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TWO PAIRS OF BIQUADRATES WITH EQUAL SUMS

Ajai Choudhry
13/4 A Clay Square, Lucknow, India
ajaic203@yahoo.com

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Abstract

In this paper we present a new method of obtaining parametric solutions of the classical diophantine equation $A^4 + B^4 = C^4 + D^4$ whose complete solution is still not known. Two methods of solving the equation, given by Euler, yield parametric solutions given by polynomials of degrees 7 and 13. Several other parametric solutions are now known, and with the exception of one solution of degree 11, all the published solutions are of degrees $6n + 1$ for some integer n. The method described in this paper yields new parametric solutions of degrees 21, 39 and 75, that is, degrees that are expressible as $6n + 3$.

1. Introduction

This paper is concerned with the classical diophantine equation,

$$A^4 + B^4 = C^4 + D^4, \tag{1}$$

for which integer solutions were first found by Euler in 1772. In fact, Euler gave two ways of solving Equation (1) leading to parametric solutions given by polynomials of degrees 7 and 13, respectively (see [5, p. 1062]). Dickson [3, pp. 644–647] mentions various methods, found subsequently by several mathematicians, of solving the diophantine Equation (1). Lander [5, pp. 1062–1065] applied geometric methods to obtain parametric solutions of (1). Zajta [7] carried out a survey of the important methods of solving Equation (1) and obtained a new parametric solution given by polynomials of degree 11. Parametric solutions of degrees 13, 19, 25 and 31 have been published by various authors ([1], [2], [5], [7]). The complete solution of Equation (1) is, however, still not known.

Guy [4, p. 212–213] states that "a method is known for generating parametric solutions of $a^4 + b^4 = c^4 + d^4$ which will generate all published solutions from the trivial one $(\lambda, 1, \lambda, 1)$; it will only generate solutions of degree $6n + 1$." He also

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mentions that there exist solutions of even degree but until now no solution of even degree has been found.

Since the existing methods of solving Equation (1) generate only solutions of odd degrees, and solutions of even degree are known to exist, there is interest in devising new ways of solving Equation (1) that may yield solutions of even degrees. Accordingly, we tried to find solutions of (1) by a new method which is presented below in Section 2. While the desired solution of even degree could not be found, the method yielded solutions of degrees 21, 39 and 75, all of these degrees being expressible as $6n+3$. As none of the known solutions are of degree $6n+3$, the new solutions obtained in this paper are interesting although solutions of even degrees remain elusive.

2. A New Method of Finding Parametric Solutions of Equation (1)

To solve Equation (1), we write

$$\begin{aligned} A &= a_0x^2 + a_1x + a_2, \quad B = b_0x^2 + b_1x + b_2, \\ C &= a_0x^2 - a_1x + a_2, \quad D = b_0x^2 - b_1x + b_2, \end{aligned} \tag{2}$$

where $a_i, b_i, i = 0, 1, 2$, and x are arbitrary parameters. With these values, Equation (1) reduces, on transposing all terms to the left-hand side and removing the common factor $8x$, to

$$\begin{aligned} &(a_0^3a_1 + b_0^3b_1)x^6 + (3a_0^2a_1a_2 + a_0a_1^3 + 3b_0^2b_1b_2 + b_0b_1^3)x^4 \\ &\quad + (3a_0a_1a_2^2 + a_1^3a_2 + 3b_0b_1b_2^2 + b_1^3b_2)x^2 + a_1a_2^3 + b_1b_2^3 = 0. \end{aligned} \tag{3}$$

We will now choose the parameters $a_i, b_i, i = 0, 1, 2$, such that the coefficients of x^6 and x^4 in Equation (3) become 0. Accordingly, we take

$$a_1 = b_0^3, \quad b_1 = -a_0^3, \quad a_2 = (a_0^8 - b_0^8)u/(3a_0b_0^2), \quad b_2 = (a_0^8 - b_0^8)(u-1)/(3a_0^2b_0), \tag{4}$$

where u is an arbitrary parameter, and now Equation (3) reduces to

$$9a_0^2b_0^2((a_0^8 - b_0^8)u + b_0^8)x^2 + (a_0^8 - b_0^8)^2(3u^2 - 3u + 1) = 0. \tag{5}$$

We now write, without any loss of generality,

$$a_0 = b_0t, \quad x = vb_0^2(t^8 - 1)/(3t((t^8 - 1)u + 1)), \tag{6}$$

when Equation (5) reduces to

$$v^2 = (3u^2 - 3u + 1)((1 - t^8)u - 1). \tag{7}$$

The birational transformation defined by

$$\begin{aligned} v &= Y/(3(t^8-1)), \qquad & u &= -(X+3)/(3(t^8-1)), \\ X &= 3(-t^8+1)u-3, \qquad & Y &= 3v(t^8-1), \end{aligned} \tag{8}$$

reduces Equation (7) to

$$Y^2 = X(X^2+3(t^8+1)X+3t^{16}+3t^8+3). \tag{9}$$

Now Equation (9) may be considered as the Weierstrass model of an elliptic curve over the function field $\mathbb{Q}(t)$. It was found by trial that a rational point P on the elliptic curve (9) is given by

$$P = ((t^4-t^2+1)(t^8-t^4+1)/t^2, (t^{18}+t^{12}+t^6+1)/t^3). \tag{10}$$

For integer values of $t > 1$, the elliptic curve (9) is in Weierstrass form with integer coefficients while the point P has rational coordinates. It, therefore, follows from the Nagell-Lutz theorem [6, p. 56] that in these cases P is not a point of finite order. Hence, for arbitrary rational values of t also, the point P cannot be of finite order. Thus, we can find infinitely many rational points on the elliptic curve (9) using the group law. In fact, the coordinates of the point $2P$ are readily found and are given by

$$((t^6-2t^4-2t^2+1)^2/(4t^2), (t^{18}-17t^{12}-17t^6+1)/(8t^3)).$$

If (X, Y) are the coordinates of any rational point on the curve (9), we find, on using the relations (2), (4), (6) and (8), that a solution of the diophantine Equation (1) may be written as follows:

$$\begin{aligned} A &= t^9X - tY + (t^{16}+t^8+1)t, \quad & B &= X + t^4Y + t^{16}+t^8+1, \\ C &= t^9X + tY + (t^{16}+t^8+1)t, \quad & D &= X - t^4Y + t^{16}+t^8+1. \end{aligned} \tag{11}$$

While the point P yields a trivial solution of the diophantine Equation (1), the point $2P$ gives a nontrivial solution, which on writing $t = p/q$ and clearing denominators, may be expressed in terms of homogeneous polynomials of degree 21 in arbitrary parameters p and q as follows:

$$A = f(p,q), B = f(q,-p), C = f(p,-q), D = f(q,p), \tag{12}$$

where

$$\begin{aligned} f(m,n) = (m-n)(m^2+mn+n^2)(2m^{18}+3m^{15}n^3 \\ +23m^{12}n^6+6m^9n^9+8m^6n^{12}-9m^3n^{15}-n^{18}). \end{aligned} \tag{13}$$

As a numerical example, taking $p = 2, q = 1$, yields the following solution of Equation (1):

$$A = 5042177, B = 575226, C = 4659327, D = 3638026.$$

The points $3P$ and $4P$ yield solutions of degrees 39 and 75, respectively. Since these solutions are cumbersome to write, we do not give them explicitly. All the computations to find the solutions were performed by me on the software MAPLE. The referee carried out the computations much further till the point $10P$, and experimentally found a pattern that when n is even, the point nP yields a parametric solution of degree $9n^2/2+3$, while for odd values of n, the point nP yields a parametric solution of degree $(9n^2-3)/2$. It may be interesting for the reader to prove that this is, in fact, true for all positive integer values of n.

3. Concluding Remarks

In an effort to find a solution of even degree of the diophantine Equation (1), we explored a new method of attacking the problem, and obtained new parametric solutions of degrees 21, 39 and 75. However, finding a solution of even degree remains an open problem.

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