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ON THE INTEGER SOLUTIONS OF THE DIOPHANTINE EQUATION $(x+y+z)^2 = xyw$

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Abstract

By using the theory of Pell equations, we prove that the Diophantine equation $(x+y+z)^2 = xyw$ has infinitely many integer solutions. Moreover, we show that every positive integer solution of this Diophantine equation can generate infinitely many different positive integer solutions by the following transformation:

$$(x,y,z,w) \to \begin{cases} \left(x, y, kxyw - 2x - 2y - z, k^2xyw^2 - 2kw(x+y+z) + w\right), \\ (x, wx - 2x - 2z - y, z, w), \\ (wy - 2y - 2z - x, y, z, w), \end{cases}$$

where $k > \frac{2}{x+y+z}$ and $k \in \mathbb{Q}^+$.

1. Introduction

In 1994, Guy [8] proposed the problem of determining which integers can be represented by

$$\frac{(x+y+z)^2}{xyz},$$

where x, y and z are positive integers. In 1997, Bremner [3] reported that all negative integers can be represented in this way by taking $(x,y,z) = (n, -n, n^3)$. In 1998, Brueggeman [4] made an interesting discovery. He constructed four binary

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trees from the root triples $(1,1,1)$, $(1,1,2)$, $(1,2,3)$ and $(1,4,5)$, such that the triple (x,y,z) gives rise to the triples $(x,z,(x+z)^2/y)$ and $(y,z,(y+z)^2/x)$. Each node (x,y,z) of these trees makes $(x+y+z)^2/xyz$ an integer. He proved that 1,2,3,4,5,6,8, and 9 are the only integers that are represented by $(x+y+z)^2/xyz$, where x, y and z are positive integers.

In 2002, Andreescu [1] investigated the Diophantine equation

$$(x+y+z)^2 = xyz \tag{1}$$

and found four different families of infinitely many positive integer solutions with z being 5,6,8, and 12. In 2013, Gopalan, Vidhyalakshmi and Kavitha [6] found finite integer solutions of Equation (1). Furthermore, in 2003, Andreescu [2] studied the positive integer solutions of the Diophantine equation $(x+y+z+t)^2 = xyzt$, but only found nine families of infinitely many positive integer solutions. In 2013, Gopalan, Vidhyalakshmi and Kavitha [7] indicated a method to generate six different infinite families of positive integral solutions of the Diophantine equation $(x+y+z+t)^2 = xyzt+1$. In 2018, Sadhasivam, Nagajothi and Vimala [10] investigated the integer solutions of the Diophantine equation

$$(x_1+x_2+x_3+\cdots+x_n)^2 = x_1x_2x_3\cdots x_n \tag{2}$$

while they only found finite families of infinitely many positive integer solutions with n being 5 and 6.

Duan and Li [5] investigated the solvability of the Diophantine equation

$$(x+y+z)^2 = axyz, a \in \mathbb{Z}, \tag{3}$$

and they found infinitely many integer solutions.

In this note, we are interested in the existence of the integer solutions of the Diophantine equation

$$(x+y+z)^2 = xyw. \tag{4}$$

We prove that Equation (4) has infinitely many integer solutions.

2. Preliminaries

To prove our results, we give the following lemmas.

Lemma 1 ([9]). *Let D be a positive integer which is not a perfect square, then the Pell equation $x^2 - Dy^2 = 1$ has infinitely many positive integer solutions. If (u,v) is the least positive integer solution of the Pell equation $x^2 - Dy^2 = 1$, then all positive integer solutions are given by*

$$x_k + y_k\sqrt{D} = (u+v\sqrt{D})^k,$$

where k is an arbitrary positive integer.

Lemma 2 ([9]). *Let D be a positive integer which is not a perfect square, N be a nonzero integer, and (u, v) is the least positive integer solution of $x^2 - Dy^2 = 1$. If (p, q) is a positive integer solution of $x^2 - Dy^2 = N$, then infinite positive integer solutions are given by*

$$x_k + y_k\sqrt{D} = (p + q\sqrt{D})(u + v\sqrt{D})^k,$$

where k is an arbitrary nonnegative integer.

3. The Main Results

Theorem 1. *For any given $w, z \in \mathbb{Z}$, if $w(w-4)$ is a positive integer, there are infinite families of infinitely many integer solutions of Equation (4).*

Remark 1. For any two integer solutions of Equation (4), we say they are in the same family if they have the same w and z.

Proof. For any given $w, z \in \mathbb{Z}$, solving Equation (4), we have

$$x = \frac{wy}{2} - y - z \pm \frac{\sqrt{w(w-4)y^2 - 4wyz}}{2}. \tag{5}$$

We will only discuss the case of minus sign. To find $x \in \mathbb{Z}$, we consider

$$w(w-4)y^2 - 4wzy = t^2,$$

then

$$(w(wy - 4y - 2z))^2 - w(w-4)t^2 = 4w^2z^2.$$

Let

$$X = w(wy - 4y - 2z) \quad \text{and} \quad Y = t = \pm(wy - 2x - 2y - 2z). \tag{6}$$

We obtain the Pell equation

$$X^2 - w(w-4)Y^2 = 4w^2z^2. \tag{7}$$

It is easy to show that $w(w-4)$ is not a perfect square. If $w(w-4)$ is a positive integer, by Lemma 1, the Pell equation $X^2 - w(w-4)Y^2 = 1$ has infinitely many positive integer solutions. Put (u, v) be the least positive integer solution of $X^2 - w(w-4)Y^2 = 1$.

According to $(x, y, z, w) = (-wz, -z, z, w)$ is a trivial solution of Equation (4), from Equation (6), we find that $(X, Y) = (-z(w-2)w, wz)$ is an integer solution of Equation (7). By Lemma 2, an infinitude of integer solutions of Equation (7) are given by

$$X_k + Y_k\sqrt{w(w-4)} = (-z(w-2)w + wz\sqrt{w(w-4)})(u + v\sqrt{w(w-4)})^k, \quad k \geq 0.$$

Thus

$$\begin{cases} X_{k+1} = 2uX_k - X_{k-1}, \\ Y_{k+1} = 2uY_k - Y_{k-1}, \end{cases}$$

where

$$\begin{aligned} X_0 &= -\, z(w-2)w, \\ X_1 &= -\, wz((w-2)u - vw(w-4)), \\ Y_0 &= wz, \\ Y_1 &= wz(u - v(w-2)). \end{aligned}$$

Using the recurrence relations of X_k and Y_k twice, we get

$$\begin{cases} X_{2k+2} = 2(2u^2-1)X_{2k} - X_{2k-2}, \\ Y_{2k+2} = 2(2u^2-1)Y_{2k} - Y_{2k-2}, \end{cases} \tag{8}$$

where

$$\begin{aligned} X_0 &= -\, z(w-2)w, \\ X_2 &= -\, wz((w-2)u^2 - 2vw(w-4)u + v^2w(w-2)(w-4)), \\ Y_0 &= wz, \\ Y_2 &= wz(u - v(w-4))(u - vw). \end{aligned}$$

From Equation (5) and Equation (6), we have

$$x = \frac{wy}{2} - y - z \pm \frac{Y}{2} \quad \text{and} \quad y = \frac{2wz + X}{w(w-4)}. \tag{9}$$

Substituting Equation (9) into Equation (8), we obtain

$$\begin{cases} x_{2k+2} = 2(2u^2-1)x_{2k} - x_{2k-2} - 8v^2wz, \\ y_{2k+2} = 2(2u^2-1)y_{2k} - y_{2k-2} - 8v^2wz, \end{cases} \tag{10}$$

where

$$\begin{aligned} x_0 &= -\, wz, \\ x_2 &= (2(w-2)uv - 2(w^2-4w+2)v^2 - 1)wz, \\ y_0 &= -\, z, \\ y_2 &= (2uvw - 2w(w-2)v^2 - 1)z. \end{aligned}$$

It follows that $x_{2k}, y_{2k} \in \mathbb{Z}$ for all $k \geq 0$.

Therefore, for any given $w, z \in \mathbb{Z}$, if $w(w-4)$ is a positive integer, there are infinite families of infinitely many integer solutions of Equation (4). $\square$

Example 1. When $w = -1, z = 1$, then Equation (7) becomes

$$X^2 - 5Y^2 = 4,$$

which has an integer solution $(X, Y) = (-3, -1)$. Note that $(u, v) = (9, 4)$ is the least positive integer solution of $X^2 - 5Y^2 = 1$. Therefore, from Equation (10), infinitely many integer solutions of Equation (4) are given by

$$\begin{cases} x_{2k+2} = 322x_{2k} - x_{2k-2} + 128, & x_0 = 1, \quad x_2 = 441, \\ y_{2k+2} = 322y_{2k} - y_{2k-2} + 128, & y_0 = -1, \quad y_2 = -169. \end{cases}$$

Theorem 2. *If Equation (4) has a positive integer solution (x_0, y_0, z_0, w_0), then it has infinitely many positive integer solutions with the same $z = z_0$ and $w = w_0$.*

Proof. Suppose that (x_0, y_0, z_0, w_0) is a positive integer solution of Equation (4). Without losing generality, we set $0 < y_0 \leq x_0$. Solving Equation (4) with $z = z_0$ and $w = w_0$, we have

$$x = \frac{w_0 y}{2} - y - z_0 \pm \frac{\sqrt{w_0(w_0 - 4)y^2 - 4w_0 z_0 y}}{2}. \tag{11}$$

We will only discuss the case of minus sign. It is necessary to take

$$w_0(w_0 - 4)y^2 - 4w_0 z_0 y = t^2.$$

By the transformation

$$X = w_0(w_0 y - 4y - 2z_0) \quad \text{and} \quad Y = t = \pm(w_0 y - 2x - 2y - 2z_0), \tag{12}$$

we get the Pell equation

$$X^2 - w_0(w_0 - 4)Y^2 = 4w_0^2 z_0^2. \tag{13}$$

From Equation (4), we have

$$w_0 = \frac{(x_0 + y_0 + z_0)^2}{x_0 y_0} > \frac{(x_0 + y_0)^2}{x_0 y_0} \geq 4.$$

Hence, $w_0(w_0 - 4)$ is a positive integer but not a perfect square. By Lemma 1, the Pell equation $X^2 - w_0(w_0 - 4)Y^2 = 1$ has infinitely many positive integer solutions. Put (u, v) be the least positive integer solution of $X^2 - w_0(w_0 - 4)Y^2 = 1$.

From the given positive integer solution (x_0, y_0, z_0, w_0), we find that

$$(X_0, Y_0) = (w_0(w_0 y_0 - 4y_0 - 2z_0), w_0 y_0 - 2x_0 - 2y_0 - 2z_0)$$

is an integer solution of Equation (13).

In order to obtain positive integer solutions of Equation (4), we need to ensure that X_0 and Y_0 are positive. Multiplying Y_0 by x_0, we have

$$x_0Y_0 = x_0y_0w_0 - 2x_0^2 - 2x_0y_0 - 2x_0z_0. \tag{14}$$

Substituting $(x_0+y_0+z_0)^2 = x_0y_0w_0$ into Equation (14), we get

$$x_0Y_0 = (y_0+z_0)^2 - x_0^2.$$

Note that

$$x_0 = \frac{w_0y_0}{2} - y_0 - z_0 - \frac{\sqrt{w_0(w_0-4)y_0^2 - 4w_0z_0y_0}}{2} \le \frac{w_0y_0}{2} - y_0 - z_0.$$

Multiplying x_0 by x_0, we get

$$x_0^2 \le \frac{x_0w_0y_0}{2} - x_0y_0 - x_0z_0. \tag{15}$$

Then substitute $(x_0+y_0+z_0)^2 = x_0y_0w_0$ into Equation (15), we obtain $x_0^2 \le (y_0+z_0)^2$. Therefore, $Y_0 \ge 0$ with $0 < x_0 \le y_0+z_0$. Moreover, it is easy to see that $X_0 = w_0Y_0 + 2w_0(x_0-y_0)$. Hence, X_0 is positive with $y_0 \le x_0$.

It is easy to provide infinitely many positive integer solutions of Equation (13) by the formula

$$X_k + \sqrt{w_0(w_0-4)}Y_k = (X_0 + \sqrt{w_0(w_0-4)}Y_0)(u + \sqrt{w_0(w_0-4)}v)^k, \ k \ge 0.$$

Then we have

$$\begin{cases} X_{k+1} = 2uX_k - X_{k-1}, \quad X_0 = X_0, \quad X_1 = uX_0 + w_0(w_0-4)vY_0, \\ Y_{k+1} = 2uY_k - Y_{k-1}, \quad Y_0 = Y_0, \quad Y_1 = vX_0 + uY_0. \end{cases}$$

Using the recurrence relations of X_k and Y_k twice, we get

$$\begin{cases} X_{2k+2} = 2(2u^2-1)X_{2k} - X_{2k-2}, \\ Y_{2k+2} = 2(2u^2-1)Y_{2k} - Y_{2k-2}, \end{cases} \tag{16}$$

where

$$\begin{aligned} X_0 &= X_0, \\ X_2 &= (2u^2-1)X_0 + 2w_0(w_0-4)uvY_0, \\ Y_0 &= Y_0, \\ Y_2 &= 2uvX_0 + (2u^2-1)Y_0. \end{aligned}$$

From Equation (6) and Equation (11), we have

$$x = \frac{w_0y}{2} - y - z_0 \pm \frac{Y}{2} \quad \text{and} \quad y = \frac{2w_0z_0 + X}{w_0(w_0-4)}. \tag{17}$$

Substituting Equation (17) into Equation (16), we obtain

$$\begin{cases} x_{2k+2} = 2(2u^2-1)x_{2k} - x_{2k-2} - 8v^2 w_0 z_0, \\ y_{2k+2} = 2(2u^2-1)y_{2k} - y_{2k-2} - 8v^2 w_0 z_0, \end{cases} \tag{18}$$

where

$$\begin{aligned} x_0 &= x_0, \\ x_2 &= 2w_0(w_0x_0 - 4x_0 - 2z_0)v^2 - 2v(w_0x_0 - 2x_0 - 2y_0 - 2z_0)u + x_0, \\ y_0 &= y_0, \\ y_2 &= 2w_0(w_0y_0 - 4y_0 - 2z_0)v^2 + 2u(w_0y_0 - 2x_0 - 2y_0 - 2z_0)v + y_0. \end{aligned}$$

It follows that $x_{2k}, y_{2k} \in \mathbb{Z}$ for all $k \geq 0$.

Since $X_{2k} > 0$ for all $k \geq 0$, from Equation (17), we have $y_{2k} > 0$ for all $k \geq 0$. Moreover, from Equation (6), we have

$$x_{2k} = \frac{(x_{2k} + y_{2k} + z_{2k})^2}{y_{2k}w_{2k}} > 0 \quad \text{for all } k \geq 0.$$

Therefore, if Equation (4) has a positive integer solution (x_0, y_0, z_0, w_0), then it has infinitely many positive integer solutions with the same $z = z_0$ and $w = w_0$. $\square$

Example 2. It is easy to check that $(x, y, z, w) = (1, 1, 1, 9)$ is a positive integer solution of Equation (4); then Equation (13) becomes $X^2 - 45Y^2 = 324$, which has an positive integer solution $(X_0, Y_0) = (27, 3)$. Note that $(u, v) = (161, 24)$ is the least positive integer solution of $X^2 - 45Y^2 = 1$. Therefore, from Equation (18), infinitely many positive integer solutions of Equation (4) are given by

$$\begin{cases} x_{2k+2} = 103682x_{2k} - x_{2k-2} - 41472, \quad x_0 = 1, \quad x_2 = 7921, \\ y_{2k+2} = 103682y_{2k} - y_{2k-2} - 41472, \quad y_0 = 1, \quad y_2 = 54289. \end{cases}$$

Duan and Li [5] investigated Equation (3) and they showed that every positive integer solution of Equation (3) can generate infinitely many different positive integer solutions of Equation (3) by the following transformation:

$$(x, y, z) \longrightarrow \begin{cases} (x, y, axy - 2x - 2y - z), \\ (x, axz - 2x - 2z - y, z), \\ (ayz - 2y - 2z - x, y, z). \end{cases}$$

We give infinitely many positive integer solutions of Equation (4) by similar transformation in Theorem 3.

Theorem 3. *Every positive integer solution of Equation (4) can generate infinitely many different positive integer solutions by the following transformation:*

$$(x, y, z, w) \longrightarrow \begin{cases} \left(x, y, kxyw - 2x - 2y - z, k^2xyw^2 - 2kw(x + y + z) + w\right), \\ (x, wx - 2x - 2z - y, z, w), \\ (wy - 2y - 2z - x, y, z, w), \end{cases}$$

where $k > \frac{2}{x+y+z}$ *and* $k \in \mathbb{Q}^+$.

Proof. Suppose that (x_0, y_0, z_0, w_0) is a positive integer solution of Equation (4). Without losing generality, we set $0 < y_0 \leq x_0$. Next, we need to show that the generated integer solutions are all positive and there are at least two solutions different from the previous. We just need to prove that the generated integer solutions are larger than the previous. For the first branch, we need to show that

$$kxyw - 2x - 2y - 2z > 0 \quad \text{and} \quad k^2xyw^2 - 2kw(x + y + z) > 0. \tag{19}$$

Substituting $(x + y + z)^2 = xyw$ into Equation (19), we have

$$(x + y + z)(k(x + y + z) - 2) > 0 \quad \text{and} \quad kw(x + y + z)(k(x + y + z) - 2) > 0.$$

Therefore, if $k > \frac{2}{x+y+z}$, then

$$kxyw - 2x - 2y - z > z \quad \text{and} \quad k^2xyw^2 - 2kw(x + y + z) + w > w.$$

For the other two branches, we show that

$$xw - 2(x + y + z) > 0 \quad \text{and} \quad yw - 2(x + y + z) \geq 0. \tag{20}$$

Multiplying Equation (20) by y and x, respectively, then

$$xyw - 2y(x + y + z) > 0 \quad \text{and} \quad xyw - 2x(x + y + z) \geq 0. \tag{21}$$

Substituting $(x + y + z)^2 = xyw$ into Equation (21), we have

$$(x + y + z)(x - y + z) > 0 \quad \text{and} \quad (x + y + z)(y + z - x) \geq 0.$$

Thus, $xw - 2(x + y + z) + y > y$ and $yw - 2(x + y + z) + x \geq x$.

That is to say, every positive integer solution of Equation (4) can generate infinitely many different positive integer solutions. □

Example 3. It is easy to check that $(x, y, z, w) = (1, 1, 1, 9)$ is a positive integer solution of Equation (4), then this solution can generate infinitely many different positive integer solutions:

$$(1,1,1,9) \begin{cases} \left(1,1,9k-5,81k^2-54k+9\right)\cdots \\ (1,4,1,9)\cdots \\ (4,1,1,9)\cdots \end{cases}$$

where $k > \frac{2}{3}$ and $k \in \mathbb{Q}^+$.

Remark 2. From Theorem 3, every positive integer solution of Equation (4) can generate a positive integer solution with different z and w, then from Theorem 2, it is easy to draw the conclusion that there are infinite families of infinitely many positive integer solutions of Equation (4).

Remark 3. `Case 1.` If $w = z$, then Equation (4) becomes Equation (1).

`Case 2.` If $w = az$, then Equation (4) becomes Equation (3).

`Case 3.` If $x = x_1, y = x_2, z = \sum_{i=3}^{n} x_i$ and $w = \prod_{i=3}^{n} x_i$, then Equation (4) becomes Equation (2). Every positive integer solution of Equation (2) can generate infinitely many different positive integer solutions by the following transformation:

$$(x_1, x_2, \cdots, x_n) -- \left(x_1, x_2, \cdots, x_{k-1}, \frac{\prod_{i=1}^{n} x_i}{x_k} - 2\left(\sum_{i=1}^{n} x_i\right) + x_k, x_{k+1}, \cdots, x_n\right),$$

where $k = 1, \cdots, n$.

Take $n = 4$ as an example, then Equation (2) becomes

$$(x_1 + x_2 + x_3 + x_4)^2 = x_1x_2x_3x_4. \tag{22}$$

Then, we need to show that every positive integer solution of Equation (22) can generate infinitely many different positive integer solutions by the following transformation:

$$(x_1,x_2,x_3,x_4) \begin{cases} \left(x_2x_3x_4 - x_1 - 2x_2 - 2x_3 - 2x_4, x_2, x_3, x_4\right), \\ \left(x_1, x_2, x_1x_2x_4 - 2x_1 - 2x_2 - x_3 - 2x_4, x_4\right), \\ \left(x_1, x_2, x_3, x_1x_2x_3 - 2x_1 - 2x_2 - 2x_3 - x_4\right), \\ \left(x_1, x_1x_3x_4 - 2x_1 - x_2 - 2x_3 - 2x_4, x_3, x_4\right). \end{cases}$$

Since x_1, x_2, x_3, and x_4 are symmetric, without losing generality, we set $0 < x_1 \leq x_2 \leq x_3 \leq x_4$. As an example, we prove $x_2x_3x_4 - x_1 - 2x_2 - 2x_3 - 2x_4$ is larger than x_1, i.e.,

$$x_2x_3x_4 - x_1 - 2x_2 - 2x_3 - 2x_4 - x_1 = x_2x_3x_4 - 2(x_1 + x_2 + x_3 + x_4) > 0. \quad (23)$$

Multiplying Equation (23) by x_1, we have

$$x_1(x_2x_3x_4 - x_1 - 2x_2 - 2x_3 - 2x_4 - x_1) = x_1x_2x_3x_4 - 2x_1(x_1 + x_2 + x_3 + x_4). \quad (24)$$

Substituting $x_1x_2x_3x_4 = (x_1 + x_2 + x_3 + x_4)^2$ into Equation (24), we get

$$x_1(x_2x_3x_4 - x_1 - 2x_2 - 2x_3 - 2x_4 - x_1) = (x_1 + x_2 + x_3 + x_4)(x_2 + x_3 + x_4 - x_1) > 0.$$

Thus, $x_2x_3x_4 - x_1 - 2x_2 - 2x_3 - 2x_4 > x_1$. Similarly, we can show that

$$x_1x_2x_4 - 2x_1 - 2x_2 - x_3 - 2x_4 > x_3 \quad \text{and} \quad x_1x_3x_4 - 2x_1 - x_2 - 2x_3 - 2x_4 > x_2.$$

Moreover, solving Equation (22), we get

$$x_4 = \frac{x_1x_2x_3 \pm \sqrt{x_1^2x_2^2x_3^2 - 4x_1^2x_2x_3 - 4x_1x_2^2x_3 - 4x_1x_2x_3^2}}{2} - x_1 - x_2 - x_3. \quad (25)$$

We will only consider the case in which the square root is subtracted.

Note that

$$x_4 \leq \frac{x_1x_2x_3}{2} - x_1 - x_2 - x_3.$$

Multiplying x_4 by x_4, we obtain

$$x_4^2 \leq \frac{x_1x_2x_3x_4}{2} - x_4(x_1 + x_2 + x_3). \quad (26)$$

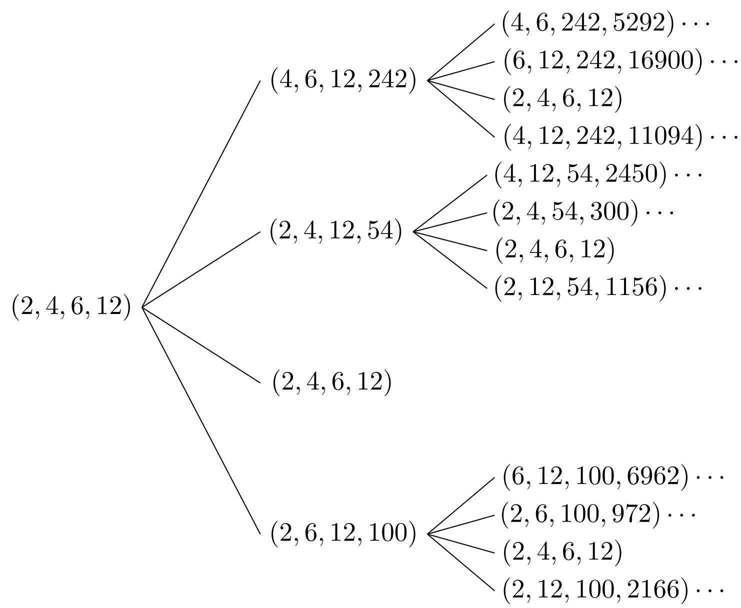
Replacing $x_1x_2x_3x_4$ by $(x_1 + x_2 + x_3 + x_4)^2$, we get

$$x_4^2 \leq (x_1 + x_2 + x_3)^2.$$

In consequence, it is easy to prove that $x_1x_2x_3 - 2x_1 - 2x_2 - 2x_3 - x_4 \geq x_4$.

That is to say, every positive integer solution of Equation (22) can generate infinitely many different positive integer solutions.

Example 4. It is easy to verify that $(2, 4, 6, 12)$ is a positive integer solution of Equation (22), then this solution can generate infinitely many different positive integer solutions:



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