A CANONICAL COLORING THEOREM FOR PIECEWISE SYNDETIC SUBSETS OF $\mathbb{N}$

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Abstract

In this short note, we first give a simple example (Theorem 1 below) of a coloring f of $\mathbb{N}$ (which uses infinitely many colors) for which no piecewise syndetic subset S of $\mathbb{N}$ exists such that f restricted to S is either constant or 1-1. We then show (Theorem 2) that for every coloring f of $\mathbb{N}$ (using finitely many or infinitely many colors) there exists a piecewise syndetic subset S of $\mathbb{N}$ on which f is either constant or is *finite-to-one*, which means that for all $y \in \mathbb{N}$, we have $|\{x \in S : f(x) = y\}| < \infty$.

–Dedicated to the memory of Allen R. Freedman

1. Introduction

We write $\mathbb{N}$ for the set of positive integers $\{1, 2, 3, \dots\}$. The "canonical van der Waerden theorem" states that for any coloring of $\mathbb{N}$ with finitely many or infinitely many colors, there exists, for every $k \in \mathbb{N}$, a k-term arithmetic progression on which the given coloring is either constant or 1-1. It is in this sense (the use of either finitely many or infinitely many colors) that we use the word "canonical" in the title of this paper. (The canonical van der Waerden theorem is first stated in [4, p. 17] as a consequence of Szemerédi's theorem [14]. For an elementary proof, see [6] or [13].)

We write $[0, d]$ for the interval $\{0, 1, 2, \dots, d\}$. Let S be an infinite subset of $\mathbb{N}$. If there exists $d \in \mathbb{N}$ such that $S + [0, d]$ contains an infinite interval, then S is *syndetic*; if $S + [0, d]$ contains arbitrarily large finite intervals of $\mathbb{N}$, then S is *piecewise syndetic*. In both cases, the minimum such d is the *gap size* associated with S. (As usual, $S + [0, d]$ denotes the set $\{s + x : s \in S, x \in [0, d]\}$.) Of course, $\mathbb{N}$ itself is syndetic, and hence piecewise syndetic, with $d = 1$. It is an elementary fact ([1], [2], [5], [7]-[13]) that for every finite coloring of $\mathbb{N}$ there is a monochromatic

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piecewise syndetic set. In fact, for any finite coloring of any piecewise syndetic subset X of $\mathbb{N}$, there is a monochromatic piecewise syndetic set. We use this, stated as Lemma 1, in the proof of Theorem 2 below. (It is also an elementary fact (see for example [13]) that every piecewise syndetic set S contains arbitrarily long (finite) arithmetic progressions.)

In the short paper [3] is a complicated coloring f of $\mathbb{N}$ (each color class is a translate of a single fixed infinite set) such that for every piecewise syndetic subset S of $\mathbb{N}$, f restricted to S is neither constant nor 1-1. Here is a simple coloring of $\mathbb{N}$ (each color class is a finite interval) which has this property.

Theorem 1. *Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence in $\mathbb{N}$ such that $\lim_{n\to\infty}(a_{n+1} - a_n) = \infty$ and define $f : \mathbb{N} \to \mathbb{N}$ by $f(x) = \min\{n \in \mathbb{N} : x < a_n\}$. Then there is no piecewise syndetic set S on which f is either constant or 1-1.*

Proof. Since each color is used only finitely many times, and a piecewise syndetic set S is infinite, f is not constant on S. Since the coloring is constant on the intervals $[a_n, a_{n+1})$, which become arbitrarily large, then no matter how large the "gap size" d associated with S is, the piecewise syndetic set S will intersect some intervals $[a_n, a_{n+1})$ more than once, so the coloring is not 1-1 on S. □

Remark. For the reader who is familiar with the relevant notions, it can be observed that Theorem 1 provides a very simple proof that there are no selective ultrafilters in the closure of the smallest ideal of $(\beta\mathbb{N}, +)$. (In fact, by [9, Corollary 8.37], there are no P-points in that closure, but that has a much more complicated proof.)

2. The Main Theorem

Lemma 1. *Let X be any piecewise syndetic subset of $\mathbb{N}$. For every finite coloring of X, there is a monochromatic piecewise syndetic set.*

(References for Lemma 1 are given in the Introduction. The simplest proof is in [9, p. 332], where it is observed that if A, B are subsets of $\mathbb{N}$ which are not piecewise syndetic then $A \cup B$ is also not piecewise syndetic.)

Definition 1. Let $f : \mathbb{N} \to \mathbb{N}$ and let $S \subseteq \mathbb{N}$. Then f is *finite-to-one* on S if for all $y \in \mathbb{N}$,

$$|\{x \in S : f(x) = y\}| < \infty.$$

Theorem 2. *Let $f : \mathbb{N} \to \mathbb{N}$ be arbitrary. Then there exists a piecewise syndetic subset S of $\mathbb{N}$ such that either f is constant on S or f is finite-to-one on S.*

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be given. Let us assume throughout the proof that f is not constant on any piecewise syndetic set. We now inductively construct pairwise disjoint intervals $B_0, B_1, \ldots, B_n, \ldots$ which have the following three properties:

1. $|B_i| = 10^i, 0 \leq i$.
2. $\max B_i < \min B_j, 0 \leq i < j$.
3. $f(B_i) \cap f(B_j) = \emptyset, 0 \leq i < j$.

We set $B_0 = \{1\}$. To choose B_1, consider the infinite sequence of intervals, each of size 10, given by

$$\begin{aligned} I_1 &= [2, 3, 4, \ldots, 11], \\ I_2 &= [3, 4, 5, \ldots, 12], \\ I_3 &= [4, 5, 6, \ldots, 13], \\ &\vdots \\ I_j &= [j+1, j+2, j+3, \ldots, j+10], \\ &\vdots . \end{aligned}$$

If there is $x_j \in I_j$ with $f(x_j) = f(1)$ for all $j \geq 1$, then the set $X = \{x_1, x_2, x_3, \ldots\}$ is a monochromatic syndetic (hence piecewise syndetic) set with gap size at most 10, contrary to our assumption on f. Hence there is an interval $B_1 = [a+1, a+2, \ldots, a+10]$ such that $f(1) \notin f(B_1)$.

These choices for B_0 and B_1 begin our induction. We have $|B_0| = 10^0, |B_1| = 10^1$, $\max B_0 < \min B_1$, and $f(B_0) \cap f(B_1) = \emptyset$.

Now suppose $n \geq 1$ and that $B_0, B_1, \ldots, B_n$ are intervals of $\mathbb{N}$ which satisfy the above properties 1 - 3. Let $C = f(B_0 \cup B_1 \cup \cdots \cup B_n)$, the set of all the colors assigned by f to the elements of $B_0 \cup B_1 \cup \cdots \cup B_n$. Consider the infinite sequence $J_1, J_2, J_3, \ldots$ of successive intervals, each of size 10^{n+1}, which immediately follow the last element, $q = \max B_n$, of B_n:

$$\begin{aligned} J_1 &= [q+1, q+10^{n+1}], \\ J_2 &= [q+2, q+1+10^{n+1}], \\ J_3 &= [q+3, q+2+10^{n+1}], \\ &\vdots \\ J_j &= [q+j, q+j-1+10^{n+1}], \\ &\vdots . \end{aligned}$$

Suppose that $C \cap f(J_j) \neq \emptyset$ for all $j \geq 1$, say $x_j \in J_j$ and $f(x_j) \in C,\ j \geq 1$. Then the set $X = \{x_1, x_2, x_3, \ldots\}$ is a piecewise syndetic (hence syndetic) set with gap size at most 10^{n+1}, and the coloring f restricted to X maps X to the finite set C. By Lemma 1 there is an f-monochromatic piecewise syndetic set, contrary to our initial assumption about the coloring f. Thus for some $j \geq 1$ we have $C_n \cap I_j = \emptyset$, and we take $B_{n+1} = I_j$. We now have that $B_1, B_2, B_3, \ldots, B_{n+1}$ satisfy the three properties listed at the beginning of the proof.

This finishes the inductive construction of the sets $B_0, B_1, B_2 \ldots,$ and we set $S = B_0 \cup B_1 \cup B_2 \cup \cdots$. The set S is certainly piecewise syndetic (the associated gap size is $d = 1$). Finally, property (3) of the sets $B_i,\ i \geq 1$, guarantees that any color that appears in B_n occurs at most 10^n times in S altogether, thus f restricted to S is finite-to-one. □

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