# $p$-ADIC VALUATION OF $\prod_{k=m+1}^{n}(k^2 - m^2)$

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**Abstract**

This paper presents the $p$-adic valuation of the sequence $C_n(m) = \prod_{k=m+1}^{n}(k^2 - m^2)$, $n = m+1, m+2, \cdots$ . An explicit formula is derived for the $p$-adic valuation of $C_n(m)$. From an applicational perspective, this study proves that $C_n(m)$ is not a square when $m = 2, 3$. Additionally, this paper provides a criterion for $C_n(m)$ being a powerful number when $n \geq 3m$.

## 1. Introduction

Let $f(x) \in \mathbb{Z}[x]$. Squares and powerful numbers in the sequence $C_n(f) = \prod_{k=1}^{n} f(k)$ have already attracted significant attention of several researchers.

In 2008, Amdeberhan, Medina, and Moll [1] presented a conjecture that $C_n(f)$ is not a square when $f(x) = x^2 + 1$ and $n > 3$. After the conjecture was proven by Cilleruelo [3], the problems related to the squares in $C_n(f)$ have been studied by numerous mathematicians.

For $f(x) = x^2 + b$ with $b \in \mathbb{Z}$, Ho [10], Hong and Liu [11], Gürel [8], and Zhang and Niu [16] studied the cases $b = 1, b = -1, b = m^2 - 1$, and $b = m^2$, respectively. Fang [7] proved that $C_n(f)$ is not a square when $f(x) = 4x^2 + 1$ and $2x^2 - 2x + 1$. Yang, Togbé, and He [14] studied the number of squares in the sequence $C_n(f)$, where $f(x) = ax^2 + b$ with $(a, b) = 1, 1 \leq a \leq 10$, and $1 \leq b \leq 20$.

Additionally, the squares in $C_n(f)$ when the degree of $f(x)$ is greater than 2 have been investigated. For $f(x) = x^3 + 1$, Gürel and Kisisel [9] proved that $C_n(f)$ is

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not a square. Generally, for irreducible $f(x)$, the number of squares in the sequence $C_n(f)$ was estimated by Cilleruelo, Luca, Quirós, and Shparlinski [4].

For $f(x) = x^d + 1$, Zhang and Wang [18], Chen and Gong [6], and Chen, Gong, and Ren [5] discussed the cases where $d$ is a prime number greater than 5, an odd prime, and an odd number, respectively, to study powerful numbers in $C_n(f)$. When $f(x) = x^d + q^d$, Niu and Liu [12] and Yang and Zhao [15] investigated the cases where $d = 3$ and $d$ is an odd integer, respectively.

Now, for $C_n(m) = \prod_{k=m+1}^{n}(k^2 - m^2)$, we discuss the $p$-adic valuations of $C_n(m)$ in this study and extend results for $C_n(1)$ by Hong and Liu [11].

For a prime $p$, let $v_p(n) = e$ if $p^e \mid n$ and $p^{e+1} \nmid n$. Let $\sum_{i=0}^{l} n_i p^i$ be the $p$-adic expansion of $n$. Then, we write $s_p(n) = \sum_{i=0}^{l} n_i$.

For positive integers $x$ and $y$, with $x \geq y$, we can define a constant $\delta_p(x, y)$ as follows. Let

$$x = a_0 + a_1 p + a_2 p^2 + \cdots + a_l p^l \tag{1}$$

and

$$y = b_0 + b_1 p + b_2 p^2 + \cdots + b_k p^k \tag{2}$$

be the $p$-adic expansions of $x$ and $y$, respectively. For convenience, let $b_i = 0$ if $i > k$. Define

$$I = \{i \mid a_i - b_i < 0,\ i = 0, 1, \cdots, l\} \cup \{i_0\} = \{i_0, i_1, i_2, \cdots, i_t\}, \tag{3}$$

$$J = \{j \mid a_j - b_j > 0,\ j = 0, 1, \cdots, l\} \cup \{j_0\} = \{j_0, j_1, j_2, \cdots, j_s\}, \tag{4}$$

where $i_0 = j_0 = -1$, $i_1 < i_2 < \cdots < i_t$, and $j_1 < j_2 < \cdots < j_s$. Then $x - y$ can be written as

$$x - y = \sum_{r=1}^{t}(a_{i_r} - b_{i_r})p^{i_r} + \sum_{r=1}^{s}(a_{j_r} - b_{j_r})p^{j_r}.$$

Let

$$I' = \{i_k \in I \mid i_{k-1} \leq j < i_k \text{ for some } j \in J\} = \{\alpha_1, \cdots, \alpha_{h_1}\} \tag{5}$$

and

$$J' = \{j_k \in J \mid j_{k-1} < i < j_k \text{ for some } i \in I\} = \{\beta_1, \cdots, \beta_{h_2}\}, \tag{6}$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_{h_1}$ and $\beta_1 < \beta_2 < \cdots < \beta_{h_2}$.

**Definition 1.** For positive integers $x \geq y$ and with the same notation as in Equations (5) and (6), we define

$$\delta_p(x, y) = \sum_{j \in J'} j - \sum_{i \in I'} i.$$

In this paper, we show the following results.

**Theorem 1.** *Let $x, y$ be positive integers with $x \geq y$. We have*

$$s_p(x-y) + s_p(y) - s_p(x) = (p-1)\delta_p(x,y).$$

By Theorem 1, the $p$-adic valuation of the binomial coefficient $\binom{x}{y}$ is just $\delta_p(x,y)$.

**Corollary 1.** *Let $x, y$ be positive integers with $x \geq y$. We have*

$$v_p(\tbinom{x}{y}) = \delta_p(x,y).$$

By Theorem 1, a formula of $v_p(C_n(m))$ can be formulated, and the asymptotic behavior of $v_p(C_n(m))$ can be studied.

**Theorem 2.** *Let $m, n$ be positive integers with $n \geq m+1$. We have*

$$v_p(C_n(m)) = \frac{2}{p-1}(n-m-s_p(n-m)) + \delta_p(n+m, 2m).$$

**Corollary 2.** *For any prime $p$, we have $v_p(C_n(m)) \sim \frac{2n}{p-1}$ as $n \to \infty$.*

Hence, we deduce the following corollary.

**Corollary 3.** *Let $m, n$ be positive integers with $n \geq m+1$. Then $C_n(m)$ is a square if and only if $\delta_p(n+m, 2m)$ is even for any prime $p$.*

From the applicational perspective of Theorem 2, when $m = 2$ or $m = 3$, we can show that a prime $p$ always exists such that $v_p(C_n(m))$ is odd. Consequently, we can deduce the following theorems.

**Theorem 3.** *When $m = 2$, we have that $C_n(2)$ is not a square if $n \geq 3$.*

**Theorem 4.** *When $m = 3$, we have that $C_n(3)$ is not a square if $n \geq 4$.*

The paper is organized as follows. In Section 2, we study the properties of $\delta_p(x,y)$ and prove Theorems 1 and 2. A comparative study of our formula for $v_p(C_n(1))$ relative to that of [11] is presented in Section 3. In Sections 4 and 5, we restrict our attention to $m = 2$ and $m = 3$ and prove Theorems 3 and 4, respectively. Finally, a criterion for $C_n(m)$ being a powerful number when $n \geq 3m$ is discussed in Section 6.

## 2. Proofs of Theorems 1 and 2

Recall the definition of $\delta_p(x,y)$. The following two examples contribute to our understanding of $I', J', \delta_p(x,y)$, and Theorem 2.

**Example 1.** Let $p = 17$,

$$x = 2 + 17 + 4 \cdot 17^2 + 2 \cdot 17^3 + 3 \cdot 17^4,$$

and

$$y = 1 + 2 \cdot 17 + 17^2 + 2 \cdot 17^4.$$

Thus,

$$x - y = 1 + (-1) \cdot 17 + 3 \cdot 17^2 + 2 \cdot 17^3 + 17^4.$$

We have $I = \{-1, 1\}$, $J = \{-1, 0, 2, 3, 4\}$, $I' = \{1\}$, and $J' = \{2\}$, which results in $\delta_{17}(x, y) = 1$. Let $x = n + m$ and $y = 2m$. It follows that $n = 177881$, $m = 83683$, and

$$n - m = 94198 = 1 + 16 \cdot 17 + 2 \cdot 17^2 + 2 \cdot 17^3 + 17^4.$$

By Theorem 2, we have $v_{17}(C_n(m)) = 11773$.

**Example 2.** Let $p = 7$,

$$x = 1 + 3 \cdot 7 + 2 \cdot 7^2 + 6 \cdot 7^3 + 4 \cdot 7^4 + 4 \cdot 7^5 + 7^6 + 2 \cdot 7^7 + 4 \cdot 7^8 + 6 \cdot 7^9 + 7^{10},$$

and

$$y = 3 + 7 + 4 \cdot 7^2 + 3 \cdot 7^3 + 2 \cdot 7^4 + 6 \cdot 7^5 + 2 \cdot 7^6 + 4 \cdot 7^7 + 4 \cdot 7^8 + 3 \cdot 7^9.$$

Thus

$$\begin{aligned} x - y = & -2 + 2 \cdot 7 + (-2) \cdot 7^2 + 3 \cdot 7^3 + 2 \cdot 7^4 + (-2) \cdot 7^5 \\ & + (-1) \cdot 7^6 + (-2) \cdot 7^7 + 3 \cdot 7^9 + 7^{10}. \end{aligned}$$

We have $I = \{-1, 0, 2, 5, 6, 7\}$, $J = \{-1, 1, 3, 4, 9\}$, $I' = \{0, 2, 5\}$, and $J' = \{1, 3, 9\}$, which results in $\delta_7(x, y) = 6$. Let $x = n + m$ and $y = 2m$. It follows that $n = 475621653$, $m = 73878187$, and

$$\begin{aligned} n - m &= 401743466 \\ &= 5 + 7 + 5 \cdot 7^2 + 2 \cdot 7^3 + 2 \cdot 7^4 + 5 \cdot 7^5 \\ &\quad + 5 \cdot 7^6 + 4 \cdot 7^7 + 6 \cdot 7^8 + 2 \cdot 7^9 + 7^{10}. \end{aligned}$$

By Theorem 2, we have $v_7(C_n(m)) = 133914476$.

**Lemma 1.** *Considering the same notation as was used in Equations (3), (4), (5), and (6), we have that $\alpha_1 = i_1$, that $\alpha_k$ is the smallest element in $I$ and is greater than $\beta_{k-1}$ for $k \geq 2$, and that $\beta_k$ is the smallest element in $J$ and is greater than $\alpha_k$ for $k \geq 1$. As an immediate consequence, we have $h_1 = h_2$.*

*Proof.* Since $i_1 > i_0$ and $i_0 = j_0 < i_1$, we have $i_1 \in I'$. Hence $\alpha_1 = i_1$.

If $\beta_1 = j_{s_1}$, we claim that $i_1 \in (j_{s_1-1}, j_{s_1})$. Otherwise, we have $i_1 < j_{s_1-1}$. It follows that $s_1 > 1$. Therefore, there exists $j_{s_1'} \in J$ such that $j_{s_1'} \le j_{s_1-1} < j_{s_1}$ and $j_{s_1'-1} < i_1 < j_{s_1'}$. Thus $j_{s_1'} \in J'$, which contradicts $j_{s_1} = \beta_1$. Hence $j_{s_1}$ is the smallest element in $J$ and is greater than $i_1$.

If $\alpha_2 = i_{t_2} > i_1$, we claim that $\beta_1 \in (i_{t_2-1}, i_{t_2})$. Otherwise, there exists $i_{t_2'} \in I$ such that $i_{t_2'} \le i_{t_2-1} < i_{t_2}$ and $i_{t_2'-1} < \beta_1 < i_{t_2'}$. It follows that $i_{t_2'} \in I'$ and $i_{t_2'} > i_1$, which contradicts $i_{t_2} = \alpha_2$. Thus, from $\beta_1 \in (i_{t_2-1}, i_{t_2})$, we have $i_{t_2}$ is the smallest element in $I$ that is greater than $\beta_1$.

Therefore, the lemma follows by induction. □

**Definition 2.** For a positive integer $n$ and a prime $p$, if

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_l p^l, \tag{7}$$

where $0 < n_l \le p-1$ and $1-p \le n_i \le p-1$ for $i = 0, 1, \cdots, l-1$, we can define

$$(n)_\Gamma = \sum_{i \in \Gamma \cap \mathbb{Z}} n_i p^i,$$

where $\Gamma \subseteq \mathbb{R}$ is an interval.

**Lemma 2.** *Considering the same notation as was used in Definition 2, we have*

$$s_p(n) = s_p((n)_{[0,t]}) + s_p((n)_{(t,l]}),$$

*where* $t = \min\{i \mid n_i > 0,\ i = 0, 1, \cdots, l\}$.

*Proof.* Since $n_t, n_l > 0$, it follows that $(n)_{[0,t]}$ and $(n)_{(t,l]}$ are both positive. Then

$$(n)_{[0,t]} = n_0' + n_1' p + \cdots + n_t' p^t \tag{8}$$

and

$$(n)_{(t,l]} = n_{t+1}' p^{t+1} + n_{t+2}' p^{t+2} + \cdots + n_l' p^l, \tag{9}$$

where $0 \le n_i' \le p-1$ for $i = 0, 1, \cdots, l$. Thus

$$s_p((n)_{[0,t]}) = n_0' + n_1' + \cdots + n_t',$$

and

$$s_p((n)_{(t,l]}) = n_{t+1}' + n_{t+2}' + \cdots + n_l'.$$

From Equations (8) and (9), we have

$$n = (n)_{[0,t]} + (n)_{(t,l]} = n_0' + n_1' p + \cdots + n_l' p^l.$$

It follows that

$$s_p(n) = n'_0 + n'_1 + \cdots + n'_l = s_p((n)_{[0,t]}) + s_p((n)_{(t,l]}).$$

□

**Lemma 3.** *Let $n$ be a positive integer. If*

$$n = n_0 + n_1 p + \cdots + n_l p^l$$

*with $0 < n_l \le p-1$ and $1-p \le n_i \le 0$ for $i = 0, 1, \cdots, l-1$, we have*

$$s_p(n) = \sum_{i=0}^{l} n_i + (p-1)(l-k),$$

*where $k = \min\{i \mid n_i \neq 0,\ i = 0, 1, \cdots, l\}$.*

*Proof.* Write

$$n = n_{i_1} p^{i_1} + n_{i_2} p^{i_2} + \cdots + n_{i_t} p^{i_t} + n_l p^l,$$

where $n_i < 0$ for $i = i_1, i_2, \cdots, i_t$ and $i_1 < i_2 < \cdots < i_t$, then the $p$-adic expansion of $n$ is

$$\begin{aligned} n =& (n_{i_1} + p)p^{i_1} + (p-1)p^{i_1+1} + \cdots + (p-1)p^{i_2-1} + (n_{i_2} + p - 1)p^{i_2} \\ &+ (p-1)p^{i_2+1} + \cdots + (p-1)p^{i_3-1} + (n_{i_3} + p - 1)p^{i_3} \\ &+ \cdots \\ &+ (n_{i_t} + p - 1)p^{i_t} + (p-1)p^{i_t+1} + \cdots + (p-1)p^{l-1} + (n_l - 1)p^l. \end{aligned}$$

Thus

$$s_p(n) = \sum_{i=0}^{l} n_i + (p-1)(l - i_1) = \sum_{i=0}^{l} n_i + (p-1)(l-k).$$

□

**Corollary 4.** *For positive integers $x \ge y$, and with the same notation as in Equations (5) and (6), when $k = 1, 2, \cdots, h_1$, we have*

$$s_p((x-y)_{[\alpha_k,\beta_k]}) + s_p((y)_{[\alpha_k,\beta_k]}) - s_p((x)_{[\alpha_k,\beta_k]}) = (p-1)(\beta_k - \alpha_k).$$

*Proof.* Observe that $1 - p \le a_i - b_i \le 0$ for $i \in [\alpha_k, \beta_k)$, $1 - p \le a_{\alpha_k} - b_{\alpha_k} < 0$, and $0 < a_{\beta_k} - b_{\beta_k} \le p - 1$. By Lemma 3, we have

$$\begin{aligned} s_p((x-y)_{[\alpha_k,\beta_k]}) =& \sum_{i \in [\alpha_k,\beta_k]} (a_i - b_i) + (p-1)(\beta_k - \alpha_k) \\ =& \, s_p((x)_{[\alpha_k,\beta_k]}) - s_p((y)_{[\alpha_k,\beta_k]}) + (p-1)(\beta_k - \alpha_k). \end{aligned}$$

□

*Proof of Theorem 1.* By Lemma 2, we have

$$\begin{aligned}
&s_p(x-y)+s_p(y)-s_p(x)\\
&=s_p((x-y)_{[0,\alpha_1)})+s_p((y)_{[0,\alpha_1)})-s_p((x)_{[0,\alpha_1)})\\
&\quad+\sum_{k=1}^{h-1}\left(s_p((x-y)_{(\beta_k,\alpha_{k+1})})+s_p((y)_{(\beta_k,\alpha_{k+1})})-s_p((x)_{(\beta_k,\alpha_{k+1})})\right)\\
&\quad+\sum_{k=1}^{h}\left(s_p((x-y)_{[\alpha_k,\beta_k]})+s_p((y)_{[\alpha_k,\beta_k]})-s_p((x)_{[\alpha_k,\beta_k]})\right)\\
&\quad+s_p((x-y)_{(\beta_{h_1},l]})+s_p((y)_{(\beta_{h_1},l]})-s_p((x)_{(\beta_{h_1},l]}).
\end{aligned}\tag{10}$$

If $i\in[0,\alpha_1)$, then $a_i-b_i\geq 0$. Thus, the $p$-adic expansion of $(x-y)_{[0,\alpha_1)}$ is $\sum_{i\in[0,\alpha_1)}(a_i-b_i)p^i$. Therefore

$$s_p((x-y)_{[0,\alpha_1)})=\sum_{i\in[0,\alpha_1)}(a_i-b_i)=s_p((x)_{[0,\alpha_1)})-s_p((y)_{[0,\alpha_1)}).$$

Hence, we have

$$s_p((x-y)_{[0,\alpha_1)})+s_p((y)_{[0,\alpha_1)})-s_p((x)_{[0,\alpha_1)})=0.$$

Additionally, by similar arguments we have

$$s_p((x-y)_{[\beta_{h_1}+1,l]})+s_p((y)_{[\beta_{h_1}+1,l]})-s_p((x)_{[\beta_{h_1}+1,l]})=0,$$

and for $1\leq k\leq h_1-1$, we have

$$s_p((x-y)_{(\beta_k,\alpha_{k+1})})+s_p((y)_{(\beta_k,\alpha_{k+1})})-s_p((x)_{(\beta_k,\alpha_{k+1})})=0.$$

Furthermore, by Equation (10) and Corollary 4, we have

$$\begin{aligned}
&s_p(x-y)+s_p(y)-s_p(x)\\
&=\sum_{k=1}^{h_1}\left(s_p((x-y)_{[\alpha_k,\beta_k]})+s_p((y)_{[\alpha_k,\beta_k]})-s_p((x)_{[\alpha_k,\beta_k]})\right)\\
&=(p-1)\sum_{k=1}^{h_1}(\beta_k-\alpha_k)=(p-1)\delta_p(x,y).
\end{aligned}$$

This completes the proof. □

*Proof of Corollary 1.* Since $v_p(n!)=\frac{n-s_p(n)}{p-1}$, by Theorem 1 we have

$$v_p(\tbinom{x}{y})=v_p\left(\frac{x!}{y!(x-y)!}\right)=\frac{1}{p-1}(s_p(x-y)+s_p(y)-s_p(x))=\delta_p(x,y).$$

This completes the proof. □

*Proof of Theorem 2.* Since $C_n(m) = \binom{n+m}{2m}((n-m)!)^2$, we have

$$\begin{aligned} v_p(C_n(m)) &= v_p(\tbinom{n+m}{2m}) + 2v_p((n-m)!) \\ &= \frac{2}{p-1}(n-m-s_p(n-m)) + \delta_p(n+m, 2m). \end{aligned}$$

The proof is done. □

**Lemma 4** ([2])**.** *For any positive integer $n$ and prime $p$, we have*

$$\frac{n}{p-1} - \frac{\log(1+n)}{\log p} \leq v_p(n!) \leq \frac{n}{p-1}.$$

**Lemma 5.** *For any positive integer $n$ and prime $p$, we have $s_p(n) \leq (p-1)\log_p(n+1)$.*

*Proof.* By Legendre's formula and Lemma 4, we have

$$s_p(n) = n - (p-1)v_p(n!) \leq (p-1)\log_p(n+1).$$

□

*Proof of Corollary 2.* By Theorem 2, we have

$$\frac{p-1}{2n}v_p(C_n(m)) = 1 + \frac{p-1}{2n}\delta_p(n+m, 2m) - \frac{1}{n}(m + s_p(n-m)). \tag{11}$$

Observe that

$$0 \leq \delta_p(x, y) \leq \sum_{k=1}^{l} k = \frac{l(l+1)}{2} \leq \frac{\log_p^2 x + \log_p x}{2}.$$

Then

$$0 \leq \frac{p-1}{2n}\delta_p(n+m, 2m) \leq \frac{(p-1)(\log_p^2 n + \log_p n)}{4n}. \tag{12}$$

By Lemma 5, we have

$$0 \leq \frac{s_p(n-m)}{n} \leq \frac{(p-1)\log_p(n+1)}{n}. \tag{13}$$

Thus, by Equations (12) and (13), the limits of $\frac{p-1}{2n}\delta_p(n+m, 2m)$ and $\frac{1}{n}(m+s_p(n-m))$ in Equation (11) are both 0 as $n$ goes to infinity. The proof is done. □

## 3. $m = 1$

The following formula of $v_p(C_n(1))$ was provided by Hong and Liu [11]:

$$v_p(C_n(1)) = \begin{cases} 2n-2-2s_2(\frac{n-1}{2})+v_2(\frac{n+1}{2}), & p=2,\ 2\nmid n; \\ 2n-4-2s_2(\frac{n}{2}-1)+v_2(\frac{n}{2}), & p=2,\ 2\mid n; \\ \frac{2}{p-1}(n-1-s_p(n-1))+v_p(n)+v_p(n+1), & p>2. \end{cases} \quad (14)$$

By Theorem 2, for $n \geq 2$, we have

$$v_p(C_n(1)) = \frac{2}{p-1}(n-1-s_p(n-1))+\delta_p(n+1,2). \quad (15)$$

For $p = 2$, we have

$$n+1 = a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + \cdots + a_l \cdot 2^l,$$

where $a_i = 0$ or 1 for $i = 0, 1, \cdots, l$. Let $k = \min\{i \mid a_i = 1,\ i = 2, 3, \cdots, l\}$.

If $a_1 = 1$, we have

$$n-1 = a_0 + a_k \cdot 2^k + \cdots + a_l \cdot 2^l.$$

It follows that $I' = \varnothing$ and $J' = \varnothing$. Hence, we have $\delta_2(n+1,2) = 0$ and $v_2(C_n(1)) = 2n-2-2s_2(n-1)$.

If $a_1 = 0$, we have

$$n-1 = a_0 - 2 + a_k \cdot 2^k + \cdots + a_l \cdot 2^l.$$

Then $I' = \{1\}, J' = \{k\}$, and $\delta_2(n+1,2) = k-1$. Thus $v_2(C_n(1)) = 2n-3-2s_2(n-1)+k$.

For an odd prime $p$, by Corollary 1, we have

$$\delta_p(n+1,2) = v_p(\tbinom{n+1}{2}) = v_p(n)+v_p(n+1).$$

Hence, we have $v_p(C_n(1)) = v_p(n)+v_p(n+1)+\frac{2}{p-1}(n-1-s_p(n-1))$.

In summary, Equation (15) is equivalent to

$$v_p(C_n(1)) = \begin{cases} 2n-2-2s_2(n-1), & p=2,\ a_1=1; \\ 2n-3-2s_2(n-1)+k, & p=2,\ a_1=0; \\ \frac{2}{p-1}(n-1-s_p(n-1))+v_p(n)+v_p(n+1), & p>2. \end{cases} \quad (16)$$

Herein, it is trivial to see that Equations (14) and (16) coincide with each other.

## 4. $m = 2$

In this section, we discuss the case for $m = 2$ and prove Theorem 3, while assuming $n \geq 3$. The objective is to show that a prime $p$ exists such that $v_p(C_n(2))$ is odd. Let $S = \{a^2 \mid a \in \mathbb{Z}\}$. Let $\mathcal{P}(n)$ denote the set of primes $p$ such that $v_p(n)$ is odd.

**Lemma 6.** *For a positive integer $n$ and a prime $p \geq 5$, we have $v_p(\prod_{i=-1}^{2}(n+i)) = v_p(n+i_0)$ for some $i_0 \in \{-1, 0, 1, 2\}$.*

*Proof.* The greatest common divisors of any two numbers in $\{n-1, n, n+1, n+2\}$ should not exceed 3. If $p \nmid n+i$ for each $i$, then $v_p(\prod_{i=-1}^{2}(n+i)) = 0$. Otherwise, there exists a unique $i_0$ such that $p \mid n+i_0$. It follows that

$$v_p((n-1)n(n+1)(n+2)) = v_p(n+i_0). \qquad \square$$

*Proof of Theorem 3.* By Theorem 2, we observe that $v_p(C_n(2))$ and $\delta_p(n+2, 4)$ have the same parity. By Corollary 1, we have

$$\delta_p(n+2, 4) = v_p\left(\binom{n+2}{4}\right) = v_p((n^2-1)(n^2+2n)) - 3v_p(2) - v_p(3).$$

We assume that $v_p\left(\binom{n+2}{4}\right)$ is even for all primes $p \geq 5$. Otherwise, there exists a prime $p \geq 5$ such that $v_p(C_n(2))$ is odd and $C_n(2)$ is not a square. Thus, this assumption and Lemma 6 imply that $p \notin \mathcal{P}(n^2-1)$ if $p \geq 5$. Similarly, $\mathcal{P}(n^2+2n) \neq \varnothing$ and $p \notin \mathcal{P}(n^2+2n)$ if $p \geq 5$. Thus, for any $p \geq 5$, we have

$$p \notin P(n-1) \cup P(n) \cup P(n+1) \cup P(n+2). \tag{17}$$

`Case 1.` $\mathcal{P}(n^2-1) = \{3\}$. If $3 \in \mathcal{P}(n^2+2n)$, then $v_3(\binom{n+2}{4}) = v_3(n^2+2n) + v_3(n^2-1) - 1$ is odd. Thus $C_n(2)$ is not a square. If $\mathcal{P}(n^2+2n) = \{2\}$, then $n$ is even. Hence $(n, n+2) = 2$. Since $\frac{n^2+2n}{2} \in S$, we either have $n \in S$ or $n+2 \in S$. Thus $n+1, n-1 \notin S$, which contradicts $\frac{n^2-1}{3} \in S$.

`Case 2.` $\mathcal{P}(n^2-1) = \{2\}$. In this case, we have that $n$ is odd and $(n+1, n-1) = 2$. Since $\frac{n^2-1}{2} \in S$, we either have $n+1 \in S$ or $n-1 \in S$. Thus $n, n+2 \notin S$, which implies that $\mathcal{P}(n), \mathcal{P}(n+2) \neq \varnothing$. If $2 \notin \mathcal{P}(n) \cup \mathcal{P}(n+2)$ and $\mathcal{P}(n) \cap \mathcal{P}(n+2) = \varnothing$, then there exists a prime $p \geq 5$ in $\mathcal{P}(n) \cup \mathcal{P}(n+2)$, which contradicts Equation (17).

`Case 3.` $\mathcal{P}(n^2-1) = \{2, 3\}$. In this case, we have that $n$ is odd and $(n, n+2) = 1$. Since $n^2+2n \notin S$, we either have $n \notin S$ or $n+2 \notin S$. Since $2 \mid n^2-1$ and $3 \mid n^2-1$, we have $2, 3 \notin \mathcal{P}(n)$. Thus, combining with Equation (17), we have $n \in S$ and $n+2 \notin S$. Since $2 \notin \mathcal{P}(n+2)$, we have $\mathcal{P}(n+2) = \{3\}$. Thus

$$v_3(\binom{n+2}{4}) = v_3(n+2) + v_3(n^2-1) - 1$$

is odd.

The proof of Theorem 3 is completed. $\square$

## 5. $m = 3$

Similar to Section 4, we discuss the proof for Theorem 4 in this section.

**Lemma 7.** *For a positive integer $n$ and a prime $p \geq 7$, we have $v_p(\prod_{i=-2}^{3}(n+i)) = v_p(n+i_0)$ for some $i_0 \in [-2, 3]$.*

*Proof.* The proof is similarly to that of Lemma 6. □

**Lemma 8.** *The following facts have been established.*

1. *The Diophantine equation $x^2 - y^2 = k$ has no solutions when $x > 3$ and $1 \leq k \leq 5$.*
2. *The Diophantine equation $ax^2 - 2y^2 = k$ has no solutions when $x$ and $y$ are odd, $a \equiv 1 \pmod 4$, and $k \equiv 1 \pmod 4$.*
3. *The Diophantine equation $ax^2 - ky^2 = -1$ has no solutions when $x$ is odd and $y$ is even for any integer $k$ and $a \equiv 1 \pmod 4$.*
4. *The Diophantine equation $x^2 - 5y^2 = k$ has no solutions for $k \equiv \pm 2 \pmod 5$.*

*Proof.* Here, the first statement is trivial. For the other statements, otherwise, we have $-2 \equiv 1 \pmod 4$, $-1 \equiv 1 \pmod 4$, or $x^2 \equiv \pm 2 \pmod 5$, respectively, which are all contradictions. □

**Lemma 9.** *For an integer $n > 6$, there exists at most one square in $\{n-2, n-1, n, n+1, n+2, n+3\}$.*

*Proof.* By trivial computations, we observe that the lemma holds for $n = 7, 8, 9,$ and 10. When $n > 10$, we assume that $n+i, n+j$ are squares when $i > j$ and $i, j \in \{-2, -1, 0, 1, 2, 3\}$. It follows that $(\sqrt{n+i}, \sqrt{n+j})$ is a solution to $x^2 - y^2 = i - j$ with $\sqrt{n+i} > 3$ and $1 \leq i - j \leq 5$. This result contradicts the first statement of Lemma 8. □

*Proof of Theorem 4.* Upon the decompositions of $C_n(3)$ when $n = 4, 5,$ and 6, the theorem holds trivially for $n \leq 6$. Hence, we have assumed $n > 6$.

By Theorem 2, it is sufficient to prove that there exists a prime $p$ such that $v_p(\binom{n+3}{6})$ is odd. By Corollary 1, we have

$$v_p(\binom{n+3}{6}) = v_p(n^2 - 4) + v_p(n^2 - 1) + v_p(n^2 + 3n) - v_p(5) - 2v_p(12). \tag{18}$$

By Lemma 7, if there exists a prime $p \geq 7$ in $\bigcup_{i=-2}^{3} \mathcal{P}(n+i)$, then $v_p(\binom{n+3}{6})$ is odd, and $C_n(3)$ is not a square. Hence, we always assume $\bigcup_{i=-2}^{3} \mathcal{P}(n+i) \subseteq \{2, 3, 5\}$.

We claim that $n^2 - 1 \notin S, n^2 - 4 \notin S$, and $n^2 + 3n \notin S$. The first two facts are trivial, while for the last fact, observe that $(n, n+3) = 1$ or 3. Thus $n^2 + 3n \in S$

implies that either $n, n+3 \in S$ or $\frac{n}{3}, \frac{n+3}{3} \in S$. This yields a solution $(\sqrt{n+3}, \sqrt{n})$ or $(\sqrt{\frac{n+3}{3}}, \sqrt{\frac{n}{3}})$ of $x^2 - y^2 = 3$ or $x^2 - y^2 = 1$, respectively, which contradicts Lemma 8.

Case 1. $\mathcal{P}(n^2-4) = \{5\}$. Since $n^2 \equiv 4 \pmod 5$, we have $5 \notin \mathcal{P}(n^2-1)$. If $3 \in \mathcal{P}(n^2-1)$, then $v_3(\binom{n+3}{6}) = v_3(n^2-1) + v_3(n^2-4) - 2$ is odd as desired. If $\mathcal{P}(n^2-1) = \{2\}$, then $n$ is odd and $(n+1, n-1) = 2$. Since $\frac{n^2-1}{2} \in S$, we either have $n+1 \in S$ or $n-1 \in S$. Thus $n+2, n-2 \notin S$. Therefore, by the fact that $(n+2, n-2) = 1$, we have $\frac{n^2-4}{5} \notin S$, which contradicts $\mathcal{P}(n^2-4) = \{5\}$.

Case 2. $\mathcal{P}(n^2-4) = \{3\}$. Since $n^2 \equiv 4 \pmod 3$, we have $\mathcal{P}(n) \cap \mathcal{P}(n+3) = \varnothing$ and $3 \notin \mathcal{P}(n) \cup \mathcal{P}(n+3)$.

Case 2.1. $n \notin S, n+3 \notin S$. If $\mathcal{P}(n) = \{5\}$ and $\mathcal{P}(n+3) = \{2\}$, then $n$ is odd and $(n+2, n-2) = 1$. It follows that either $n+2 \in S$ or $n-2 \in S$ since $\frac{n^2-4}{3} \in S$. Therefore, we either have a solution $(\sqrt{n+2}, \sqrt{\frac{n}{5}})$ of $x^2 - 5y^2 = 2$ or a solution $(\sqrt{n-2}, \sqrt{\frac{n}{5}})$ of $x^2 - 5y^2 = -2$, which is a contradiction. Thus $\mathcal{P}(n) = \{2\}$ and $v_2(n)$ is odd. Additionally, since $2 \notin \mathcal{P}(n^2-4)$, we observe that $v_2(n^2-4)$ is even. It follows that $v_2(\binom{n+3}{6}) = v_2(n) + v_2(n^2-4) - 4$ is odd.

Case 2.2. $n \in S$ or $n+3 \in S$. In this case, we have $n+2 \notin S$ and $n-2 \notin S$. If $n$ is odd, then $(n+2, n-2) = 1$. Since $\frac{n^2-4}{3} \in S$, we either have $n+2 \in S$ or $n-2 \in S$, which is a contradiction. Hence $n$ is even. If $n \equiv 2 \pmod 4$, then $(\frac{n+2}{4}, \frac{n-2}{4}) = 1$. Since $\frac{n^2-4}{3} \in S$, we have $n+2 \in S$ or $n-2 \in S$, which is a contradiction. Therefore, we have $n \equiv 0 \pmod 4$ and $(\frac{n+2}{2}, \frac{n-2}{2}) = 1$. Thus, either $\frac{n-2}{2} \in S$ or $\frac{n+2}{2} \in S$ since $\frac{n^2-4}{3} \in S$.

If $n \in S$, then $\mathcal{P}(n+3) = \{5\}$ since $2 \notin \mathcal{P}(n+3)$. Therefore $\frac{n+3}{5} \in S$. Hence, we either have an odd solution $(\sqrt{\frac{n+3}{5}}, \sqrt{\frac{n-2}{2}})$ of $5x^2 - 2y^2 = 5$ or an odd solution $(\sqrt{\frac{n+3}{5}}, \sqrt{\frac{n+2}{2}})$ of $5x^2 - 2y^2 = 1$, which is a contradiction.

Additionally, if $n+3 \in S$, then we either have an odd solution $(\sqrt{n+3}, \sqrt{\frac{n-2}{2}})$ of $x^2 - 2y^2 = 5$ or an odd solution $(\sqrt{n+3}, \sqrt{\frac{n+2}{2}})$ of $x^2 - 2y^2 = 1$, which is a contradiction.

Case 3. $\mathcal{P}(n^2-4) = \{2\}$. In this case, we have that $n$ is even. If $n \equiv 0 \pmod 4$, we have $v_2(n^2-4) = 2$, which implies that $2 \notin \mathcal{P}(n^2-4)$. Therefore $n \equiv 2 \pmod 4$ and $(\frac{n-2}{4}, \frac{n+2}{4}) = 1$. Since $\frac{n^2-4}{2} \in S$, we either have $n-2 \in S$ or $n+2 \in S$. Thus $n-1, n+1, n+3 \notin S$, and $\mathcal{P}(n-1), \mathcal{P}(n+1)$ and $\mathcal{P}(n+3)$ are all non-empty sets. Since any two of $n-1, n+1$, and $n+3$ are coprime, we have that $\mathcal{P}(n-1), \mathcal{P}(n+1)$, and $\mathcal{P}(n+3)$ are disjoint from each other. Therefore, one of $\mathcal{P}(n-1), \mathcal{P}(n+1)$, and $\mathcal{P}(n+3)$ contains 2, which contradicts that $n$ is even.

Case 4. $\mathcal{P}(n^2-4) = \{3, 5\}$. In this case, we have $n^2 \equiv 4 \pmod 3$ and $n^2 \equiv 4$

(mod 5). It follows that $3 \notin \mathcal{P}(n) \cup \mathcal{P}(n+3)$ and $5 \notin \bigcup_{i=-1}^{1} \mathcal{P}(n+i)$. Additionally, we have $\mathcal{P}(n) \cap \mathcal{P}(n+3) = \varnothing$ since $(n, n+3) = 1$.

If either $n-1 \in S$ or $n+1 \in S$, then we have $n \notin S$ and $n+3 \notin S$. Thus $\mathcal{P}(n) = \{2\}$, which implies that $n$ is even. Hence, we have $\mathcal{P}(n+3) = \{5\}$ since $2, 3 \notin \mathcal{P}(n+3)$. It follows that $v_5(n^2-4)$ and $v_5(n+3)$ are both odd. Thus $v_5(\binom{n+3}{6}) = v_5(n^2-4) + v_5(n+3) - 1$ is odd as desired.

If $n-1, n+1 \notin S$, then we have that $\mathcal{P}(n^2-1) \neq \{2\}$. Otherwise, we have that $n$ is odd and $\frac{n^2-1}{2} \in S$, which implies that either $n-1 \in S$ or $n+1 \in S$. If $\mathcal{P}(n^2-1) = \{2, 3\}$, then $n$ is odd and $\frac{n^2-1}{6} \in S$. Hence, we obtain a solution $(\sqrt{\frac{n^2-4}{15}}, \sqrt{\frac{n^2-1}{6}})$ of $5x^2 - 2y^2 = -1$, where $x$ is odd and $y$ is even. This is a contradiction. If $\mathcal{P}(n^2-1) = \{3\}$, then $\frac{n^2-1}{3} \in S$. Hence, we claim that $n$ is odd. Otherwise, we have $(n+1, n-1) = 1$, which implies that either $n-1 \in S$ or $n+1 \in S$. Hence, we get a solution $(\sqrt{\frac{n^2-4}{15}}, \sqrt{\frac{n^2-1}{3}})$ of $5x^2 - y^2 = -1$, where $x$ is odd and $y$ is even. This is also a contradiction.

`Case 5.` $\mathcal{P}(n^2-4) = \{2, 5\}$. In this case, we have that $n$ is even and $n^2 \equiv 4 \pmod 5$. It follows that $2, 5 \notin \mathcal{P}(n^2-1)$. Therefore $\mathcal{P}(n^2-1) = \{3\}$ and $3 \notin \mathcal{P}(n^2+3n)$. We have $v_3(\binom{n+3}{6}) \equiv v_3(n^2-1) \equiv 1 \pmod 2$. Hence $v_3(\binom{n+3}{6})$ is odd as desired.

`Case 6.` $\mathcal{P}(n^2-4) = \{2, 3\}$. In this case, we have that $n$ is even and $n^2 \equiv 1 \pmod 3$. It follows that $3 \notin \mathcal{P}(n)$, and $2, 3 \notin \mathcal{P}(n+3)$. Since $2 \in \mathcal{P}(n^2-4)$, we have $n \equiv 2 \pmod 4$. Therefore $(n-2, n+2) = 4$.

`Case 6.1.` $n-1 \in S$ or $n+1 \in S$. In this case, we have $n \notin S$ and $n+3 \notin S$. Hence $\mathcal{P}(n+3) = \{5\}$ and $5 \notin \mathcal{P}(n)$. Therefore $\mathcal{P}(n) = \{2\}$ since $3 \notin \mathcal{P}(n)$. Since $\frac{n^2-4}{6} \in S$, we have $\frac{n^2-4}{6 \cdot 16} \in S$, which implies that $\frac{n-2}{8} \in S$ or $\frac{n+2}{8} \in S$ since $\frac{n-2}{4}, \frac{n+2}{4} \notin S$ and $(\frac{n-2}{4}, \frac{n+2}{4}) = 1$. Thus we either have $\frac{n-2}{2} \in S$ or $\frac{n+2}{2} \in S$. Hence, we get a solution $(\sqrt{\frac{n}{2}}, \sqrt{\frac{n-2}{2}})$ or $(\sqrt{\frac{n+2}{2}}, \sqrt{\frac{n}{2}})$ of $x^2 - y^2 = 1$, which is a contradiction.

`Case 6.2.` $n-1 \notin S, n+1 \notin S$. Since $\mathcal{P}(n-1) \cap \mathcal{P}(n+1) = \varnothing$ and $2 \notin \mathcal{P}(n-1) \cup \mathcal{P}(n+1)$, we either have $\mathcal{P}(n-1) = \{3\}$, $\mathcal{P}(n+1) = \{5\}$ or $\mathcal{P}(n-1) = \{5\}$, $\mathcal{P}(n+1) = \{3\}$. Since $(n-1, n+1) = 1$, we have $\mathcal{P}(n^2-1) = \mathcal{P}(n-1) \cup \mathcal{P}(n+1)$. Thus $5 \in \mathcal{P}(n^2-1)$ and $n^2 \equiv 1 \pmod 5$, which implies that $5 \notin \mathcal{P}(n^2+3n)$. Since $3 \nmid n$, we also have $3 \notin \mathcal{P}(n^2+3n)$. Hence $\mathcal{P}(n^2+3n) = \{2\}$ and $\frac{n^2+3n}{2} \in S$. It follows that $\frac{n}{2} \in S$ and $n+3 \in S$ since $(\frac{n}{2}, n+3) = 1$. Therefore, we have $n-2 \notin S$ and $n+2 \notin S$. By the same argument on $\frac{n^2-4}{6}$ in Case 6.1, we either have $\frac{n-2}{2} \in S$ or $\frac{n+2}{2} \in S$. Therefore, we get a solution $(\sqrt{\frac{n}{2}}, \sqrt{\frac{n-2}{2}})$ or $(\sqrt{\frac{n+2}{2}}, \sqrt{\frac{n}{2}})$ of $x^2 - y^2 = 1$, which is also a contradiction.

`Case 7.` $\mathcal{P}(n^2-4) = \{2, 3, 5\}$. In this case, we have that $n$ is even, $n^2 \equiv 4 \pmod 3$, and $n^2 \equiv 4 \pmod 5$. It follows that $2, 5 \notin \mathcal{P}(n^2-1)$, which implies that

$\mathcal{P}(n^2-1)=\{3\}$. Therefore, either $n-1\in S$ or $n+1\in S$ since $(n-1,n+1)=1$. Hence, we have $n\notin S$ and $n+3\notin S$. Since $3,5\notin\mathcal{P}(n)$, we have $\mathcal{P}(n)=\{2\}$, which implies that $\mathcal{P}(n+3)=\{5\}$ since $3\notin\mathcal{P}(n+3)$ and $(n,n+3)=1$. Therefore $v_5(\binom{n+3}{6})=v_5(n^2-4)+v_5(n+3)-1$ is odd.

The proof of Theorem 4 is completed. □

## 6. Criterion for $C_n(m)$ Being a Powerful Number

The criterion for $C_n(m)$ being a powerful number has been discussed in this section. A relevant example has also been provided to conclude the findings.

**Theorem 5.** *Let $m,n$ be positive integers with $n\geq m+1$. If there are no primes in $(n-m,n+m]$, then $C_n(m)$ is a powerful number.*

*Proof.* Recall that

$$v_p(C_n(m))=2v_p((n-m)!)+v_p(\tbinom{n+m}{2m}). \tag{19}$$

If there are no primes in $(n-m,n+m]$, then for any prime divisor $p$ of $C_n(m)$, we have $p\leq n-m$, which implies that $p|(n-m)!$. By Equation (19), we have $v_p(C_n(m))\geq 2v_p((n-m)!)\geq 2$. Thus $C_n(m)$ is a powerful number. □

**Corollary 5.** *Let $m,n$ be positive integers with $n\geq 3m$. Then $C_n(m)$ is a powerful number if and only if there are no primes in $(n-m,n+m]$.*

*Proof.* By Theorem 5, the sufficiency has been established. Conversely, assume that there exists a prime $p\in(n-m,n+m]$. Since $p>n-m\geq 2m$, we have $v_p((n-m)!)=0$ and $v_p((2m)!)=0$. Hence $v_p(\binom{n+m}{2m})=0$. Let $p=n+i$ for some $i\in(-m,m]$. Since the greatest common divisors of any two numbers in $\{n-m+1,n-m+2,\cdots,n+m\}$ should not exceed $2m-1$, we have $p\nmid n+j$ for $j\neq i, j\in(-m,m]$. Thus, by Equation (19) we have

$$v_p(C_n(m))=\sum_{k=1-m}^{m}v_p(n+k)=v_p(n+i)=1.$$

It follows that $C_n(m)$ is not a powerful number. □

**Remark 1.** From Corollary 5, it is easy to prove that if $C_{n_0}(m_0)$ is not a powerful number with $m_0\leq\frac{n_0}{3}$, then $C_{n_0}(m)$ is not a powerful number for all $m\in(m_0,\frac{n_0}{3}]$.

**Lemma 10** ([13])**.** *For every positive integer $n$, there exists a prime $p\in[n,\frac{9(n+3)}{8}]$.*

**Corollary 6.** *When $m\geq 3$, we have that $C_n(m)$ is not a powerful number if $n\in[\frac{5m+18}{4},17m-36]$ .*

*Proof.* Since $m \geq 3$, we have $\frac{5m+18}{4} \leq 3m$. Recall that

$$v_p(C_n(m)) = v_p((n-m)!) + v_p((2m+1)\cdots(n+m)). \tag{20}$$

For $\frac{5m+18}{4} \leq n < 3m$, we have $n+m \geq \frac{9(2m+4)}{8}$. By Lemma 10, there exists a prime $p \in [2m+1, n+m]$. Since $p \geq 2m+1 > n-m$, we have $v_p((n-m)!) = 0$. Since the greatest common divisors of any two numbers in $\{2m+1, 2m+2, \cdots, n+m\}$ should not exceed $2m-1$, we also have $v_p((2m+1)\cdots(n+m)) = 1$. Thus, by Equation (20) we have $v_p(C_n(m)) = 1$, which implies that $C_n(m)$ is not a powerful number for $n \in [\frac{5m+18}{4}, 3m)$.

For $3m \leq n \leq 17m-36$, we have $\frac{9(n-m+4)}{8} \leq n+m$. By Lemma 10, there exists a prime $p \in [n-m+1, n+m]$. Therefore, by Corollary 5, we have that $C_n(m)$ is not a powerful number. □

**Example 3.** Let $n \leq 1000$. The following table lists $N$ for $2 \leq m \leq 10$, where $N = \#\{n \in [m+1, 1000] \mid C_n(m) \text{ is a powerful number}\}$.

| $m$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $N$ | 402 | 219 | 124 | 60 | 28 | 10 | 6 | 2 | 0 |

Table 1: $N$ for $2 \leq m \leq 10$.

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