) \in D_m$ such that $\rho + \kappa = \xi$. Recall that $e_1 \leq m$ and $\frac{m}{2} + 1 \leq s_3$, so

$$k = e_1 - (s_3 - 1) \leq m - s_3 + 1 \leq (2s_3 - 2) - s_3 + 1 < s_3.$$

Thus, $\rho + \kappa = \xi$ is a rainbow solution. Both cases contradict that c is rainbow-free.

Now consider when $s_3 < \frac{m}{2} + 1$. Lemma 1 implies that $4 \leq s_3$, so $7 \leq m \leq n$. Note that if $\chi = (x_1, x_2) \in D_{m+3}$ is such that $2 \leq x_1$, then there exists a jump from α to χ implying that $c(\chi) \in c(D_m)$. Since D_{m+3} must contribute a color, and the only way it can contribute is if $x_1 = 1$, D_{m+3} must contribute $c((1, 4))$. If $5 \leq b_2$ (and $3 \leq b_1$), then there is a jump from $(1, 4)$ to β. So there exists some $\delta'' \in D_{m-1}$ with $c(\delta'') \in \{c((1, 4)), c(\beta)\}$, which is a contradiction by Lemmas 4 and 10. Thus, $b_2 \in \{3, 4\}$ but if $b_2 = 3$ there is not a jump from α to β so $\beta = (2, 4)$. Finally, $\alpha + \beta = (3, 6)$. So $c(3, 6) \in \{c(\alpha), c(\beta)\}$, but $(3, 6) \in D_{m+3}$, which is a contradiction by Lemmas 4 and 10. Therefore, contradictions are obtained in every situation when $a = m + 1$ and $b = m + 2$.

The case when $a = m - 1$ and $b = m - 2$ is very similar. Essentially, reflect the coordinates and diagonals from the argument above across the main diagonal. The analysis forces $\alpha = (2, 1)$ or $\delta = (2, 1)$, and $\rho = (s_3 - 2, s_3 - 1)$. However, the cases at this point differ slightly. Instead of scrutinizing $\frac{m}{2} + 1 \leq s_3$ and $s_3 < \frac{m}{2} + 1$, the cases that are considered are $\frac{m+1}{2} + 1 \leq s_3$ and $s_3 < \frac{m+1}{2} + 1$. In any event, contradictions are established in all situations, so there are no jumps from α to β with $c(\alpha), c(\beta) \notin c(D_m)$ and $c(\alpha) \neq c(\beta)$. □

Lemma 12 will bound s_3 which in turn will bound s_2 in the proof of Theorem 3.

Lemma 12. *Suppose c is a exact, rainbow-free $(m+n+1)$-coloring of $[m]\times[n]$ for eq with $3 \le m \le n$ and $|c(D_m)| = 3$. If $i, j < s_3$, then $c((i,j)) \in c(D_m)$.*

Proof. If $(i,j) \in D_m$, then $c((i,j)) \in c(D_m)$. So suppose $(i,j) \in D_k$, with $k \neq m$, and, for the sake of contradiction, suppose that $c((i,j)) \notin c(D_m)$. By Lemma 10, D_k contributes $c((i,j))$. Define $\alpha \in D_a$ so that $(i,j) + \alpha = (s_3, s_3)$ so $c(\alpha) \in \{c((i,j)), 3\}$. Lemma 8 implies that α and (i,j) are in different diagonals, and Lemma 10 gives $c(\alpha) \neq c((i,j))$. Thus, $c(\alpha) = 3$.

Let $\beta = (b_1, b_2) \in D_a$ be distinct from α. It will be shown that $c(\beta) \in c(D_m)$. First, suppose there exists a $\gamma = (g_1, g_2) \in D_m$ such that $\alpha + \gamma = \beta$. If $g_1 < s_3$, then $c(\gamma) \in \{1,2\}$. So $c(\gamma) \neq c(\alpha)$ implying that $c(\beta) \in c(D_m)$. If $s_3 \le g_1$, then $s_3 \le g_2$ which implies that $i < s_3 \le g_1 < b_1$ and $j < s_3 \le g_2 < b_2$. So there is a jump from (i,j) to β, and Lemma 11 implies that $c(\beta) \in c(D_m)$.

Otherwise, suppose that $\beta + \gamma = \alpha$. Then $g_1 < s_3$, so $c(\gamma) \in \{1,2\}$ which implies that $c(\alpha) \neq c(\gamma)$ and $c(\beta) \in c(D_m)$. Thus, $c(D_a) \setminus c(D_m) = \emptyset$, contradicting Lemma 10. Therefore, $c((i,j)) \in c(D_m)$. □

Theorem 3 gives that there are no rainbow-free $(m+n+1)$-colorings of $m \times n$ with no more than 3 colors in the main diagonal. Assuming that there is such a coloring, this proof shows that there must exist a vertical and horizontal pair that intersects W. This gives a contributing disjoint corner, which contradicts Lemma 9.

Theorem 3. *If c is an exact $(m+n+1)$-coloring of $[m]\times[n]$ with $3 \le m \le n$ and $|c(D_m)| \le 3$, then $[m]\times[n]$ contains a rainbow solution to eq.*

Proof. Assume, for the sake of contradiction, that c is rainbow-free. Then Corollary 2 implies that $|c(D_m)| = 3$. Lemma 10 implies every off-diagonal is a contributing diagonal, which means that D_{m-1} and D_{m-2} both contribute a color. Suppose that $\alpha = (a_1, a_1 - 1) \in D_{m-1}$ and D_{m-1} contributes $c(\alpha)$. Additionally, suppose that $\beta = (b_1, b_1 - 2) \in D_{m-2}$ and D_{m-2} contributes $c(\beta)$. Now $P = \{\alpha, \beta\}$ is a consecutive contributing pair of elements and Lemma 11 indicates P is a vertical or horizontal pair. Further, Lemma 12 implies that $s_3 \le a_1, b_1$. Note that

$$2 \le s_2 \le \frac{s_3}{2} \le \frac{a_1}{2}$$

with the middle inequality being a result of Lemma 1. Continuing,

$$s_2 \le \frac{a_1}{2} = a_1 - \frac{a_1}{2} < a_1 - 1 < a_1.$$

So $\alpha = (a_1, a_1 - 1) \in W_2$ and P intersects W. A similar argument indicates a vertical or horizontal pair, call it P', exists and intersects D_{m+1}, D_{m+2}, and W_2.

For the sake of contradiction, assume that P is a vertical pair. It follows that $a_1 - 1 = b_1 - 2$. Define K as the minimum k such that D_k contributes the color of an element of the $a_1 - 1$ column. More precisely,

$$K = \min\{k : D_k \text{ contributes } c((x, a_1 - 1)) \text{ for some } x\}.$$

Notice that K exists and $K \leq m-2$. By construction, D_K contributes $c((x, a_1-1))$ for $x = m - K + a_1 - 1$.

By Lemmas 10 and 11, D_{K-1} must contribute $c((x+1, a_1-1))$ or $c((x, a_1-2))$. By the minimality of K, D_{K-1} must contribute $c((x, a_1-2))$. This implies that $P_1 = \{(x, a_1-2), (x, a_1-1)\}$ is a horizontal pair. Since $s_2 < a_1-1 < m-K+a_1-1 = x$, it follows that P_1 intersects W_2. However, now either P_1 and P' are contributing disjoint corners or P and P' are contributing disjoint corners. Both situations contradict Lemma 9. Thus, P must be a horizontal pair.

Note that D_{m+1} must contribute $c(\gamma)$ for some $\gamma = (g_1, g_1+1) \in D_{m+1}$. Then $s_3 \leq g_1 + 1$ by Lemma 12. Define L as the maximum ℓ such that D_ℓ contributes the color of an element of the g_1 row. Specifically,

$$L = \max\{\ell : D_\ell \text{ contributes } c((g_1, y)) \text{ for some } y\}.$$

Again, notice that $m+1 \leq L$. So D_L contributes $c((g_1, y))$ for $y = L - m + g_1$.

Lemma 11 and the maximality of L imply that D_{L+1} must contribute $c((g_1 - 1, y))$. Now, using a similar argument as before, $P_2 = \{(g_1, y), (g_1-1, y)\}$ is a vertical pair that intersects W_2. Finally, since $s_2 \geq 2$, Lemma 1 implies that

$$s_2 < s_3 - 1 \leq g_1 < L - m + g_1 = y.$$

Thus, $(g_1, y) \in W_2$ implying that P_2 and P form a contributing disjoint corner, which contradicts Lemma 9. Therefore, $[m] \times [n]$ contains a rainbow solution. □

4. Four or More Colors in Main Diagonal

The main result of Section 3 states that every exact $(m+n+1)$-coloring of $[m] \times [n]$ will contain a rainbow solution to *eq* if there are three colors in the main diagonal. Section 4 culminates with Theorem 4, which will show that if an exact $(m+n+1)$-coloring of $[m] \times [n]$ contains a jump, then a rainbow solution to *eq* exists. To start, a useful set of diagonals is defined.

For each $\delta \in [m] \times [n]$, define the set of off-diagonals

$$\mathbb{D}_\delta := \{D_g \neq D_m \ : \text{for all } \gamma \in D_g, \text{ either } \gamma + \delta \text{ or } \gamma - \delta \text{ is in } [m] \times [n]\}.$$

Lemma 13 will give an upper bound for the number of colors that can exist in $[m] \times [n]$ that do not appear in the main diagonal. The upper bound is written with respect to the cardinality of some $\mathbb{D}_\delta$ with $c(\delta) \notin c(D_m)$. The proof proceeds with two cases when there are at least $\frac{|\mathbb{D}_\delta|}{3}$ non-contributing diagonals in $\mathbb{D}_\delta$ and when there are less than $\frac{|\mathbb{D}_\delta|}{3}$ non-contributing diagonals in $\mathbb{D}_\delta$. The former case is immediate. In the latter case, it is shown that there are at least $\frac{2|\mathbb{D}_\delta|}{3} - 3$ contributing diagonals that are each associated with some non-contributing diagonal and at most two of these contributing diagonals are associated with the same non-contributing diagonal. This leads to the conclusion that there are at least $\frac{|\mathbb{D}_\delta|}{3} - \frac{3}{2}$ non-contributing diagonals in $[m] \times [n]$. Subtracting this from the total number of off-diagonals gives the upper bound.

Lemma 13. *Let c be an exact, rainbow-free $(m+n+1)$-coloring of $[m] \times [n]$ for eq with $3 \le m \le n$. For each δ such that $c(\delta) \notin c(D_m)$,*

$$|c([m] \times [n]) \setminus c(D_m)| \le m + n - \frac{1}{2} - \frac{|\mathbb{D}_\delta|}{3}.$$

Proof. Assume that $\delta \in D_t$ and suppose there are k non-contributing diagonals in $\mathbb{D}_\delta$. First, consider when $k \ge \frac{|\mathbb{D}_\delta|}{3}$. Since there are $m+n-2$ off-diagonals and at least $\frac{|\mathbb{D}_\delta|}{3}$ are non-contributing, it follows that

$$|c([m] \times [n]) \setminus c(D_m)| \le m + n - 2 - \frac{|\mathbb{D}_\delta|}{3} < m + n - \frac{1}{2} - \frac{|\mathbb{D}_\delta|}{3}.$$

Now suppose that $k < \frac{|\mathbb{D}_\delta|}{3}$. This implies there are at least $\frac{2|\mathbb{D}_\delta|}{3}$ contributing diagonals in $\mathbb{D}_\delta$. Note that at most one of these contributing diagonals contains $c(\delta)$. Define D_x as the diagonal in $[m] \times [n]$ that contributes $c(\delta)$. This means there are at least $\frac{2|\mathbb{D}_\delta|}{3} - 1$ contributing diagonals in $\mathbb{D}_\delta$ that do not contain $c(\delta)$. Let D_g be one of those diagonals. Then there exists $\gamma \in D_g$ such that D_g contributes $c(\gamma)$ with $c(\gamma) \notin c(D_m) \cup \{c(\delta)\}$. Since $D_g \in \mathbb{D}_\delta$, either $\gamma + \delta \in [m] \times [n]$ or $\gamma - \delta \in [m] \times [n]$ implying either $\{\gamma, \delta, \gamma + \alpha\}$ or $\{\delta, \gamma - \delta, \gamma\}$ is a solution in $[m] \times [n]$, respectively. Specifically, $\gamma + \delta \in D_{g+t-m}$ and $\gamma - \delta \in D_{g-t+m}$ by Lemma 8.

Since c is rainbow-free, either $c(\gamma + \delta)$ or $c(\gamma - \delta)$ is in $\{c(\gamma), c(\delta)\}$. This implies either $c(D_{g+t-m})$ or $c(D_{g-t+m})$ contains $c(\gamma)$ or $c(\delta)$. If D_{g+t-m} or D_{g-t+m} contains $c(\gamma)$, then, since D_g contributes $c(\gamma)$, either D_{g+t-m} or D_{g-t+m} is non-contributing. If D_{g+t-m} or D_{g-t+m} contains $c(\delta)$, then D_{g+t-m} or D_{g-t+m} is contributing if and only if $D_{g+t-m} = D_x$ or $D_{g-t+m} = D_x$. This analysis shows that if $D_g \in \{D_{x+t-m}, D_{x-t+m}\}$, then D_g does not necessarily correspond to a non-contributing diagonal. Applying a similar analysis to D_x indicates that if $D_x \in \mathbb{D}_\delta$, D_x does not necessarily correspond to a non-contributing diagonal.

Define G to be the set of all contributing diagonals in $\mathbb{D}_\delta \setminus \{D_x, D_{x+t-m}, D_{x-t+m}\}$. Thus, each $D_g \in G$ corresponds to at least one non-contributing off-diagonal, namely D_{g+t-m} or D_{g-t+m}, and

$$\frac{2|\mathbb{D}_\delta|}{3} - 3 \le |G|.$$

If G is empty, then $|\mathbb{D}_\delta| \le 9/2$ so that

$$\begin{aligned}|c([m]\times[n]) \setminus c(D_m)| &\le m+n-2\\ &\le m+n-\frac{1}{2}-\frac{|\mathbb{D}_\delta|}{3}.\end{aligned}$$

Assume G is non-empty and let $D_g \in G$ so that D_{g+t-m} or D_{g-t+m} is non-contributing. Without loss of generality, suppose that D_{g+t-m} is non-contributing. For the sake of contradiction, suppose that $D_h, D_{h'} \in G \setminus \{D_g\}$ are distinct with $D_{g+t-m} \in \{D_{h+t-m}, D_{h-t+m}\} \cap \{D_{h'+t-m}, D_{h'-t+m}\}$. Since $g \notin \{h, h'\}$, it must be that $g+t-m = h-t+m$ and $g+t-m = h'-t+m$ implying that $h = h'$, a contradiction. Thus, there is at most one diagonal D_h in G such that $D_{g+t-m} \in \{D_{h+t-m}, D_{h-t+m}\}$. In other words, at most two diagonals in G correspond to the same non-contributing off-diagonal.

Since $D_g \in G$ was arbitrary, it can be concluded that at least $\frac{|G|}{2}$ diagonals are non-contributing. It follows that

$$\begin{aligned}|c([m]\times[n]) \setminus c(D_m)| &\le m+n-2-\frac{|G|}{2}\\ &\le m+n-2-\frac{\frac{2|\mathbb{D}_\delta|}{3}-3}{2}\\ &= m+n-\frac{1}{2}-\frac{|\mathbb{D}_\delta|}{3}.\end{aligned}$$

□

As stated, the main goal of this section is to show that a jump with distinctly colored elements yields a rainbow solution. Propositions 1 and 2 will give lower and upper bounds, respectively, on the jump distance, which will leave only a few cases to analyze.

Proposition 1. *Let c be an exact, rainbow-free $(m+n+1)$-coloring of $[m]\times[n]$ for eq with $3 \le m \le n$. If there exists a $\delta = (d_1, d_2)$ such that $c(\delta) \notin c(D_m)$, then*

$$\frac{4m+9-6\lfloor \log_2(m)+1\rfloor}{4} \le d_1+d_2.$$

Proof. Define

$$S_\delta := \{\gamma \in [m]\times[n] \mid \gamma+\delta, \gamma-\delta \notin [m]\times[n]\}.$$

Define $\delta' := (m-d_1+1, d_2) \in D_{d_1+d_2-1}$, and notice that $\delta' \pm \delta \notin [m]\times[n]$ (see Figure 3). Since there are $m+n-2$ off-diagonals and S_δ intersects $2(d_1+d_2-1)$ diagonals, it follows that

$$|\mathbb{D}_\delta| \ge m+n-2-2(d_1+d_2-1) = m+n-2d_1-2d_2.$$

Therefore, Lemma 13 implies that

$$\begin{aligned}|c([m]\times[n])\setminus c(D_m)| &\le m+n-\frac{1}{2}-\frac{|\mathbb{D}_\delta|}{3}\\ &\le m+n-\frac{1}{2}-\frac{m+n-2d_1-2d_2}{3}\\ &=\frac{4m+4n-3+4d_1+4d_2}{6}.\end{aligned} \tag{5}$$

Because c is a rainbow-free, Theorem 2 implies that $|c(D_m)| \le \lfloor \log_2(m)+1 \rfloor$. Further, c is an exact $(m+n+1)$-coloring, so

$$m+n+1-\lfloor \log_2(m)+1\rfloor \le |c([m]\times[n])\setminus c(D_m)|. \tag{6}$$

Inequalities (5) and (6) give that

$$m+n+1-\lfloor \log_2(m)+1\rfloor \le |c([m]\times[n])\setminus c(D_m)| \le \frac{4m+4n-3+4d_1+4d_2}{6}.$$

Isolating d_1+d_2 and using $m \le n$ gives that

$$\frac{4m+9-6\lfloor \log_2(m)+1\rfloor}{4} \le \frac{2m+2n+9-6\lfloor \log_2(m)+1\rfloor}{4} \le d_1+d_2. \tag{7}$$

□

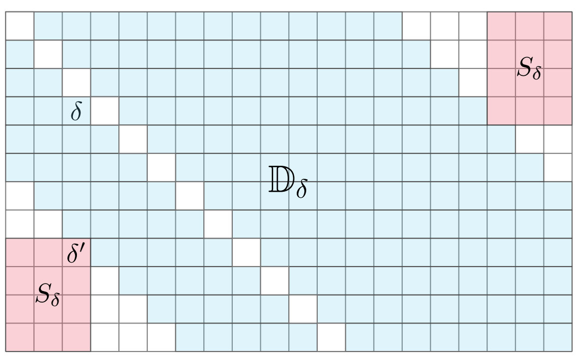

Figure 3: This is a visualization of δ, δ', S_δ (in red), and $\mathbb{D}_\delta$ (in cyan) from Proposition 1.

Lemmas 14, 15, and 16 will support Proposition 2. The assumptions for all of these results are that there is a jump from $\alpha=(a_1,a_2)$ to $\beta=(b_1,b_2)$, that $\alpha \in D_a$ and $\beta \in D_b$, that $c(\alpha), c(\beta) \notin c(D_m)$, and that $c(\alpha) \neq c(\beta)$. Also, it is assumed that $\alpha + (d_1, d_2) = \beta$. The goal is to find a relationship between the jump distance, $d_1 + d_2$, and the number of contributing diagonals. The main challenge will arise during the analysis of the diagonals that surround D_a and D_b. Technically, this set of surrounding diagonals is S which was defined by Equation (1) and visualized in Figure 1. During the analysis of S, many iterations of jumps will need to be considered. For example, if γ is an element in one of the diagonals of S, then there must be a jump from α to γ (or from γ to β), which introduces another element ζ where $\alpha + \zeta = \gamma$ (or $\gamma + \zeta = \beta$). Lemma 14 will start investigating where ζ could be in the $\alpha + \zeta = \gamma$ case, and Lemma 15 will start investigating where ζ could be in the $\gamma + \zeta = \beta$ case. Lemma 16 will add more clarification about where the ζ elements can be.

Lemma 14. *Let c be an exact, rainbow free $(m+n+1)$-coloring of $[m] \times [n]$ for eq with $3 \leq m \leq n$, and suppose there is a jump from $\alpha \in D_a$ to $\beta \in D_b$ such that $\alpha + \delta = \beta$ for some $\delta \in D_t$. If $c(\alpha), c(\beta) \notin c(D_m)$ and $c(\alpha) \neq c(\beta)$, then there is at most one diagonal D_g that contains an element γ with $c(\gamma) \notin c(D_m \cup \{\alpha, \beta\})$ such that $\alpha + \zeta = \gamma$ for some $\zeta \in D_z$ with $z \in \{a, b, t, m, g\}$. Moreover, if such a D_g exists, then $z = a$.*

Proof. Assume there exist diagonals D_g and $D_{g'}$ such that for some $\gamma \in D_g$ and $\gamma' \in D_{g'}$, the colors $c(\gamma), c(\gamma') \notin c(D_m \cup \{\alpha, \beta\})$. Also, assume there is a $\zeta \in D_z$ and $\zeta' \in D_{z'}$ such that $\alpha + \zeta = \gamma$ and $\alpha + \zeta' = \gamma'$ with $z \in \{a, b, t, m, g\}$ and $z' \in \{a, b, t, m, g'\}$.

Since $a \neq m$, it follows that $z \neq g$ and $z' \neq g'$, so $z, z' \in \{a, b, t, m\}$. Additionally, $c(\gamma) \neq c(\alpha)$ and $c(\gamma') \neq c(\alpha)$ imply that $c(\zeta) \in \{c(\alpha), c(\gamma)\}$ and $c(\zeta') \in \{c(\alpha), c(\gamma')\}$. So $z, z' \notin \{m, b\}$. Since $c(\gamma), c(\gamma') \notin c(D_m \cup \{\alpha, \beta\})$, Lemma 4 implies that $g \neq b$ and $g' \neq b$. Lemma 8 gives that $z \neq t$ and $z' \neq t$. So, it must be the case that $z = z' = a$. Finally, Lemma 8 gives that $g = g' = 2a - m$, that is D_g is unique. ☐

Lemma 15. *Let c be an exact, rainbow free $(m+n+1)$-coloring of $[m] \times [n]$ for eq with $3 \leq m \leq n$, and suppose there is a jump from $\alpha \in D_a$ to $\beta \in D_b$ such that $\alpha + \delta = \beta$ for some $\delta \in D_t$. If $c(\alpha), c(\beta) \notin c(D_m)$ and $c(\alpha) \neq c(\beta)$, then there is at most one diagonal D_g that contains an element γ with $c(\gamma) \notin c(D_m \cup \{\alpha, \beta\})$ such that $\gamma + \zeta = \beta$ for some $\zeta \in D_z$ with $z \in \{a, b, t, m, g\}$. Moreover, if such a D_g exists, then $z = g$.*

Proof. Assume there exist diagonals D_g and $D_{g'}$ such that for some $\gamma \in D_g$ and $\gamma' \in D_{g'}$, the colors $c(\gamma), c(\gamma') \notin c(D_m \cup \{\alpha, \beta\})$. Also, assume there is a $\zeta \in D_z$ and $\zeta' \in D_{z'}$ such that $\gamma + \zeta = \beta$ and $\gamma' + \zeta' = \beta$ with $z \in \{a, b, t, m, g\}$ and $z' \in \{a, b, t, m, g'\}$.

Since $c(\gamma) \neq c(\beta)$ and $c(\gamma') \neq c(\beta)$, it follows that $c(\zeta) \in \{c(\beta), c(\gamma)\}$ and $c(\zeta') \in \{c(\beta), c(\gamma')\}$. So $z, z' \notin \{a, m\}$. Also $g \neq m$ and $g' \neq m$ imply that $z \neq b$

and $z' \neq b$. Since $c(\gamma), c(\gamma') \notin c(D_m \cup \{\alpha, \beta\})$, Lemma 4 implies that $g \neq a$ and $g' \neq a$. Lemma 8 gives that $z \neq t$ and $z' \neq t$. Therefore, $z = g$ and $z' = g'$. By Lemma 8, $b = z + g - m = 2g - m$ and $b = z' + g' - m = 2g' - m$. So $g = g'$ showing that g is unique. □

Lemma 16. *Let c be an exact, rainbow free $(m+n+1)$-coloring of $[m] \times [n]$ for eq with $3 \leq m \leq n$, and suppose there is an element $\alpha \in D_a$ and an element $\beta \in D_b$ such that $c(\alpha) \neq c(\beta)$ and $c(\alpha), c(\beta) \notin c(D_m)$. Let*

$$S' \subseteq \{D_x \mid c(D_x) \backslash c(D_m \cup D_a \cup D_b) \neq \emptyset\}$$

and

$$G := \{D_x \in S' \mid \text{if } c(D_x) \backslash c(D_m) = c(D_y) \backslash c(D_m) \text{ for some } D_y \in S', \text{ then } x \leq y\}.$$

If $D_{g_i} \in G$ and $\gamma_i \in D_{g_i}$ with $c(\gamma_i) \notin c(D_m \cup \{\alpha, \beta\})$ and there is a corresponding $\zeta_i \in D_{z_i}$ such that either $\alpha + \zeta_i = \gamma_i$ or $\gamma_i + \zeta_i = \beta$, then $D_{z_i} \notin G \setminus \{D_{g_i}\}$. Moreover, if $i \neq j$, then $z_i \neq z_j$.

Proof. For the sake of contradiction, suppose $D_{z_i} \in G \setminus \{D_{g_i}\}$. By definition of G there is a $\rho \in D_{z_i}$ such that $c(\rho) \notin c(D_m \cup \{\alpha, \beta\})$. Since $z_i \neq g_i$, it follows that $c(\rho) \neq c(\gamma_i)$. Further, $\alpha + \zeta_i = \gamma_i$ or $\gamma_i + \zeta_i = \beta$ implies that $c(\zeta_i) \in c(\{\alpha, \gamma_i, \beta\})$. Thus, $c(\zeta_i) \neq c(\rho)$ and $c(\zeta_i), c(\rho) \notin c(D_m)$, contradicting Lemma 4. Therefore, $D_{z_i} \notin G \setminus \{D_{g_i}\}$.

Suppose $\gamma_j \in D_{g_j}$ with $c(\gamma_j) \notin c(D_m \cup \{\alpha, \beta\})$ have corresponding $\zeta_j \in D_{z_j}$ as in the statement of the lemma for some $j \neq i$. If $\alpha + \zeta_i = \gamma_i$ and $\alpha + \zeta_j = \gamma_j$, then $g_i \neq g_j$ and Lemma 8 imply $z_i \neq z_j$. Similarly, if $\gamma_i + \zeta_i = \beta$ and $\gamma_j + \zeta_j = \beta$, then $z_i \neq z_j$. On the other hand, if, without loss of generality, $\alpha + \zeta_i = \gamma_i$ and $\gamma_j + \zeta_j = \beta$, then $c(\zeta_i) \in \{c(\gamma_i), c(\alpha)\}$ and $c(\zeta_j) \in \{c(\gamma_j), c(\beta)\}$. This implies that $c(\zeta_i) \neq c(\zeta_j)$, so Lemma 4 implies that $z_i \neq z_j$. □

Proposition 2. *Let c be an exact, rainbow-free $(m+n+1)$-coloring of $[m] \times [n]$ for eq with $3 \leq m \leq n$. If $\alpha + (d_1, d_2) = \beta$ corresponds to a jump from α to β where $c(\alpha), c(\beta) \notin c(D_m)$ and $c(\alpha) \neq c(\beta)$, then $d_1 + d_2 \leq 2\log_2(m) + 1$.*

Proof. Let $\alpha = (a_1, a_2) \in D_a$, $\beta = (b_1, b_2) \in D_b$, and $\delta = (d_1, d_2) \in D_t$. Since c is rainbow-free, $c(\delta) \in \{c(\alpha), c(\beta)\}$. Define S as in Equation (1). By Lemmas 7 and 6, every element of every diagonal in S has a solution with α or β. In particular, if γ is such an element, then there is either a jump from α to γ or from γ to β.

Further, define

$$S' := \{D_x \in S \mid c(D_x) \backslash c(D_m \cup D_a \cup D_b) \neq \emptyset\}$$

and

$$G := \{D_x \in S' \mid \text{if } c(D_x)\backslash c(D_m) = c(D_y)\backslash c(D_m) \text{ for some } D_y \in S', \text{ then } x \le y\}.$$

Also, define

$$T := \{D_x \in [m] \times [n] \mid D_x \notin S \cup \{D_m, D_a, D_b\}\}$$

and

$$T' := \left\{ D_x \in T \mid c(D_x) \not\subseteq c\left(D_m \cup D_a \cup D_b \cup \bigcup_{D_s \in S} D_s \right) \right\}.$$

Accounting for the facts that $D_a, D_b, D_m \notin S$ and $\delta = \beta - \alpha$,

$$|S| \ge (m + b_2 - a_1) - (m + a_2 - b_1) - 1 - 3 = d_1 + d_2 - 4.$$

Suppose $|G| = k$, and reindex $G = \{D_{g_1}, \ldots, D_{g_k}\}$. Note that for each $D_{g_i} \in G$, there exists $\gamma_i \in D_{g_i}$ with $c(\gamma_i) \notin c(D_m \cup \{\alpha, \beta\})$ and $c(\gamma_i) \neq c(\gamma_j)$ when $i \neq j$. In addition, as $D_{g_i} \in S$, there exists a diagonal D_{z_i} with $\zeta_i \in D_{z_i}$ such that $\alpha + \zeta_i = \gamma_i$ or $\gamma_i + \zeta_i = \beta$. Define $Z := \{D_{z_i} \mid \zeta_i \in D_{z_i} \text{ for } 1 \le i \le k\}$. By Lemmas 14 and 15, at least $k - 2$ diagonals D_{z_i} of Z are not in $\{D_a, D_b, D_t, D_m, D_{g_i}\}$. Define

$$Z' := \begin{cases} \{D_{z_i} \in Z \mid D_{z_i} \notin \{D_a, D_b, D_t, D_m, D_{g_i}\}\} & \text{if } a = t, \\ \{D_{z_i} \in Z \mid D_{z_i} \notin \{D_a, D_b, D_t, D_m, D_{g_i}\}\} \cup \{D_t\} & \text{if } a \neq t. \end{cases}$$

For all $D_{z_i}, D_{z_j} \in Z \cap Z'$, Lemma 16 implies that $D_{z_i} \notin G$ and $z_i \neq z_j$ when $i \neq j$. It follows that

$$Z' = \begin{cases} \{D_{z_i} \in Z \mid D_{z_i} \notin \{D_a, D_b, D_t, D_m\} \cup G\} & \text{if } a = t, \\ \{D_{z_i} \in Z \mid D_{z_i} \notin \{D_a, D_b, D_t, D_m\} \cup G\} \cup \{D_t\} & \text{if } a \neq t. \end{cases}$$

Lemmas 14 and 15 imply $k - 1 \le |Z'|$.

Note that $c(\zeta_i) \in \{c(\alpha), c(\beta), c(\gamma_i)\}$, so $c(\zeta_i) \notin c(D_m)$. This implies that

$$|c(D_{z_i}) \setminus c(D_m \cup D_a \cup D_b \cup D_{g_i})| = 0$$

by Lemma 4. Similarly, $c(\delta) \in \{c(\alpha), c(\beta)\}$, so

$$|c(D_t) \setminus c(D_m \cup D_a \cup D_b)| = 0.$$

Thus,

$$c\left(\bigcup_{D_x \in Z'} D_x \right) \subseteq c\left(D_m \cup D_a \cup D_b \cup \bigcup_{D_{g_i} \in G} D_{g_i} \right).$$

By definition, for all $D_x \in Z'$, $D_x \notin G \cup \{D_a, D_b, D_m\}$, so $Z' \subseteq (T \setminus T') \cup (S \setminus G)$.

If $k \leq \frac{|S|}{2}$, then there are at least $\frac{|S|}{2}$ off-diagonals in $S \setminus G$. Therefore, there are at least $\frac{|S|}{2}$ off-diagonals in $(T \setminus T') \cup (S \setminus G)$. If $k \geq \frac{|S|+1}{2}$, then $|Z'| \geq k-1 \geq \frac{|S|-1}{2}$. Since $Z' \subseteq (T \setminus T') \cup (S \setminus G)$, there are at least $\frac{|S|-1}{2}$ off-diagonals in $(T \setminus T') \cup (S \setminus G)$.

Recall that $|S| \geq d_1 + d_2 - 4$. This implies that there are at least

$$\frac{d_1 + d_2 - 5}{2}$$

off-diagonals in $(T \setminus T') \cup (S \setminus G)$.

Because there are a total of $m + n - 2$ off-diagonals and Lemma 4 implies that each off-diagonal can contain at most one color not in the main diagonal, there are at most

$$m + n - 2 - \frac{d_1 + d_2 - 5}{2}$$

colors in $c([m] \times [n]) \backslash c(D_m)$. Therefore, there are at least

$$m + n + 1 - \left(m + n - 2 - \frac{d_1 + d_2 - 5}{2}\right) = 3 + \frac{d_1 + d_2 - 5}{2}$$

colors in $c(D_m)$. By Theorem 2,

$$|c(D_m)| \leq \log_2(m) + 1,$$

so

$$3 + \frac{d_1 + d_2 - 5}{2} \leq \log_2(m) + 1.$$

Therefore,

$$d_1 + d_2 \leq 2\log_2(m) + 1.$$

□

Combining the inequalities in Propositions 1 and 2 gives the following Corollary.

Corollary 4. *Let c be an exact, rainbow-free $(m+n+1)$-coloring of $[m] \times [n]$ for eq with $m \leq n$. If $11 \leq m$ or $14 \leq n$, then there are no jumps between elements α and β such that $c(\alpha), c(\beta) \notin c(D_m)$ and $c(\alpha) \neq c(\beta)$. Furthermore, when $8 \leq m \leq 10$, it follows that $5 \leq d_1 + d_2 \leq 7$.*

Proof. Assume there is a jump from α to β. In particular, $\alpha + (d_1, d_2) = \beta$. Then, the inequalities from Propositions 1 and 2 must be satisfied, so

$$\frac{4m + 9 - 6\lfloor \log_2(m) + 1 \rfloor}{4} \leq d_1 + d_2 \leq 2\log_2(m) + 1. \tag{8}$$

This can only happen if $m \leq 10$. Further, Inequality (7) implies that $n \leq 13$, and if $8 \leq m \leq 10$, Inequality (8) implies that $5 \leq d_1 + d_2 \leq 7$. □

Lemma 17 will be used to generalize Corollary 4 into Theorem 4. The proof of Lemma 17 will use analysis similar to the proof of Proposition 2. But the additional constraint that $|c(D_m)| = 4$, deduced using Corollary 4, will lead to deeper case analysis.

Lemma 17. *Let c be an exact, rainbow free $(m + n + 1)$-coloring of $[m] \times [n]$ for eq with $3 \le m \le n$, and suppose there is a jump from $\alpha \in D_a$ to $\beta \in D_b$ such that $\alpha + \delta = \beta$ for some $\delta = (d_1, d_2) \in D_t$. If $c(\alpha), c(\beta) \notin c(D_m)$ and $c(\alpha) \neq c(\beta)$, then $2m - t = b$.*

Proof. Corollary 2 and Lemma 11 imply $4 \le |c(D_m)|$. Lemma 2 and Corollary 4 further restrict to the situation to $8 \le m \le 10$, $n \le 13$, $5 \le d_1 + d_2 \le 7$, and $|c(D_m)| = 4$. For the sake of contradiction, assume $2m - t \neq b$. First, note that if $2m-t = a$, then subtracting $m-t$ from both sides gives $m = a+t-m$. By Lemma 8, $m = b$, a contradiction. Thus, $2m - t \notin \{a, b\}$.

Define $\varsigma := (s_4, s_4) - \delta$. Notice $d_1 + d_2 \le 7$ and $8 \le s_4$ imply that $\varsigma \in [m] \times [n]$ and more specifically, $\varsigma \in D_{2m-t}$ by Lemma 8. It will be shown that one of D_a, D_b, D_{2m-t} is non-contributing. Note that $\delta + \varsigma = (s_4, s_4)$ implies that $c(\varsigma) \in \{c(\delta), 4\} \subset \{c(\alpha), c(\beta), 4\}$. If $c(\varsigma) \in \{c(\alpha), c(\beta)\}$, then at least one of D_a, D_b, D_{2m-t} is non-contributing. Otherwise, suppose $c(\varsigma) = 4$. It will be shown that D_{2m-t} is non-contributing, the desired result. Let $\rho = (p_1, p_2) \in D_{2m-t}$. If $p_1 < s_4 - d_1$, then there exists some $k < s_4$ such that $\rho + (k, k) = \varsigma$ implying that $c(\rho) \in c(D_m)$. Second, suppose $s_4 - d_1 < p_1 < 2s_4 - d_1$. Then there exists some $k < s_4$ such that $(k, k) + \varsigma = \rho$ implying that $c(\rho) \in c(D_m)$. Finally, suppose $2s_4 - d_1 \le p_1$. This implies that $2s_4 - d_2 \le p_2$. Additionally, notice that $p_1 \le m \le 10$ and $8 \le s_4$ by Theorem 1. So $d_1 \ge 2s_4 - p_1 \ge 16 - 10 \ge 6$. However, this implies that $d_1 + d_2 \le 7$ which further implies that $d_2 \le 1$. Thus, $p_2 \ge 2s_4 - d_2 \ge 16 - 1 = 15 > n$. Therefore, no such ρ exists. This proves that $c(D_{2m-t}) \subseteq c(D_m)$ which implies that D_{2m-t} is non-contributing.

Since $|c(D_m)| = 4$, every diagonal D_k with $k \notin \{m, a, b, 2m-t\}$ must contribute a color distinct from $c(\alpha)$ and $c(\beta)$. Consider D_t. Since $\alpha+\delta = \beta$, it follows that $c(\delta) \in \{c(\alpha), c(\beta)\}$, and Lemma 4 implies that $c(D_t) \subseteq c(D_m)\cup\{c(\alpha), c(\beta)\}$. So D_t cannot contribute a color distinct from $c(\alpha)$ and $c(\beta)$. Thus, $t \in \{m, a, b, 2m-t\}$. However, $a \neq b$ implies that $t \neq m$ which further implies that $t \neq 2m - t$. Additionally, $t \neq b$ because $a \neq m$. So $t = a$.

Define S as in Equation (1), and note that

$$|S| = \begin{cases} d_1 + d_2 - 3 & \text{if } m \notin \{D_x \,|\, m + a_2 - b_1 < x < m + b_2 - a_1\}, \\ d_1 + d_2 - 4 & \text{if } m \in \{D_x \,|\, m + a_2 - b_1 < x < m + b_2 - a_1\}. \end{cases}$$

Since $5 \le d_1 + d_2$, it follows that $1 \le |S|$. It is claimed that $1 \le |S\backslash\{D_{2m-t}\}|$. If $D_{2m-t} \notin S$, then there exists some $D_g \in S\backslash\{D_{2m-t}\}$. So assume $D_{2m-t} \in S$. By definition of S,

$$m + a_2 - b_1 < 2m - t < m + a_1 - b_2$$

which implies that

$$m + a_2 - b_1 < a + b - (2m - t) < m + a_1 - b_2.$$

Thus, $D_{a+b-(2m-t)} \in S \backslash \{D_{2m-t}\}$ because if $a + b - (2m - t) \in \{a, b, m, 2m - t\}$, it can be concluded that either $b = 2m - t$ or $a = m$, both of which are contradictions. So there exists some $D_g \in S$ such that $g \neq 2m - t$.

Since $g \notin \{m, a, b, 2m - t\}$, it follows that D_g must contribute $c(\gamma)$ for some $\gamma \in D_g$. By Lemmas 6 and 7, there exists a $\zeta \in D_z$ such that $\alpha + \zeta = \gamma$ or $\gamma + \zeta = \beta$. In either case, $c(\zeta) \in \{c(\alpha), c(\beta), c(\gamma)\}$ implying that if $z \notin \{m, a, b, 2m - t, g\}$, then z is non-contributing, a contradiction. Lemmas 14 and 15 imply that

- $z = a$ and $\alpha + \zeta = \gamma$,
- $z = g$ and $\gamma + \zeta = \beta$, or
- $z = 2m - t$.

If $z = a$, then $z = t$ and Lemma 8 implies that $g = a + z - m = a + t - m = b$, contradicting that $D_g \in S$. If $z = g$, then Lemma 8 implies that

$$b = z + g - m = 2g - m.$$

Additionally, since $t = a$, it follows that $b = t + a - m = 2a - m$. So $g = a$, again contradicting that $D_g \in S$. Thus, $z = 2m - t$. If $\alpha + \zeta = \gamma$, then Lemma 8 implies that

$$g = a + z - m = a + (2m - t) - m = a + (2m - a) - m = m,$$

contradicting that $D_g \in S$. So $\gamma + \zeta = \beta$. This means that $c(\zeta) \in \{c(\beta), c(\gamma)\}$. Since $c(D_{2m-t}) \subseteq c(D_m) \cup \{c(\alpha), c(\beta)\}$, it follows that $c(\zeta) = c(\beta)$. Recall that $c(\beta) \in c(D_{2m-t})$ only if $c(\delta) = c(\beta)$. Since $\delta \in D_a$, it follows that $c(\alpha), c(\beta) \in c(D_a)$, contradicting Lemma 4. Therefore, $b = 2m - t$. □

Theorem 4 states that there are no jumps between distinctly colored elements whose colors do not appear in the main diagonal. First, it is shown that $|D_m| = 4$ and one of D_t, D_a, D_b is non-contributing; all other off-diagonals are contributing by Lemma 5. Next, some $\gamma \in D_g$ and $\zeta \in D_z$ are found such that $\alpha + \zeta = \gamma$ or $\gamma + \zeta = \beta$. The fact that D_z must be contributing means that $z \in \{g, a, b, t\}$. Lemmas 14 and 15 imply that either $\alpha + \zeta = \gamma$ and $z = a$, or that $\gamma + \zeta = \beta$ and $z = g$. In either case, a contradiction is found.

Theorem 4. *Let c be an exact, rainbow free $(m + n + 1)$-coloring of $[m] \times [n]$ for eq with $3 \leq m \leq n$, and suppose there is a jump from $\alpha \in D_a$ to $\beta \in D_b$ such that $\alpha + \delta = \beta$ for some $\delta = (d_1, d_2) \in D_t$. Then $c(\alpha) \in c(D_m)$ or $c(\beta) \in c(D_m)$ or $c(\alpha) = c(\beta)$.*

Proof. For the sake of contradiction, assume $c(\alpha), c(\beta) \notin c(D_m)$ and $c(\alpha) \neq c(\beta)$. Corollary 2 and Lemma 11 imply $4 \leq |c(D_m)|$. Lemma 2 and Corollary 4 further restrict to the situation to $8 \leq m \leq 10$, $n \leq 13$, $5 \leq d_1 + d_2 \leq 7$, and $|c(D_m)| = 4$. Additionally, Lemma 17 implies that $2m - t = b$.

Since $a = b$, $a = t$, or $b = t$ implies that $m \in \{t, a, b\}$, it follows that a, b, and t are pairwise distinct. Specifically, $t < m < b < a$ or $a < b < m < t$. Since $\alpha + \delta = \beta$, it follows that $c(\delta) \in \{c(\alpha), c(\beta)\}$. Now, Lemma 5 implies that two of D_t, D_a, D_b contribute $c(\alpha)$ and $c(\beta)$ and the third is non-contributing. Thus, every diagonal D_k with $k \notin \{m, t, a, b\}$ must contribute a color distinct from $c(\alpha)$ and $c(\beta)$.

Define S as in Equation (1). Note that

$$|S| = \begin{cases} d_1 + d_2 - 3 & \text{if } m \notin \{D_x \,|\, m + a_2 - b_1 < x < m + b_2 - a_1\}, \\ d_1 + d_2 - 4 & \text{if } m \in \{D_x \,|\, m + a_2 - b_1 < x < m + b_2 - a_1\}, \end{cases}$$

and observe $1 \leq |S|$. It is claimed that $1 \leq |S \setminus \{D_t\}|$. Indeed, if $D_t \in S$, then $|S| \geq 3$ since if $t < m < b$, it follows that

$$m + a_2 - b_1 < t \leq b - 2 = m - b_1 + b_2 - 2,$$

so that

$$2 < b_2 - a_2 = d_2 < d_1.$$

This implies that $d_1 + d_2 \geq 7$ from which it follows that $d_1 + d_2 = 7$. A similar case holds for $b < m < t$.

Let $D_g \in S \setminus \{D_t\}$, and note that D_g must contribute a color. Say D_g contributes $c(\gamma)$ for some $\gamma \in D_g$. By Lemmas 6 and 7, there exists a $\zeta \in D_z$ such that $\alpha + \zeta = \gamma$ or $\gamma + \zeta = \beta$. Note that $c(\zeta) \in \{c(\gamma), c(\alpha), c(\beta)\}$, so $z \in \{g, a, b, t\}$. By Lemmas 14 and 15, this can only happen if $\alpha + \zeta = \gamma$ and $z = a$ or $\gamma + \zeta = \beta$ and $z = g$.

`Case 1.` Suppose $\alpha + \zeta = \gamma$ and $z = a$.

Lemma 8 implies that

$$g = 2a - m = m - 2a_1 + 2a_2.$$

Since $D_g \in S$, it follows that $g < m + b_2 - a_1$ which implies that

$$a_2 - a_1 < b_2 - a_2 = d_2.$$

Similarly the lower bound implies that $a_1 - a_2 < d_1$. If $t < m < a$, then $d_2 < d_1$ and $a_1 < a_2$. Since $d_1 + d_2 = 7$, it follows that $0 < a_2 - a_1 < 3$. Similarly, if $a < m < t$, then $0 < a_1 - a_2 < 3$. Since b is between m and a, it must be true that $a = m + 2$ or $a = m - 2$. Thus, $b = m + 1$ and $g = m + 4$ or $b = m - 1$ and $g = m - 4$, respectively.

First, suppose $a = m + 2$. Since $D_{m+4} \in S$, it follows that $D_{g'} := D_{m+3} \in S$. So there exists a $\gamma' \in D_{g'}$ such that D_{m+3} contributes $c(\gamma')$. By Lemmas

6 and 7, there exists a jump $\zeta' \in D_{z'}$ from α to γ' or from γ' to β with $c(\zeta') \in \{c(\gamma'), c(\alpha), c(\beta)\}$. By Lemmas 14 and 15, this can only happen if ζ' is a jump from α to γ' and $z' = a$, or ζ' is a jump from γ' to β and $z' = g'$. If $z' = a = m + 2$, then $g' = m + 4$ since ζ' is a jump from α to $\gamma' \in D_{g'}$. Likewise, if $z' = g' = m + 3$, then $b = m + 6$. In either case, a contradiction arises.

Similar contradictions can be found when $a = m - 2$.

Case 2. Suppose $\gamma + \zeta = \beta$ and $z = g$.

Then $b = 2g - m$. Additionally, since $t = 2m - b$ and $b = a + t - m$, it follows that $a = 2b - m$. Combining these equations yields $a = 4g - 3m$ which implies that

$$g = \frac{3m + a}{4} = m + \frac{a_2 - a_1}{4}. \tag{9}$$

In parallel to Case 1, since $D_g \in S$, it follows that $g < m + b_2 - a_1$ implying that $a_2 - a_1 < 4b_2 - 4a_1$ and

$$3(a_1 - a_2) < 4(b_2 - a_2) = 4d_2.$$

Similarly, the lower bound on S implies that $3(a_2 - a_1) < 4d_1$. If $a < m < t$, then $d_1 < d_2$ and $a_2 < a_1$. Since $d_1 + d_2 = 7$, it follows that

$$0 < a_1 - a_2 < \frac{4}{3}d_2 \leq 8.$$

Likewise, if $t < m < a$, then $0 < a_2 - a_1 < 8$. Note that Equation (9) implies that $a_1 - a_2 = 4(m - g)$, i.e. four divides $a_1 - a_2$. So $a = m + 4$ or $a = m - 4$. In the former case, $D_{m+3} \in S$ with $m + 3 \notin \{g, t\}$, and, in the latter, $D_{m-3} \in S$ with $m - 3 \notin \{g, t\}$. Lemmas 6, 7, 14 and 15 yield similar contradictions to those in Case 1.

Therefore, $c(\alpha) \in c(D_m)$ or $c(\beta) \in c(D_m)$ or $c(\alpha) = c(\beta)$. In other words, there are no jumps between distinctly colored elements. □

5. Rainbow Number of $[m] \times [n]$ for $x_1 + x_2 = x_3$

This final section uses definitions that were introduced in Section 2 which can be visualized with the help of Figure 2. In particular, definitions of W, Y, P_h and P_v are used.

Three lemmas and the final results are presented. The first lemma gives a lower bound on the number of consecutive contributing diagonals and, provided a horizontal or vertical pair intersects W, the second gives upper bounds on vertical and horizontal pairs, respectively. These results are leveraged against the third lemma, a relatively unconditional and straightforward upper bound on the total number of horizontal and vertical pairs, to yield Theorem 5. This theorem will give $\text{rb}([m]\times[n], eq) = m+n+1$ for all $8 \leq m \leq n$. Along with several earlier statements to cover the cases of $2 \leq m \leq 7$, Theorem 5 will subsequently determine the final result and the result claimed to be true in the exposition of this paper: Theorem 6.

Lemma 18. *If c is an exact, rainbow-free $(m+n+1)$-coloring of $[m] \times [n]$ for eq with $3 \leq m \leq n$, then there are at least $m+n-2\log_2(m/s_2)-2$ pairs of consecutive, contributing off-diagonals.*

Proof. Define $\ell = |c(D_m)|$ and notice that Corollary 3 gives at least $m + n - \log_2(m/s_2) - 1$ contributing off-diagonals and Lemma 5 gives at most $\ell - 3$ non-contributing off-diagonals. If all of the contributing off-diagonals were consecutive, then there would be $m + n - \log_2(m/s_2) - 2$ pairs of consecutive, contributing off-diagonals. However, every non-contributing diagonal, including every non-contributing off-diagonal and D_m, when inserted between two consecutive, contributing off-diagonals, can reduce the count of pairs of consecutive, contributing off-diagonals by at most one. Finally, since Lemma 2 implies that $\ell \leq \log_2(m/s_2)+2$, there are at least

$$(m + n - \log_2(m/s_2) - 2) - (\ell - 2) \geq m + n - 2\log_2(m/s_2) - 2$$

pairs of consecutive, contributing off-diagonals. □

Lemma 19. *Let c be an exact, rainbow-free $(m+n+1)$-coloring of $[m] \times [n]$ for eq with $4 \leq m \leq n$. If there is a horizontal pair P_h intersecting W, then there are at most $2s_2 - 2$ vertical pairs in $[m] \times [n]$. Likewise, if there is a vertical pair P_v intersecting W, then there are at most $2s_2 - 2$ horizontal pairs in $[m] \times [n]$.*

Proof. Note that if $|c(D_m)| = 3$ then c is not rainbow-free by Theorem 3 so assume $|c(D_m)| \geq 4$. Then, Lemma 2 gives $s_2 \leq m/4$. Further, Lemma 18 implies that at least $m + n - 2\log_2(m/s_2) - 2$ pairs of consecutive, contributing off-diagonals. At most $2(2s_2 - 2)$ of these pairs are entirely contained in Y so at least $m + n - 2\log_2(m/s_2) - 2 - 2(2s_2 - 2)$ pairs of consecutive, contributing off-diagonals intersect W. Using that $4 \leq m \leq n$, $m/s_2 \leq m/2$, and $4s_2 \leq m$ along with the fact that $\log_2(x)$ is increasing gives

$$m + n - 2\log_2(m/s_2) - 2 - 2(2s_2 - 2) \geq m - 2\log_2(m) + 4 \geq 4.$$

That is, there are at least 4 pairs of consecutive, contributing off-diagonals that intersect W. Further, Theorem 4 indicates that these pairs are either horizontal or vertical. If any of these pairs that intersect W is a horizontal pair, there cannot also be a vertical pair that intersects W without creating a contributing disjoint corner which, by Lemma 9, implies that c is not rainbow-free. Thus, all vertical pairs must be contained completely within Y. Only half of the pairs in Y can be vertical so, in this case, there are at most $2s_2 - 2$ vertical pairs in $[m] \times [n]$. A similar result is obtained when it is assumed that a vertical pair intersects W. □

Lemma 20. *Let c be an exact, rainbow-free (m+n+1)-coloring of $[m] \times [n]$ for eq with $3 \leq m \leq n$. Then, there are at most $n - 1$ possible horizontal pairs, and there are at most $m - 1$ vertical pairs.*

Proof. Lemma 4 and Theorem 4 imply that two distinct horizontal pairs cannot both intersect the ith and $(i+1)$th columns and two distinct vertical pairs cannot both intersect the jth and $(j+1)$th rows. Since there are $n-1$ pairs of consecutive columns and $m - 1$ pairs of consecutive rows, the desired result is obtained. □

Theorem 5. *If c is an exact $(m + n + 1)$-coloring of $[m] \times [n]$ with $8 \leq m \leq n$, then $[m] \times [n]$ contains a rainbow solution to eq.*

Proof. Assume, for the sake of contradiction, that c is rainbow-free. By Theorem 3, $|c(D_m)| \geq 4$. Now, Lemma 2 implies that $4 \leq \log_2(m/s_2) + 2$ so that $m \geq 4s_2$.

`Case 1.` There exists a horizontal pair intersecting W.

By Lemma 18, there exist at least $m + n - 2\log_2(m/s_2) - 2$ pairs of consecutive contributing off-diagonals. By Theorem 4, there are at least $m + n - 2\log_2(m/s_2) - 2$ consecutive contributing pairs of elements of which each must be a vertical or horizontal pair. Since Lemma 19 implies there at most $2s_2 - 2$ vertical pairs, there must be at least $m + n - 2\log_2(m/s_2) - 2s_2$ horizontal pairs in $[m] \times [n]$.

Using that $m/s_2 \leq m/2$, $s_2 \leq m/4$, $8 \leq m$ along with the fact that $\log_2(x)$ is increasing gives

$$m + n - 2\log_2(m/s_2) - 2s_2 \geq n + m/2 - 2\log_2(m) + 2 \geq n > n - 1,$$

contradicting Lemma 20.

`Case 2.` There exists a vertical pair intersecting W.

Using an argument similar to the first case shows that there must be at least $m + n - 2\log_2(m/s_2) - 2s_2$ vertical pairs $[m] \times [n]$. Again,

$$m + n - 2\log_2(m/s_2) - 2s_2 > n - 1 \geq m - 1,$$

contradicting Lemma 20.

Since both cases give a contradiction, it follows that c is not rainbow-free. □

Combining Lemma 3, Corollary 1, Theorem 3, and Theorem 5 gives Theorem 6.

Theorem 6. *If* $2 \leq m \leq n$, *then* $\text{rb}([m] \times [n], eq) = m + n + 1$.

Proof. First, Corollary 1 supplies the desired result when $m = 2$. Now, Lemma 3 shows that $m + n + 1 \leq \text{rb}([m] \times [n], eq)$ for $3 \leq m \leq n$. For $3 \leq m \leq 7$, at most three colors are in the main diagonal so Theorem 3 implies that $\text{rb}([m] \times [n], eq) = m+n+1$. Finally, for $8 \leq m \leq n$, Theorem 5 gives $\text{rb}([m] \times [n], eq) = m+n+1$. □

6. Future Work

A generalization of the rainbow-free, exact $(m+n)$-coloring of $[m] \times [n]$ with respect to equation $x_1 + x_2 = x_3$ also provides a lower bound for $[m_1] \times [m_2] \times [m_3]$. In particular, let $c((i, j, k)) = red$ if $1 \leq i < m_1$, $1 \leq j < m_2$, and $1 \leq k < m_3$, and color every other element distinctly. Now, every solution to $x_1 + x_2 = x_3$ has at least two elements that are colored *red*, so c is rainbow-free. The inclusion-exclusion principle shows that c uses $(m_1m_2 + m_2m_3 + m_1m_3 + 1) - (m_1 + m_2 + m_3) + 1$ colors. Therefore,

$$(m_1m_2+m_2m_3+m_1m_3)-(m_1+m_2+m_3)+3 \leq \text{rb}\left([m_1] \times [m_2] \times [m_3], x_1 + x_2 = x_3\right).$$

This leads to the following conjecture.

Conjecture 1. Assume $2 \leq m_1 \leq m_2 \leq m_3$. Then

$$(m_1m_2+m_2m_3+m_1m_3)-(m_1+m_2+m_3)+3 = \text{rb}\left([m_1] \times [m_2] \times [m_3], x_1 + x_2 = x_3\right).$$

This coloring can generalize, as can the counting principle, to get a lower-bound on $\text{rb}\left(\prod_{i=1}^{k}[m_i], x_1 + x_2 = x_3\right)$. The authors' intuition indicates that the lower bound is actually the rainbow number and believe that some of the results from this paper can be generalized to higher dimensional integer arrays.

Another equation studied when looking at rainbow solutions is $x_1 + x_2 = 2x_3$. This is because sets that satisfy that equation are also 3-term arithmetic progressions. The authors in [4] determined the anti-van der Waerden number for $[n]$.

Theorem 7 ([4])**.** *Let* $7 \cdot 3^{m-2} \leq n \leq 21 \cdot 3^{m-2}$. *Then*

$$\text{aw}([n], 3) = \begin{cases} m + 2 & \text{if } n = 3^m \\ m + 3 & \text{otherwise.} \end{cases}$$

Note that in this paper $\text{rb}([n], x_1 + x_2 = 2x_3) = \text{aw}([n], 3)$. Assume that the common difference for the arithmetic progressions in $[m] \times [n]$ is $\delta = (d_1, d_2)$. Note that δ can partition $[m] \times [n]$ into δ-diagonals whose elements create arithmetic progressions with respect to δ. Enumerate these δ-diagonals as D_i with $1 \leq i \leq k$

where k is a function of d_1 and d_2. As an example, consider $[5] \times [7]$ and $\delta = (1, 2)$. Here, $\{(2, 1), (3, 3), (4, 5), (5, 7)\}$ is a δ-diagonal of length four, $\{(4, 1), (5, 3)\}$ is a δ-diagonal of length two, and $\{(5, 1)\}$ is a δ-diagonal of length one.

Define $\mathrm{rb}([m] \times [n], \delta, 3) = r$ to be the smallest number of colors such that every exact r-coloring of $[m] \times [n]$ has a rainbow 3-term arithmetic progression with common difference δ.

Conjecture 2. Consider $[m]\times[n]$ with $\delta = (d_1, d_2)$. Assume that D_i, for $1 \le i \le k$, are the δ-diagonals of $[m] \times [n]$ with respect to δ. Then,

$$\mathrm{rb}([m] \times [n], \delta, 3) = 1 + \sum_{i=1}^{k} \left(\mathrm{aw}([|D_i|], 3) - 1\right).$$

Conjecture 2 may be helpful when determining $\mathrm{rb}([m] \times [n], x_1 + x_2 = 2x_3)$.

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