#A49 INTEGERS 23 (2023)

EXTREMAL SEQUENCES RELATED TO THE JACOBI SYMBOL

Santanu Mondal
Department of Mathematics, Ramakrishna Mission Vivekananda Educational and Research Institute, Dist. Howrah, India
santanu.mondal.math18@gm.rkmvu.ac.in

Krishnendu Paul
Department of Mathematics, Ramakrishna Mission Vivekananda Educational and Research Institute, Dist. Howrah, India
krishnendu.p.math18@gm.rkmvu.ac.in

Shameek Paul
Department of Mathematics, Ramakrishna Mission Vivekananda Educational and Research Institute, Dist. Howrah, India
shameek.paul@rkmvu.ac.in

Received: 1/4/22, Revised: 4/22/23, Accepted: 7/1/23, Published: 7/21/23

Abstract

Given $A \subseteq \mathbb{Z}_n$, the A-weighted zero-sum constant C_A is defined to be the smallest natural number k such that any sequence of k elements in $\mathbb{Z}_n$ has an A-weighted zero-sum subsequence of consecutive terms. A sequence of length $C_A - 1$ in $\mathbb{Z}_n$ which does not have any A-weighted zero-sum subsequence of consecutive terms is called a C-extremal sequence for A. For n odd, let $S(n)$ be the set of all units in $\mathbb{Z}_n$ whose Jacobi symbol with respect to n is one. Given a prime divisor p of n, let $L(n;p)$ be the set of all units $\mathbb{Z}_n$ whose Jacobi symbol with respect to n is the same as their Legendre symbol with respect to p. We characterize the C-extremal sequences for $S(n)$ and $L(n;p)$. Given $A \subseteq \mathbb{Z}_n$, the A-weighted Davenport constant D_A is defined to be the smallest natural number k such that any sequence of k elements in $\mathbb{Z}_n$ has an A-weighted zero-sum subsequence. A sequence of length $D_A - 1$ in $\mathbb{Z}_n$ which does not have any A-weighted zero-sum subsequence is called a D-extremal sequence for A. We characterize the D-extremal sequences for $S(n)$ and $L(n;p)$.

1. Introduction

This paper is a complementary paper to [7]. For $a, b \in \mathbb{Z}$, we denote the set $\{x \in \mathbb{Z} : a \leq x \leq b\}$ by $[a, b]$. Let $U(n)$ denote the group of units in the ring $\mathbb{Z}_n$.

DOI: 10.5281/zenodo.8174508

Let $U(n)^2 = \{ x^2 : x \in U(n) \}$. Let Q_p denote the set $U(p)^2$ when p is an odd prime. We say that $\Omega(n) = k$ if n is a product of k primes.

Definition 1.1. For a subset $A \subseteq \mathbb{Z}_n$, the *A-weighted Davenport constant D_A*, is defined to be the least positive integer k such that any sequence in $\mathbb{Z}_n$ of length k has an A-weighted zero-sum subsequence.

Adhikari and Rath [3] gave the previous definition. Mondal, K. Paul, and S. Paul [5] gave the following definition.

Definition 1.2. For a subset $A \subseteq \mathbb{Z}_n$, the *A-weighted zero-sum constant C_A*, is defined to be the least positive integer k such that any sequence in $\mathbb{Z}_n$ of length k has an A-weighted zero-sum subsequence of consecutive terms.

Griffiths [4, Theorem 1.2] along with Yuan and Zeng [8, Theorem 1.2], and Mondal, K. Paul, and S. Paul [5, Corollary 4] showed the next result.

Theorem 1.3 ([4],[5])**.** *Let n be an odd number. Then we have $D_{U(n)} = \Omega(n)+1$ and $C_{U(n)} = 2^{\Omega(n)}$.*

Let m be a divisor of n. We refer to the homomorphism $f_{n,m} : \mathbb{Z}_n \to \mathbb{Z}_m$ given by $a + n\mathbb{Z} \mapsto a + m\mathbb{Z}$ as the natural map. Clearly this map is onto. As the image of $U(n)$ under $f_{n,m}$ is contained in $U(m)$, we get a map $U(n) \to U(m)$ which we also refer to as the natural map and denote by $f_{n,m}$.

For $x \in \mathbb{Z}_n$ the Jacobi symbol $(\frac{x}{n})$ is defined in [7]. For a prime divisor p of n, let $(\frac{x}{p})$ denote the Legendre symbol of $f_{n,p}(x) \in \mathbb{Z}_p$.

$$\text{Let } S(n) = \left\{ x \in U(n) : \left(\frac{x}{n}\right) = 1 \right\} \text{ and } L(n;p) = \left\{ x \in U(n) : \left(\frac{x}{n}\right) = \left(\frac{x}{p}\right) \right\}.$$

When p is an odd prime, we see that $S(p) = Q_p$. Adhikari and Rath [3, Theorem 2], and Mondal, K. Paul, and S. Paul [5, Theorem 4] showed the next result.

Theorem 1.4 ([3],[5])**.** *Let p be an odd prime. Then we have $C_{Q_p} = D_{Q_p} = 3$.*

Mondal, K. Paul, and S. Paul [7, Theorems 3.3 and 3.4] showed the next result.

Theorem 1.5 ([7])**.** *When n is a squarefree number which is not a prime and every prime divisor of n is at least seven, we have $D_{S(n)} = \Omega(n)+1$ and $C_{S(n)} = 2^{\Omega(n)}$.*

The next two results follow from Theorems 5.2, 5.3, 5.4 and 5.5 of [7].

Theorem 1.6 ([7])**.** *When n is a squarefree number which is not a product of two primes and every prime divisor of n is at least seven, we have $D_{L(n;p)} = \Omega(n)+1$ and $C_{L(n;p)} = 2^{\Omega(n)}$ where p is a prime divisor of n.*

Theorem 1.7 ([7])**.** *Let $n = pq$ where p and q are distinct primes which are at least seven. Then we have that $D_{L(n;\,p)} = 4$ and $C_{L(n;\,p)} = 6$.*

Adhikari, Molla, and Paul [2] gave the following definition.

Definition 1.8. A sequence S in $\mathbb{Z}_n$ of length $D_A - 1$ which has no A-weighted zero-sum subsequence is called a *D-extremal sequence for A*.

Mondal, K. Paul, and S. Paul [6] gave the following definition.

Definition 1.9. A sequence S in $\mathbb{Z}_n$ of length $C_A - 1$ which has no A-weighted zero-sum subsequence of consecutive terms is called a *C-extremal sequence for A*.

Assume that n is a squarefree number such that every prime divisor of n is at least seven. Let p be a prime divisor of n. For a sequence S in $\mathbb{Z}_n$ we have shown the following:

- Suppose $\Omega(n) \neq 1$. Then S is a C-extremal sequence for $S(n)$ if and only if S is a C-extremal sequence for $U(n)$.
- Suppose $\Omega(n) \neq 1, 2$. Then S is a D-extremal sequence for $S(n)$ if and only if S is a D-extremal sequence for $U(n)$.
- Suppose $\Omega(n) \neq 2$. Then S is a C-extremal sequence for $L(n;p)$ if and only if S is a C-extremal sequence for $U(n)$.
- Suppose $\Omega(n) \neq 2, 3$. Then S is a D-extremal sequence for $L(n;p)$ if and only if S is a D-extremal sequence for $U(n)$.

Remark 1.10. When n is odd, Adhikari, Molla, and Paul have characterized the D-extremal sequences for $U(n)$ in [2, Theorem 6]. Mondal, K. Paul, and S. Paul have characterized the C-extremal sequences for $U(n)$ in [6, Theorems 5 and 6].

If p is a prime divisor of n, we use the notation $v_p(n) = r$ to mean that $p^r \mid n$ and $p^{r+1} \nmid n$. Let $S = (x_1, \ldots, x_l)$ be a sequence in $\mathbb{Z}_n$ and let p be a prime divisor of n such that $v_p(n) = r$. We denote the image of $x \in \mathbb{Z}_n$ under f_{n,p^r} by $x^{(p)}$ and we denote the sequence $(x_1^{(p)}, \ldots, x_l^{(p)})$ in $\mathbb{Z}_{p^r}$ by $S^{(p)}$.

2. Some Results about $S(n)$-Weighted Zero-Sum Sequences

From this point onwards, we assume that n is odd.

The set $S(n) = \{\, x \in U(n) : (\frac{x}{n}) = 1\}$ was considered as a weight-set in Section 3 of [1]. From Proposition 2.2 of [7] we see that $S(n) = U(n)$ if n is a square, and $S(n)$ is a subgroup of index two in $U(n)$, otherwise.

The next four results are Lemmas 2.5, 2.6, 3.1 and 3.2 of [7]. They are used in the next section.

Lemma 2.1 ([7])**.** *Let d be a proper divisor of n such that d is not a square. Suppose d is coprime with m where $m = n/d$. Then we have that $U(m) \subseteq f_{n,m}\big(S(n)\big)$.*

Lemma 2.2 ([7])**.** *Let S be a sequence in $\mathbb{Z}_n$ and d be a proper divisor of n which divides every element of S. Suppose that d is coprime with $m = n/d$. Let S' be the image of S under the map $f_{n,m}$. Let $A \subseteq \mathbb{Z}_n$ and $B \subseteq f_{n,m}(A)$. Suppose S' is a B-weighted zero-sum sequence. Then S is an A-weighted zero-sum sequence.*

Lemma 2.3 ([7])**.** *Let n be an odd, squarefree number. Suppose S is a sequence in $\mathbb{Z}_n$ such that at most one term of S is a unit, and for every prime divisor q of n at least two terms of S are coprime to q. Then S is an $S(n)$-weighted zero-sum sequence.*

Lemma 2.4 ([7])**.** *Let n be a squarefree number whose every prime divisor is at least seven. Suppose S is a sequence in $\mathbb{Z}_n$ such that for every prime divisor q of n at least two terms of S are coprime to q, and there is a prime divisor p of n such that at least three terms of S are coprime to p. Then S is an $S(n)$-weighted zero-sum sequence.*

3. D-Extremal Sequences for $S(n)$

Remark 3.1. As $S(n) \subseteq U(n)$, an $S(n)$-weighted zero-sum subsequence is also a $U(n)$-weighted zero-sum subsequence. So if n is such that $D_{S(n)} = D_{U(n)}$, then every D-extremal sequence for $U(n)$ is also a D-extremal sequence for $S(n)$. Also, if n is such that $C_{S(n)} = C_{U(n)}$, then every C-extremal sequence for $U(n)$ is also a C-extremal sequence for $S(n)$.

Theorem 3.2. *Let n be a squarefree number such that every prime divisor of n is at least seven. Suppose $\Omega(n) \geq 3$ and S is a sequence in $\mathbb{Z}_n$. Then S is a D-extremal sequence for $S(n)$ if and only if S is a D-extremal sequence for $U(n)$.*

Proof. As $\Omega(n) \geq 3$, by Theorems 1.3 and 1.5 we see that $D_{S(n)} = D_{U(n)}$. So by Remark 3.1 it is enough to show that every D-extremal sequence for $S(n)$ is a D-extremal sequence for $U(n)$.

Let $S = (x_1, \ldots, x_k)$ be a D-extremal sequence for $S(n)$. By Theorem 1.5 we have $D_{S(n)} = \Omega(n) + 1$. Hence, it follows that $k = \Omega(n)$. Clearly, all the terms of S are non-zero. We have three cases to consider.

`Case 1.` There is a prime divisor p of n such that at most one term of S is coprime to p.

Suppose all terms of S are divisible by p. Let $m = n/p$ and S' be the image of S under $f_{n,m}$. By Theorem 1.3 we have $D_{U(m)} = \Omega(m) + 1$. As S' has length $\Omega(n) = \Omega(m) + 1$, we see that S' has a $U(m)$-weighted zero-sum subsequence. As n

is squarefree, it follows that p is coprime to m. So by Lemmas 2.1 and 2.2, we get the contradiction that S has an $S(n)$-weighted zero-sum subsequence.

So in this case, exactly one term of S is not divisible by p. Let us assume that that term is x_1. Let $T = (x_2, \ldots, x_k)$ and T' be the image of T under $f_{n,m}$. Suppose T' has a $U(m)$-weighted zero-sum subsequence. By Lemmas 2.1 and 2.2, we get the contradiction that S has an $S(n)$-weighted zero-sum subsequence. Thus, the sequence T' in $\mathbb{Z}_m$ does not have any $U(m)$-weighted zero-sum subsequence. As $D_{U(m)} = \Omega(m) + 1$ and the length of T' is $\Omega(m)$, it follows that T' is a D-extremal sequence for $U(m)$. So from Theorem 5 of [2] we see that S is a D-extremal sequence for $U(n)$.

`Case 2.` For every prime divisor q of n, exactly two terms of S are coprime to q.

Suppose S has at most one unit. By Lemma 2.3 we get the contradiction that S is an $S(n)$-weighted zero-sum sequence. So we can assume that S has at least two units. By the assumption in this case, we see that S has exactly two units and the other terms of S are divisible by n. As the length of S is $\Omega(n)$, which is at least three, we get the contradiction that some term of S is zero.

`Case 3.` For every prime divisor q of n at least two terms of S are coprime to q, and there is a prime divisor p of n such that at least three terms of S are coprime to p.

In this case, by Lemma 2.4 we get the contradiction that S is an $S(n)$-weighted zero-sum sequence. □

Theorem 3.3. *Let n be a squarefree number such that every prime divisor of n is at least seven. Suppose $\Omega(n) = 2$. Then a sequence S in $\mathbb{Z}_n$ is a D-extremal sequence for $S(n)$ if and only if S is either a D-extremal sequence for $U(n)$ or $S = (x_1, x_2)$ where x_1 and $-x_2$ are in different cosets of $S(n)$ in $U(n)$.*

Proof. From Theorems 1.3 and 1.5 we have $D_{U(n)} = D_{S(n)}$. So from Remark 3.1 we see that if S is a D-extremal sequence for $U(n)$, then S is a D-extremal sequence for $S(n)$.

Let $S = (x_1, x_2)$ where x_1 and $-x_2$ are in different cosets of $S(n)$ in $U(n)$. Suppose T is an $S(n)$-weighted zero-sum subsequence of S. Then we see that T must be S itself. So there exist $a, b \in S(n)$ such that $ax_1 + bx_2 = 0$ and hence there exists $c \in S(n)$ such that $-x_2 = cx_1$. As x_1 and $-x_2$ are in different cosets of $S(n)$ in $U(n)$, we get the contradiction that $c \notin S(n)$. Thus, it follows that S does not have any $S(n)$-weighted zero-sum subsequence. From Theorem 1.5, we have that $D_{S(n)} = 3$ and so we see that S is a D-extremal sequence for $S(n)$.

Conversely, suppose S is a D-extremal sequence for $S(n)$. By Theorem 1.5 we have $D_{S(n)} = \Omega(n) + 1 = 3$. Hence, it follows that S has length 2. Let $S = (x_1, x_2)$. We have two cases to consider.

`Case 1.` For every prime divisor q of n, exactly two terms of S are coprime to q.

As x_1 and x_2 are coprime to every prime divisor of n, it follows that $x_1, x_2 \in U(n)$. As n is squarefree, from Proposition 2.2 of [7] we get that $S(n)$ has index two in $U(n)$. Suppose either $x_1, -x_2 \in S(n)$ or $x_1, -x_2 \in U(n) \setminus S(n)$. Then we see that $a = -x_2 x_1^{-1} \in S(n)$. As $1 \in S(n)$ and $ax_1 + x_2 = 0$, we get the contradiction that S is an $S(n)$-weighted zero-sum sequence. Thus, the sequence $S = (x_1, x_2)$ where x_1 and $-x_2$ are in different cosets of $S(n)$ in $U(n)$.

Case 2. The assumption in Case 1 does not hold.

We use similar arguments as in the proof of Theorem 3.2 to conclude that S is a D-extremal sequence for $U(n)$. □

Remark 3.4. For a prime p, we have that $S(p) = Q_p$. From Corollary 2 of [6], we can see that the D-extremal sequences for Q_p are precisely those which are of the form (x_1, x_2) where x_1 and $-x_2$ are in different cosets of Q_p in $U(p)$.

4. C-Extremal Sequences for $S(n)$

Theorem 4.1. *Let n be a non-prime squarefree number such that every prime divisor of n is at least seven. Then a sequence S in $\mathbb{Z}_n$ is a C-extremal sequence for $S(n)$ if and only if S is a C-extremal sequence for $U(n)$.*

Proof. By Theorems 1.3 and 1.5 we get $C_{U(n)} = C_{S(n)}$. So by Remark 3.1 it is enough to show that every C-extremal sequence for $S(n)$ is a C-extremal sequence for $U(n)$.

Suppose a sequence $S = (x_1, \ldots, x_l)$ in $\mathbb{Z}_n$ is a C-extremal sequence for $S(n)$. By Theorem 1.5 we have $C_{S(n)} = 2^{\Omega(n)}$ and so we see that $l = 2^{\Omega(n)} - 1$. Clearly, all the terms of S must be non-zero. We have three cases to consider.

Case 1. There is a prime divisor p of n such that at most one term of S is not divisible by p.

Suppose all the terms of S are divisible by p. Let $m = n/p$ and S' be the image of S under $f_{n,m}$. By Theorem 1.3 we have $C_{U(m)} = 2^{\Omega(m)}$. As S' has length $l = 2^{\Omega(n)} - 1$ and as $\Omega(n) = \Omega(m) + 1$, we get that $l > 2^{\Omega(m)}$. Hence, it follows that S' has a $U(m)$-weighted zero-sum subsequence of consecutive terms. Thus by Lemmas 2.1 and 2.2, we get the contradiction that S has an $S(n)$-weighted zero-sum subsequence of consecutive terms.

Thus, in this case, we see that exactly one term x^* of S is coprime to p. Suppose $x^* \neq x_{k+1}$ where $k + 1 = (l + 1)/2$. Then there is a subsequence T of consecutive terms of S of length at least $k + 1$ such that p divides every term of T. As we have $l + 1 = 2^{\Omega(n)}$, we see that $k + 1 = (l + 1)/2 = 2^{\Omega(n)-1} = 2^{\Omega(m)}$. So by a similar argument as in the previous paragraph, we get the contradiction that T (and hence S) has an $S(n)$-weighted zero-sum subsequence of consecutive terms. Thus, we see that $x^* = x_{k+1}$.

Let $S_1 = (x_1, \ldots, x_k)$ and $S_2 = (x_{k+2}, \ldots, x_l)$. Let S_1' and S_2' be the images of the sequences S_1 and S_2 respectively under the map $f_{n,m}$. Suppose S_1' has a $U(m)$-weighted zero-sum subsequence of consecutive terms. By Lemma 2.1, we have $U(m) \subseteq f_{n,m}\big(S(n)\big)$. As p divides every term of S_1, by Lemma 2.2 we get the contradiction that S_1 (and hence S) has an $S(n)$-weighted zero-sum subsequence of consecutive terms. Thus, the sequence S_1' does not have any $U(m)$-weighted zero-sum subsequence of consecutive terms. As S_1' has length $k = 2^{\Omega(m)} - 1 = C_{U(m)} - 1$, it follows that S_1' is a C-extremal sequence for $U(m)$.

A similar argument shows that S_2' is also a C-extremal sequence for $U(m)$. Thus, from Theorem 5 of [6] it follows that S is a C-extremal sequence for $U(n)$.

`Case 2.` For every prime divisor q of n, exactly two terms of S are coprime to q.

If S has at most one unit, by Lemma 2.3 we get the contradiction that S is an $S(n)$-weighted zero-sum sequence. So we can assume that S has at least two units. By the assumption in this case, we see that S has exactly two units and the other terms of S are divisible by n. As the length of S is $2^{\Omega(n)} - 1$ and as $\Omega(n) \geq 2$, we see that S has at least three terms. Thus, we get the contradiction that S has a term which is zero.

`Case 3.` For every prime divisor q of n at least two terms of S are coprime to q, and there is a prime divisor p of n such that at least three terms of S are coprime to p.

In this case, by Lemma 2.4 we get the contradiction that S is an $S(n)$-weighted zero-sum sequence. □

Remark 4.2. For a prime p, we have that $S(p) = Q_p$. The C-extremal sequences for Q_p have been characterized in Corollary 2 of [6]. They are the sequences which are of the form (x_1, x_2) where x_1 and $-x_2$ are in different cosets of Q_p in $U(p)$.

5. Some Results about the Weight-Set $L(n; p)$

In [7] we considered the subset $L(n; p)$ of $\mathbb{Z}_n$. Let us recall the definition.

Definition 5.1. For a prime divisor p of n, let

$$L(n;p) = \left\{ a \in U(n) \;\middle|\; \left(\frac{a}{n}\right) = \left(\frac{a}{p}\right) \right\}.$$

Remark 5.2. From Proposition 4.2 of [7], we see that $L(n; p) = U(n)$ if n has a unique prime divisor p such that $v_p(n)$ is odd, and $L(n; p)$ is a subgroup of $U(n)$ having index two, otherwise.

The next five results are Lemmas 4.4, 4.5, 4.7 and 5.1 and Observation 4.6 of [7]. They will be used in the next section.

Lemma 5.3 ([7]). *Let p', p be prime divisors of n. Suppose p is coprime to $m = n/p$. Then we have that $S(m) \subseteq f_{n,m}\big(L(n;p')\big)$.*

Lemma 5.4 ([7]). *Let p' be a prime divisor of n which is coprime to $m = n/p'$. Then we have that $U(p') \subseteq f_{n,p'}\big(L(n;p')\big)$.*

Lemma 5.5 ([7]). *Let n be squarefree and p' be a prime divisor of n. Suppose the map $\psi : U(n) \to U(m) \times U(p')$ is the isomorphism given by the Chinese remainder theorem where $m = n/p'$. Then we have that $S(m) \times U(p') \subseteq \psi\big(L(n;p')\big)$.*

Lemma 5.6 ([7]). *Let n be squarefree and p' be a prime divisor of n. Let S be a sequence in $\mathbb{Z}_n$ such that for every prime divisor q of n, at least two terms of S are coprime to q. Let $m = n/p'$ and S' be the image of S under $f_{n,m}$. Suppose at most one term of S' is a unit, or there is a prime divisor p of m such that at least three terms of S are coprime to p. Then S is an $L(n;p')$-weighted zero-sum sequence.*

Observation 5.7 ([7]). *Let $n = m_1 m_2$ where m_1 and m_2 are coprime. Let $A \subseteq \mathbb{Z}_n$ and S be a sequence in $\mathbb{Z}_n$. Let S_i denote the image of the sequence S under f_{n,m_i} for each $i \in [1,2]$. Let $\psi : U(n) \to U(m_1) \times U(m_2)$ be the isomorphism given by the Chinese remainder theorem. Suppose $A_1 \subseteq U(m_1)$ and $A_2 \subseteq U(m_2)$ are such that $A_1 \times A_2 \subseteq \psi(A)$. If S_1 is an A_1-weighted zero-sum sequence and S_2 is an A_2-weighted zero-sum sequence, then S is an A-weighted zero-sum sequence.*

6. D-Extremal Sequences for $L(n;p)$

Remark 6.1. Let p be a prime divisor of n. As $L(n;p) \subseteq U(n)$, an $L(n;p)$-weighted zero-sum subsequence is also a $U(n)$-weighted zero-sum subsequence. So if n is such that $D_{L(n;p)} = D_{U(n)}$, then every D-extremal sequence for $U(n)$ is a D-extremal sequence for $L(n;p)$. Also, if n is such that $C_{L(n;p)} = C_{U(n)}$, then every C-extremal sequence for $U(n)$ is a C-extremal sequence for $L(n;p)$.

Theorem 6.2. *Let n be a squarefree number such that every prime divisor of n is at least seven. Suppose p' is a prime divisor of n and $\Omega(n) \neq 2, 3$. Then S is a D-extremal sequence for $L(n;p')$ if and only if S is a D-extremal sequence for $U(n)$.*

Proof. As $\Omega(n) \neq 2$, by Theorems 1.3 and 1.6 we have that $D_{L(n;p)} = D_{U(n)}$. So by Remark 6.1 it is enough to show that every D-extremal sequence for $L(n;p')$ is a D-extremal sequence for $U(n)$.

Let S be a D-extremal sequence for $L(n;p')$. If $\Omega(n) = 1$, then $n = p'$. As $L(n;p') = U(n)$, it follows that S is a D-extremal sequence for $U(n)$. So we may assume that $\Omega(n) \geq 4$. By Theorem 1.6 we have $D_{L(n;p')} = \Omega(n) + 1$. Thus S must have length $\Omega(n)$. Let $S = (x_1, \ldots, x_k)$ where $k = \Omega(n)$. Clearly, all the terms of S are non-zero. We have three cases to consider.

Case 1. There is a prime divisor p of n such that at most one term of S is coprime to p.

Suppose all terms of S are divisible by p. Let $m = n/p$ and S' be the image of S under $f_{n,m}$. By Theorem 1.5 we have $D_{S(m)} = \Omega(m) + 1$. As S' has length $\Omega(n) = \Omega(m) + 1$, we see that S' has an $S(m)$-weighted zero-sum subsequence. As n is squarefree, so p is coprime to m. So from Lemmas 2.2 and 5.3 we get the contradiction that S has an $L(n; p')$-weighted zero-sum subsequence.

So in this case, exactly one term of S is not divisible by p. Let us assume that that term is x_1. Let $T = (x_2, \ldots, x_k)$ and T' be the image of T under $f_{n,m}$. Suppose T' has an $S(m)$-weighted zero-sum subsequence. By Lemma 5.3 we have that $S(m) \subseteq f_{n,m}\big(L(n; p')\big)$ and so by Lemma 2.2 we get the contradiction that S has an $L(n; p')$-weighted zero-sum subsequence. Thus, the sequence T' in $\mathbb{Z}_m$ does not have any $S(m)$-weighted zero-sum subsequence. As $D_{S(m)} = \Omega(m) + 1$ and the length of T' is $\Omega(m)$, it follows that T' is a D-extremal sequence for $S(m)$.

As $\Omega(n) \geq 4$, we have that $\Omega(m) \geq 3$ and so from Theorem 3.2, we get that T' is a D-extremal sequence for $U(m)$. Thus, by Theorem 6 of [2] we see that S is a D-extremal sequence for $U(n)$.

Case 2. For every prime divisor q of n/p' exactly two terms of S are coprime to q, and at least two terms of S are coprime to p'.

Let $m = n/p'$ and S' be the image of S under $f_{n,m}$. Suppose at most one term of S' is a unit. By Lemma 5.6 we see that S is an $L(n; p')$-weighted zero-sum sequence. Suppose at least two terms of S' are units. By the assumption in this case we see that exactly two terms of S' are units, say x'_{j_1} and x'_{j_2} and the other terms of S' are zero.

It follows that all terms of S are divisible by m except x_{j_1} and x_{j_2}. As the sequence S has length at least four, we can find a subsequence T of S having length at least two which does not contain the terms x_{j_1} and x_{j_2}. If x_j is divisible by p' where $j \neq j_1, j_2$, we get the contradiction that $x_j = 0$. So it follows that all the terms of the sequence $T^{(p')}$ are non-zero. As $T^{(p')}$ has length at least two, from [4, Lemma 2.1] we see that $T^{(p')}$ is a $U(p')$-weighted zero-sum sequence. Also all the terms of T are divisible by m. Hence, by taking $d = m$ in Lemma 2.2 and by Lemma 5.4 we get the contradiction that T is an $L(n; p')$-weighted zero-sum subsequence of S.

Case 3. For every prime divisor q of n at least two terms of S are coprime to q, and there is a prime divisor p of n/p' such that at least three terms of S are coprime to p.

In this case, by Lemma 5.6 we get the contradiction that S is an $L(n; p')$-weighted zero-sum sequence. □

Lemma 6.3. *Let $n = p'p\,q$ where $p', p\,, q$ are distinct primes and $m = n/p$. Then we have $U(m) \subseteq f_{n,m}\big(L(n; p')\big)$.*

Proof. As p is coprime with m, by the Chinese remainder theorem we have an isomorphism $\psi : U(n) \to U(m) \times U(p)$. Let $b \in U(m)$. There exists $c \in U(p)$ such that $\left(\dfrac{c}{p}\right) = \left(\dfrac{b}{q}\right)$. Let $a \in U(n)$ such that $\psi(a) = (b, c)$. Then $a \in L(n; p')$ as

$$\left(\frac{a}{n}\right) = \left(\frac{b}{m}\right)\left(\frac{c}{p}\right) = \left(\frac{b}{p'q}\right)\left(\frac{b}{q}\right) = \left(\frac{b}{p'}\right) = \left(\frac{a}{p'}\right).$$

As $f_{n,m}(a) = b$, we get that $b \in f_{n,m}\big(L(n; p')\big)$. □

Theorem 6.4. *Let n be squarefree such that every prime divisor of n is at least seven. Let p' be a prime divisor of n and $m = n/p'$. Suppose $\Omega(n) = 3$ and S is a sequence in $\mathbb{Z}_n$. Then the sequence S is a D-extremal sequence for $L(n; p')$ if and only if either S is a D-extremal sequence for $U(n)$ or S is a permutation of a sequence (x_1, x_2, x_3) where the image of the sequence (x_2, x_3) under $f_{n,m}$ is a D-extremal sequence for $S(m)$ and x_1 satisfies one of the following conditions:*

- *The term x_1 is a non-zero multiple of m.*
- *The term x_1 is the only term of S which is coprime to p'.*

Proof. From Theorems 1.3 and 1.6 we have $D_{U(n)} = D_{L(n;\,p')}$. From Remark 6.1 we see that if S is a D-extremal sequence for $U(n)$, then S is a D-extremal sequence for $L(n; p')$. For any $a \in U(n)$ we have $(\frac{a}{n}) = (\frac{a}{m})(\frac{a}{p})$ and so $f_{n,\,m}\left(L(n; p')\right) \subseteq S(m)$. Let $S = (x_1, x_2, x_3)$ where the image of the sequence (x_2, x_3) under $f_{n,m}$ is a D-extremal sequence for $S(m)$.

Consider the case when the term x_1 is a non-zero multiple of m. Suppose T is an $L(n; p')$-weighted zero-sum subsequence of S. As $f_{n,\,m}\left(L(n; p')\right) \subseteq S(m)$ and $f_{n,m}(x_1) = 0$, we get the contradiction that the image of (x_2, x_3) under $f_{n,m}$ has an $S(m)$-weighted zero-sum subsequence.

Consider the case when the term x_1 is the only term of S which is coprime to p'. Suppose T is an $L(n; p')$-weighted zero-sum subsequence of S. Then we see that T cannot contain x_1. As $f_{n,\,m}\left(L(n; p')\right) \subseteq S(m)$, we get the contradiction that the image of (x_2, x_3) under $f_{n,m}$ has an $S(m)$-weighted zero-sum subsequence.

So we see that the sequences of the other two types are also D-extremal sequences for $L(n; p')$. Thus, we have shown that the reverse implication in the statement of Theorem 6.4 is true. We now proceed to prove the forward implication.

Suppose the sequence $S = (x_1, x_2, x_3)$ is a D-extremal sequence for $L(n; p')$. Clearly, all the terms of S must be non-zero. We have three cases to consider.

`Case 1.` There is a prime divisor p of n such that at most one term of S is coprime to p.

Suppose all the terms of S are divisible by p. We use a similar argument as in Case 1 of Theorem 6.2 to get the contradiction that S has an $L(n; p')$-weighted zero-sum subsequence. So in this case, exactly one term of S is not divisible by p. Let us assume that that term is x_1. Let $m = n/p$ and T' be the image of $T = (x_2, x_3)$ under $f_{n,m}$. By a similar argument as in Case 1 of Theorem 6.2, we see that T' is a D-extremal sequence for $S(m)$.

Suppose we have that $p \neq p'$. We claim that T' is a D-extremal sequence for $U(m)$. As $\Omega(m) = 2$, by Theorem 1.3 we have $D_{U(m)} = 3$. As T' has length two, it is enough to show that T' does not have any $U(m)$-weighted zero-sum subsequence. As n is squarefree and $\Omega(n) = 3$, by Lemma 6.3 we have $U(m) \subseteq f_{n,m}\big(L(n;p')\big)$. So if T' has a $U(m)$-weighted zero-sum subsequence, by Lemma 2.2 we get the contradiction that T (and hence S) has an $L(n;p')$-weighted zero-sum subsequence. Hence, it follows that our claim is true.

Thus, by Theorem 6 of [2] we see that S is a D-extremal sequence for $U(n)$ when $p \neq p'$.

`Case 2.` For every prime divisor q of n/p' exactly two terms of S are coprime to q, and at least two terms of S are coprime to p'.

Let $m = n/p'$ and $S' = (x'_1, x'_2, x'_3)$ be the image of S under $f_{n,m}$. Suppose at most one term of S' is a unit. By Lemma 5.6, we see that S is an $L(n;p')$-weighted zero-sum sequence. So we can assume that at least two terms of S' are units. By the assumption in this case, we see that exactly two terms of S are units and the other term is zero. Let us assume that $x'_1 = 0$ and the terms x'_2 and x'_3 are units. Thus, it follows that the term x_1 is a non-zero multiple of m.

If (x'_2, x'_3) has an $S(m)$-weighted zero-sum subsequence, then the sequence S' is an $S(m)$-weighted zero-sum sequence as $x'_1 = 0$. From [4, Lemma 2.1], we see that $S^{(p')}$ is a $U(p')$-weighted zero-sum sequence. Let $\psi : U(n) \to U(m) \times U(p')$ be the isomorphism given by the Chinese remainder theorem. From Lemma 5.5, we have $S(m) \times U(p') \subseteq \psi\big(L(n;p')\big)$. So from Observation 5.7 we get the contradiction that S is an $L(n;p')$-weighted zero-sum sequence.

Hence, the sequence (x'_2, x'_3) does not have any $S(m)$-weighted zero-sum subsequence. By Theorem 1.5 we have $D_{S(m)} = 3$. So it follows that the sequence (x'_2, x'_3) is a D-extremal sequence for $S(m)$.

`Case 3.` For every prime divisor q of n at least two terms of S are coprime to q, and there is a prime divisor p of n/p' such that at least three terms of S are coprime to p.

In this case, by Lemma 5.6 we get the contradiction that S is an $L(n;p')$-weighted zero-sum sequence. □

Theorem 6.5. *Let $n = p'q$ where p' and q are distinct primes which are at least seven. Suppose S is a sequence in $\mathbb{Z}_n$. Then S is a D-extremal sequence for $L(n;p')$ if and only if S is a permutation of a sequence (x_1, x_2, x_3) where the image of the sequence (x_2, x_3) under $f_{n,q}$ is a D-extremal sequence for Q_q and x_1 satisfies one of the following conditions:*

- *The term x_1 is a non-zero multiple of q.*

- *The term x_1 is the only term of S which is coprime to p'.*

We omit the proof of Theorem 6.5 to avoid making the paper lengthy.

7. C-Extremal Sequences for $L(n;p)$

Theorem 7.1. *Let n be a squarefree number such that every prime divisor of n is at least seven. Suppose p' is a prime divisor of n and $\Omega(n) \neq 2$. Then the C-extremal sequences for $L(n;p')$ are the same as the C-extremal sequences for $U(n)$.*

Proof. As $\Omega(n) \neq 2$, by Theorems 1.3 and 1.6 we have $C_{L(n;\,p')} = C_{U(n)}$. So by Remark 6.1 it is enough to show that every C-extremal sequence for $L(n;p')$ is a C-extremal sequence for $U(n)$.

Let S be a C-extremal sequence for $L(n;p')$. When n is a prime, then $n = p'$ and $L(n;p') = U(n)$. So S is a C-extremal sequence for $U(n)$. Thus, we may assume that $\Omega(n) \geq 3$. By Theorem 1.6 we have $C_{L(n;\,p')} = 2^{\Omega(n)}$. So $S = (x_1, \ldots, x_l)$ where $l = 2^{\Omega(n)} - 1$. Clearly, all the terms of S must be non-zero. We have three cases to consider.

`Case 1.` There is a prime divisor p of n such that at most one term of S is coprime to p.

Suppose the 'middle' term x_{k+1} is divisible by p where $k+1 = (l+1)/2$. Then we can find a subsequence T having consecutive terms of S of length $k+1$ such that all the terms of T are divisible by p. Let $m = n/p$ and T' be the image of T under $f_{n,m}$.

As $\Omega(m) = \Omega(n) - 1 \geq 2$ and T' has length $2^{\Omega(m)}$, by Theorem 1.5 we see that T' has an $S(m)$-weighted zero-sum subsequence of consecutive terms. By Lemma 5.3 we have that $S(m) \subseteq f_{n,m}\big(L(n;p')\big)$. So by Lemma 2.2 we get the contradiction that T (and hence S) has an $L(n;p')$-weighted zero-sum subsequence of consecutive terms.

Thus, we see that the term x_{k+1} is not divisible by p. Let $S_1 = (x_1, \ldots, x_k)$ and $S_2 = (x_{k+2}, \ldots, x_l)$. Let S_1' and S_2' be the images of S_1 and S_2 respectively under $f_{n,m}$. Suppose S_1' has an $S(m)$-weighted zero-sum subsequence of consecutive terms. As we have that $S(m) \subseteq f_{n,m}\big(L(n;p')\big)$, by Lemma 2.2 we get the contradiction that S_1 (and hence S) has an $L(n;p')$-weighted zero-sum subsequence of consecutive terms.

So the sequence S_1' does not have any $S(m)$-weighted zero-sum subsequence of consecutive terms. From Theorem 1.5 as $\Omega(m) \geq 2$ we have that $C_{S(m)} = 2^{\Omega(m)}$. As S_1' has length $k = 2^{\Omega(m)} - 1$, it follows that S_1' is a C-extremal sequence for $S(m)$. As $\Omega(m) \geq 2$, from Theorem 4.1 we see that S_1' is a C-extremal sequence for $U(m)$.

By a similar argument we see that S_2' is also a C-extremal sequence for $U(m)$. So by Theorem 5 of [6] it follows that S is a C-extremal sequence for $U(n)$.

Case 2. For every prime divisor q of n/p' exactly two terms of S are coprime to q, and at least two terms of S are coprime to p'.

We use a similar argument as the one given in the same case in the proof of Theorem 6.2. We just observe that as the sequence S has length at least seven, we can find a subsequence T having consecutive terms of S and having length at least two, which does not contain the terms x_{j_1} and x_{j_2}.

Case 3. For every prime divisor q of n at least two terms of S are coprime to q, and there is a prime divisor p of n/p' such that at least three terms of S are coprime to p.

In this case, by Lemma 5.6 we get the contradiction that S is an $L(n;p')$-weighted zero-sum sequence. □

Theorem 7.2. *Let $n = p'q$ where p' and q are distinct primes which are at least seven. Suppose $S = (x_1, x_2, x_3, x_4, x_5)$ is a sequence in $\mathbb{Z}_n$. Then S is a C-extremal sequence for $L(n;p')$ if and only if S has either of the following two forms:*

- *The terms x_1, x_3 and x_5 are non-zero multiples of q and the image of the sequence (x_2, x_4) under $f_{n,q}$ is a C-extremal sequence for Q_q.*
- *The term x_3 is the only term of S which is coprime to p' and the images of the sequences (x_1, x_2) and (x_4, x_5) under $f_{n,q}$ are C-extremal sequences for Q_q.*

We omit the proof of Theorem 7.2 to avoid making the paper lengthy.

8. Concluding Remarks

When $\Omega(n) = 2$, from Theorem 3.3 we see that there exist D-extremal sequences for $S(n)$ which are not D-extremal sequences for $U(n)$. When $\Omega(n) = 3$, from Theorem 6.4 we see that there exist D-extremal sequences for $L(n;p')$ which are not D-extremal sequences for $U(n)$.

Let $n = p'q$ where p' and q are distinct primes which are at least seven. From Theorems 1.3 and 1.7, we have $D_{U(n)} = 3$ and $D_{L(n;\,p')} = 4$, and so we cannot compare the D-extremal sequences for $U(n)$ with the D-extremal sequences for $L(n;p')$. For such an n, as $C_{U(n)} = 4$ and $C_{L(n;\,p')} = 6$, we cannot compare the C-extremal sequences for $U(n)$ with the C-extremal sequences for $L(n;p')$. The following questions can be investigated as well as their analogues for the constants $C_{S(n)}$ and $C_{U(n)}$.

- Can we determine the value of $D_{S(n)}$ and characterize the D-extremal sequences for $S(n)$ when n is a non-prime, squarefree number which is not coprime with thirty?
- If n is a non-squarefree number such that $D_{U(n)} = D_{S(n)}$, can we say that a D-extremal sequence for $S(n)$ is also a D-extremal sequence for $U(n)$?

Acknowledgements. Santanu Mondal would like to acknowledge CSIR, Govt. of India for a research fellowship whose file number is 09/934(0013)/2019-EMR-I. We thank the referee for taking the time to go through this paper and for the corrections and suggestions.

References

[1] S. D. Adhikari, C. David, and J. J. Urroz, Generalizations of some zero-sum theorems, *Integers* **8** (2008), #A52.

[2] S. D. Adhikari, I. Molla, and S. Paul, Extremal sequences for some weighted zero-sum constants for cyclic groups, *CANT IV, Springer Proc. Math. Stat.* **347** (2021), 1-10.

[3] S. D. Adhikari and P. Rath, Davenport constant with weights and some related questions, *Integers* **6** (2006), #A30.

[4] S. Griffiths, The Erdős-Ginzberg-Ziv theorem with units, *Discrete Math.* **308** (2008), 5473-5484.

[5] S. Mondal, K. Paul, and S. Paul, On a different weighted zero-sum constant, *Discrete Math.* **346** (2023), 113350.

[6] S. Mondal, K. Paul, and S. Paul, Extremal sequences for a weighted zero-sum constant, *Integers* **22** (2022), #A93.

[7] S. Mondal, K. Paul, and S. Paul, Zero-sum constants related to the Jacobi symbol, e-print is available at arxiv:2111.14477v3.

[8] P. Yuan and X. Zeng, Davenport constant with weights, *European J. Combin.* **31** (2010), 677-680.