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PERIODICITY AND THE INDEX OF INTEGER PARTITIONS

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Abstract

Seaweed subalgebras of $\mathfrak{sl}(n)$, referred to here simply as seaweed algebras, are defined by an ordered pair of compositions of n. The index theory of these algebras has recently been used to define statistics on integer partitions. Given a partition λ, one natural choice of such statistic, denoted here by $\text{ind}(\lambda)$, arises from taking the index of the seaweed algebra defined by the pair consisting of λ and its weight $w(\lambda)$. Here, we examine ind restricted to partitions whose parts come from the set $\{1, 2, d\}$ for $d > 1$ odd. In particular, we find (partial) formulae for (1) the number of such partitions with fixed ind value, and (2) when $d \equiv 3 \pmod 4$, the difference between the number of such partitions with ind even and the number with ind odd. Interestingly, we find that for (1) the corresponding sequences of values are eventually periodic and for (2) the sequences of values are periodic. Periodic phenomena are not new to integer partition statistics arising from the index theory of seaweed algebras. Consequently, the addition of the results established in this paper further suggest the existence of a more general theorem involving index, integer partitions, and periodicity.

1. Introduction

In recent work, the index theory of seaweed subalgebras of $\mathfrak{sl}(n)$ is used to generate statistics on integer partitions (see [1, 3, 4, 8]). Fixing a natural choice of such a statistic, denoted here by ind and defined in Section 2 below, sequences of values of two types have been of interest. Letting $\mathcal{P}(n, S)$ denote the collection of partitions of n with parts coming from a set $S \subset \mathbb{Z}_{>0}$, values of the following forms have been investigated: (1) the number of partitions $\lambda \in \mathcal{P}(n, S)$ with $\text{ind}(\lambda) = i$ for fixed $i \geq 0$ and (2) the difference between the number of partitions $\lambda \in \mathcal{P}(n, S)$ with $\text{ind}(\lambda)$ even and the number with $\text{ind}(\lambda)$ odd. Here, for $d > 1$ odd, we establish the following, which constitute the main results of the paper:

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(1) the number of partitions $\lambda \in \mathcal{P}(n, \{1, 2, d\})$ satisfying $\text{ind}(\lambda) = i$ for $i \geq 0$ is eventually periodic (see Theorem 2);

(2) the difference between the number of partitions $\lambda \in \mathcal{P}(n, \{1, 2, d\})$ with $\text{ind}(\lambda)$ even and the number with $\text{ind}(\lambda)$ odd is periodic for $d \equiv 3 \pmod 4$ (see Theorem 3).

Interestingly, periodic results like (1) are not new. In [3], it was shown that the number of partitions $\lambda \in \mathcal{P}(n, \{1, 2, \ldots, k\})$ for $1 \leq k \leq 7$ with $\text{ind}(\lambda) = 0$ is eventually periodic. With the addition of the periodicity results established in this paper, it seems reasonable to conjecture that there exists a more general relationship between index, integer partitions, and periodicity.

In general, the "index" of a Lie algebra $\mathfrak{g}$ is a nonnegative-integer algebraic invariant which is notoriously difficult to compute.[1] In the case of seaweed subalgebras of $\mathfrak{sl}(n)$, referred to here simply as seaweed algebras, Dergachev and Kirillov [5] provided a combinatorial formula for the index in terms of a corresponding graph, called a meander. The authors of [5] define both seaweed algebras and their meanders in terms of a corresponding ordered pair of compositions (see Section 2.1 for details). Given the composition definition of seaweed algebras, one can use the corresponding index theory to define a natural statistic on integer partitions. In particular, given a partition λ of n one can define the index of λ, denoted $\text{ind}(\lambda)$, to be the index of the seaweed algebra defined by the pair of compositions λ and n.[2]

Applying the statistic ind to partitions with largest part less than or equal to 7, the authors of [3] established the following theorem.

Theorem 1 ([3, Theorem 16]). *For $d \in \mathbb{Z}_{>0}$, let $\mathcal{P}_0(n, d)$ denote the collection of partitions λ of n with largest part less than or equal to d and $\text{ind}(\lambda) = 0$. If $d \in \{1, 2, 3, 4, 5, 6, 7\}$, then the values of $|\mathcal{P}_0(n, d)|$ are eventually periodic. More precisely,*

- $\{|\mathcal{P}_0(n, 1)|\}_{n\geq 3}$ *is periodic with repeating sequence* 0.
- $\{|\mathcal{P}_0(n, 2)|\}_{n\geq 5}$ *is periodic with repeating sequence* $1, 0$.
- $\{|\mathcal{P}_0(n, 3)|\}_{n\geq 13}$ *is periodic with repeating sequence* $2, 0$.
- $\{|\mathcal{P}_0(n, 4)|\}_{n\geq 17}$ *is periodic with repeating sequence* $4, 2, 3, 0$.

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[1]The *index* of a Lie algebra was first introduced by Dixmier [6] and is defined as

$$\text{ind}(\mathfrak{g}) = \min_{F \in \mathfrak{g}^*} \dim(\ker(B_F)),$$

where B_F is the associated skew-symmetric *Kirillov form* defined by $B_F(x, y) = F([x, y])$ for all $x, y \in \mathfrak{g}$.

[2]In [3], a more general integer partition statistic construction is described. Let $\mathcal{P}(n)$ denote the collection of partitions of n and $\mathcal{C}(n)$ denote the collection of compositions of n. Given a mapping $f : \mathcal{P}(n) \to \mathcal{C}(n)$ for all $n \in \mathbb{Z}_{>0}$, one can define a statistic $\text{ind}_f(\lambda)$ on partitions λ equal to the index of the seaweed algebra associated with the pair of compositions λ and $f(\lambda)$.

- $\{|\mathcal{P}_0(n,5)|\}_{n\geq 21}$ *is periodic with repeating sequence* $7,3,5,3$.
- $\{|\mathcal{P}_0(n,6)|\}_{n\geq 37}$ *is periodic with repeating sequence*

$$14,5,9,3,11,5,11,3,12,5,8,3.$$

- $\{|\mathcal{P}_0(n,7)|\}_{n\geq 41}$ *is periodic with repeating sequence*

$$19,9,18,7,19,9,17,7,20,9,17,7.$$

For each sequence above, starting from a smaller value of n does not result in a periodic sequence.

In this article, focusing on partitions whose parts come from the set $\{1,2,d\}$ for $d>1$ odd, we establish the following two results.

Theorem 2. *For $d>1$ odd and $i\geq 0$, the sequence of values*

$$\{|\{\lambda\in\mathcal{P}(n,\{1,2,d\})\mid \mathrm{ind}(\lambda)=i\}|\}_{n\geq k}$$

is periodic if and only if $k\geq 4(i+1)d+1$ with

$$|\{\lambda\in\mathcal{P}(n,\{1,2,d\})\mid \mathrm{ind}(\lambda)=i\}|=\begin{cases}2(i+1), & n \text{ odd}\\ 0, & n \text{ even}\end{cases}$$

for $n\geq 4(i+1)d+1$.

Theorem 3. *If $d\equiv 3 \pmod 4$,*

$$\mathcal{P}_d^E(n)=|\{\lambda\in\mathcal{P}(n,\{1,2,d\})\mid \mathrm{ind}(\lambda)\equiv 0 \pmod 2\}|,$$

and

$$\mathcal{P}_d^O(n)=|\{\lambda\in\mathcal{P}(n,\{1,2,d\})\mid \mathrm{ind}(\lambda)\equiv 1 \pmod 2\}|,$$

then the sequence $\{\mathcal{P}_d^E(n)-\mathcal{P}_d^O(n)\}_{n\geq 1}$ is periodic with each period having the following form:

$$\underbrace{1,0,0,-1,\ldots,1,0,0,-1,1,0}_{d-1},\underbrace{1,0,1,0,\ldots,1,0,1,0,1}_{d},\underbrace{1,0,0,1,\ldots,1,0,0,1,1,0,0}_{d},\underbrace{0,\ldots,0}_{d},-1.$$

Note that all three theorems listed above contain the following three ingredients: index, a restricted class of integer partitions, and periodicity. Consequently, one is led to conjecture that there may exist a more general theorem relating index, restricted classes of integer partitions, and periodic phenomena for which the above theorems are special cases.

The structure of the paper is as follows. In Section 2, we provide necessary preliminaries from the theory of seaweed algebras as well as integer partitions. Sections 3 and 4 contain the proofs of the two main theorems. In Section 5, directions for further research are discussed.

2. Preliminaries

In Section 2.1, we define seaweed subalgebras of $\mathfrak{sl}(n)$ and discuss their index theory, and Section 2.2 consists of the necessary preliminaries from the theory of integer partitions.

2.1. Seaweed Algebras

Seaweed subalgebras of $\mathfrak{sl}(n)$ – the set of all $n \times n$ complex-valued matrices of trace zero – can be naturally defined in terms of two compositions of the positive integer n. Recall that a *composition* of n is a finite sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that $n = \sum_{i=1}^{m} \lambda_i$. We denote the composition defined by the sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ by $\lambda_1|\lambda_2|\cdots|\lambda_m$. If V is an n-dimensional vector space with a basis $\{e_1, \ldots, e_n\}$, let $a_1|\ldots|a_m$ and $b_1|\ldots|b_t$ be two compositions of n and consider the flags

$$\{0\} \subset V_1 \subset \cdots \subset V_{m-1} \subset V_m = V \quad \text{and} \quad V = W_0 \supset W_1 \supset \cdots \supset W_t = \{0\},$$

where $V_i = \operatorname{span}\{e_1, \ldots, e_{a_1+\cdots+a_i}\}$ and $W_j = \operatorname{span}\{e_{b_1+\cdots+b_j+1}, \ldots, e_n\}$. The subalgebra of $\mathfrak{sl}(n)$ preserving these flags is called a *seaweed subalgebra of* $\mathfrak{sl}(n)$ and is said to have type $\dfrac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$. Here we will refer to seaweed subalgebras of $\mathfrak{sl}(n)$ simply as *seaweed algebras*.

Example 1. The evocative "seaweed" is descriptive of the shape of the algebra when exhibited in matrix form. For example, the seaweed algebra of type $\frac{2|4}{1|2|3}$ consists of trace-zero matrices of the form depicted in Figure 1, where the $*$'s indicate the possible non-zero entries from the complex numbers.

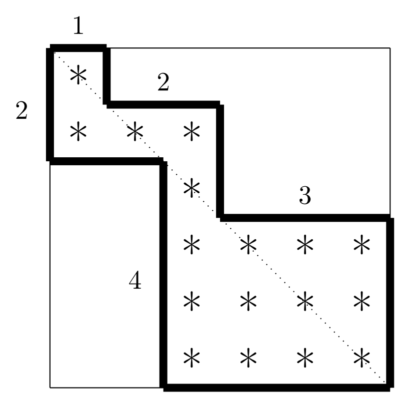

Figure 1: Seaweed algebra of type $\frac{2|4}{1|2|3}$

Recall that the index of a Lie algebra $\mathfrak{g}$ is defined by

$$\text{ind}(\mathfrak{g}) = \min_{F \in \mathfrak{g}^*} \dim(\ker(B_F)),$$

where B_F is the skew-symmetric *Kirillov form* defined by $B_F(x, y) = F([x, y])$ for all $x, y \in \mathfrak{g}$. As stated in the introduction, Dergachev and A. Kirillov [5] provided a combinatorial formula for the index of seaweed algebras in terms of an associated planar graph, called a meander.

Let $\mathfrak{g}$ be the seaweed algebra of type $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$. The *meander* of $\mathfrak{g}$, also referred to here as the *meander of type* $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$, consists of $n = \sum_{i=1}^{m} a_i = \sum_{i=1}^{t} b_i$ vertices and is constructed as follows. Label the n vertices of our meander as $v_1, v_2, \ldots, v_n$ from left to right along a horizontal line. We then place edges above the horizontal line, called *top edges*, according to $a_1|\cdots|a_m$ in the following manner. Partition the set of vertices by grouping together the first a_1 vertices, then the next a_2 vertices, and so on, lastly grouping together the final a_m vertices. We call each set of vertices formed a *block*. For each block in the set partition determined by $a_1|\cdots|a_m$, add an edge from the first vertex of the block to the last vertex of the block, then add an edge between the second vertex of the block and the second to last vertex of the block, and so on within each block, assuming that the vertices being connected are distinct. More explicitly, given vertices v_j, v_k in a block of size a_i, there is an edge between them if and only if $j + k = 2(a_1 + a_2 + \cdots + a_{i-1}) + a_i + 1$. In the same way, place *bottom edges* below the horizontal line of vertices according to the blocks in the set partition determined by $b_1|\cdots|b_t$ (see Figure 2).

Example 2. The meander associated with the seaweed algebra of type $\frac{17|3}{10|4|6}$ is illustrated in Figure 2 below.

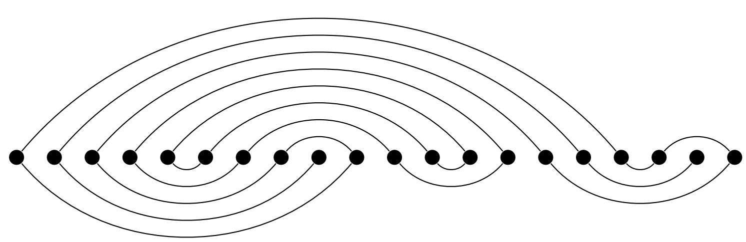

Figure 2: Meander of type $\frac{17|3}{10|4|6}$

Every meander consists of a disjoint union of cycles and paths. The main result of [5] is that the index of a seaweed algebra can be computed by counting the number and type of these components in its associated meander.

Theorem 4 ([5])**.** *If $\mathfrak{g}$ is a seaweed algebra, then*

$$\operatorname{ind}(\mathfrak{g}) = 2C + P - 1,$$

where C is the number of cycles and P is the number of paths in the associated meander.

Example 3. Let $\mathfrak{g}$ be the seaweed algebra of type $\frac{17|3}{10|4|6}$. As can be seen from Figure 2, the meander associated with $\mathfrak{g}$ consists of three cycles and a single path. Thus, $\operatorname{ind}(\mathfrak{g}) = 2(3) + 1 - 1 = 6$.

While Theorem 4 is an elegant combinatorial result, it is difficult to apply in practice. However, in [2] the authors show that any meander can be contracted, or "wound down," to the empty meander through a unique sequence of graph-theoretic moves, called the meander's *signature*.[3] The signature provides a fast algorithm for the computation of the index of seaweed algebras. Since we will need the explicit graph-theoretic moves involved in forming a meander's signature in what follows, we review the winding-down process below.

Lemma 1 ([2, Lemma 4])**.** *Given a meander M of type $\dfrac{a_1|a_2|...|a_m}{b_1|b_2|...|b_t}$, create a meander M' by exactly one of the following moves.*

1. *Vertical Flip (F_v): If $a_1 < b_1$, then M' has type*

$$\frac{b_1|b_2|...|b_t}{a_1|a_2|...|a_m}.$$

2. *Component Elimination ($C(c)$): If $a_1 = b_1 = c$, then M' has type*

$$\frac{a_2|a_3|...|a_m}{b_2|b_3|...|b_t}.$$

3. *Rotation Contraction (R): If $b_1 < a_1 < 2b_1$, then M' has type*

$$\frac{b_1|a_2|a_3|...|a_m}{(2b_1 - a_1)|b_2|...|b_t}.$$

4. *Block Elimination (B): If $a_1 = 2b_1$, then M' has type*

$$\frac{b_1|a_2|..|a_m}{b_2|b_3|...|b_t}.$$

[3]The signature is essentially a graph-theoretic recasting of Panyushev's reduction algorithm described in [7].

5. *Pure Contraction* (P)*: If* $a_1 > 2b_1$*, then* M' *has type*

$$\frac{(a_1 - 2b_1)|b_1|a_2|a_3|...|a_m}{b_2|b_3|...|b_t}.$$

For all moves except the Component Elimination move, M *and* M' *have the same number of paths and the same number of cycles, i.e., the corresponding seaweed algebras have the same index.*

Example 4. In this example, the meander of type $\frac{17|3}{10|4|6}$ is wound down to the empty meander using the moves detailed in Lemma 1.

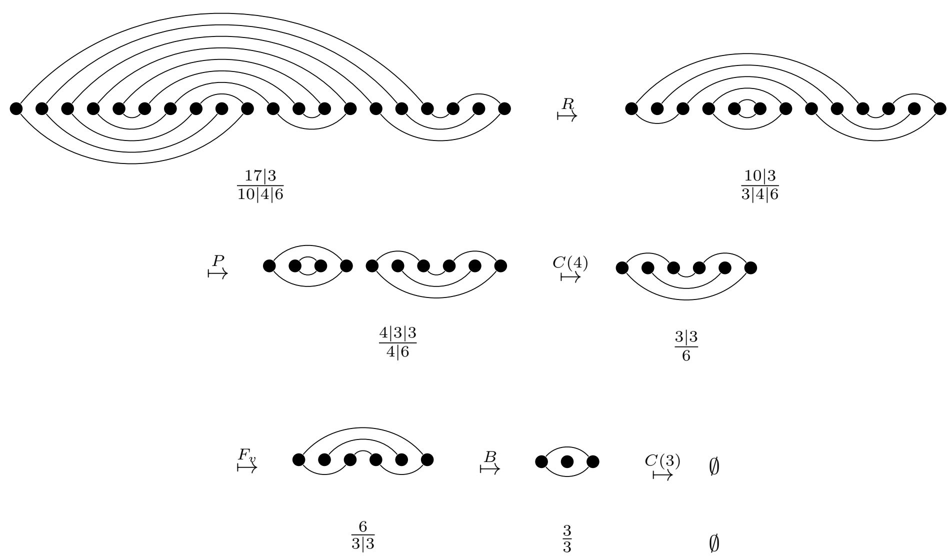

Figure 3: Winding down the meander $\dfrac{17|3}{10|4|6}$

In what follows, it is helpful to add a fifth index preserving transformation, F_h, called a *horizontal flip*, which takes the meander of type $\dfrac{a_1|a_2|...|a_m}{b_1|b_2|...|b_t}$ to the meander of type $\dfrac{a_m|...|a_2|a_1}{b_t|...|b_2|b_1}$.

One combination of the above graph-theoretic moves is applied often enough below to warrant the addition of notation. Let M be the meander of type $\frac{a_1|a_2|\cdots|a_m}{n}$.

Define

$$UF(i,M)=\begin{cases}(F_h)(F_v)(P)^i(F_v)(F_h), & i=1 \text{ and } n>2a_m\\ (F_h)(F_v)(P)^{i-1}(B)(F_v)(F_h), & i=1 \text{ and } n=2a_m\\ (F_h)(F_v)(P)^i(F_v)(F_h), & i>1 \text{ and } n-2\sum_{j=0}^{i-2}a_{m-j}>2a_{m-i+1}\\ (F_h)(F_v)(P)^{i-1}(B)(F_v)(F_h), & i>1 \text{ and } n-2\sum_{j=0}^{i-2}a_{m-j}=2a_{m-i+1}\\ \text{undefined}, & \text{otherwise},\end{cases}$$

where F_v is applied regardless of whether, in the notation of Lemma 1, $a_1 < b_1$ or not.

Example 5. Applying $UF(3,M)$ to the meander M of type $\frac{3|3|3|2|2|2|2|2}{19}$ results in the meander of type $\frac{3|3|3|2|2}{2|2|2|7}$. See Figure 4 below.

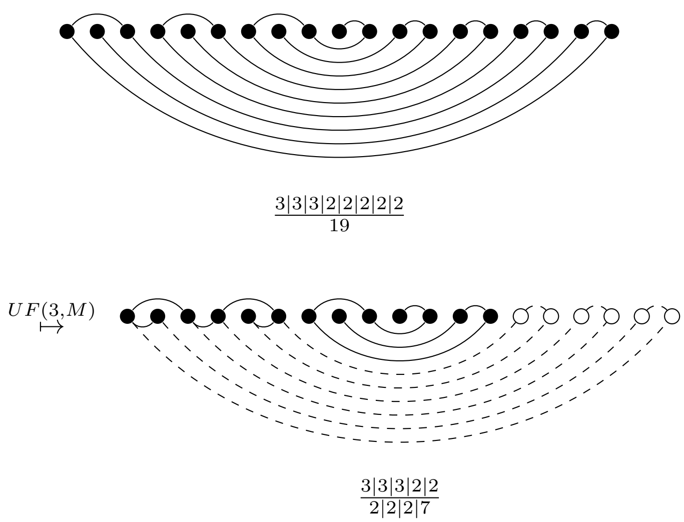

Figure 4: Applying $UF(3,M)$ to the meander M of type $\frac{3|3|3|2|2|2|2|2}{19}$

2.2. Integer Partitions

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_m)$ of a positive integer n is a finite sequence of weakly decreasing positive integers such that $n = \sum_{i=1}^{m} \lambda_i$. The λ_i are called the *parts* of the partition and $w(\lambda) = n$ is the *weight* of the partition. We often employ the notation for partitions that makes explicit the number of times a particular integer occurs as a part of a partition; that is, if $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition, we write

$$\lambda = 1^{f_1}2^{f_2}3^{f_3}\cdots,$$

where exactly f_i of the λ_j are equal to i. As noted in the introduction, for $S \subset \mathbb{Z}_{>0}$ we let $\mathcal{P}(n, S)$ denote the collection of partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $w(\lambda) = n$ and $\lambda_i \in S$ for $1 \leq i \leq m$.

An integer partition *statistic* is a function from the collection of all partitions into $\mathbb{Z}_{\geq 0}$. As noted in [3], the index theory of seaweed algebras provides a natural framework for generating integer partition statistics. In this paper, we define the index of a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ as follows. If $\mathfrak{g}$ is the seaweed algebra of type $\frac{\lambda_1|\cdots|\lambda_m}{\sum_{i=1}^{m} \lambda_i}$, then

$$\text{ind}(\lambda) = \text{ind}(\mathfrak{g}).$$

3. Fixed Index Periodicity

In this section, we show that the number of partitions $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ of fixed index $i \geq 0$ is eventually periodic for $d > 1$ odd. In particular, we prove the following.

Theorem 2. *For $d > 1$ odd and $i \geq 0$, the sequence of values*

$$\{|\{\lambda \in \mathcal{P}(n, \{1, 2, d\}) \mid \text{ind}(\lambda) = i\}|\}_{n \geq k}$$

is periodic if and only if $k \geq 4(i + 1)d + 1$ with

$$|\{\lambda \in \mathcal{P}(n, \{1, 2, d\}) \mid \text{ind}(\lambda) = i\}| = \begin{cases} 2(i+1), & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

for $n \geq 4(i + 1)d + 1$.

Example 6. For $d = 3$ and $i = 0$ we have

$$\{|\{\lambda \in \mathcal{P}(n, \{1, 2, 3\}) \mid \text{ind}(\lambda) = 0\}|\}_{n \geq 13} = 2, 0, 2, 0, 2, 0, \ldots$$

and $|\{\lambda \in \mathcal{P}(12, \{1, 2, 3\}) \mid \text{ind}(\lambda) = 0\}| = 1$.

Remark 1. Considering a subset of the cases involved in the proof of Theorem 2, it follows that the sequence of values

$$\{|\{\lambda \in \mathcal{P}(n, \{1, 2\}) \mid \text{ind}(\lambda) = i\}|\}_{n \geq k}$$

is periodic if and only if $k \geq 4(i + 1) + 1$ with

$$|\{\lambda \in \mathcal{P}(n, \{1, 2\}) \mid \text{ind}(\lambda) = i\}| = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

for $n \geq 4(i + 1) + 1$.

Remark 2. Experimental evidence suggests that a similar, but more complicated, result holds for the values $|\{\lambda \in \mathcal{P}(n, \{1, 2, d\}) \mid \operatorname{ind}(\lambda) = i\}|$ with $d > 2$ even. In particular, it appears that the corresponding sequence is eventually periodic with period d. For example, taking $d = 8$ and $i = 0$, the resulting sequence appears to be periodic for $n \geq 4$ with the sequence of values $1, 1, 0, 1, 0, 2, 1, 2$.

The following lemmas will prove helpful in establishing Theorem 2.

Lemma 2 ([2, Theorem 10])**.** *Let M be the meander of type $\frac{a|a|\ldots|a|b}{n}$. If a is even and $\gcd(a, b) = 1$, then M consists of a single path.*

Lemma 3. *Let M be the meander of type $\frac{a_1|\ldots|a_m}{\sum_{j=1}^{m} a_j}$. If*

$$|\{a_j \mid a_j \text{ is odd}\}| > 2i + 2,$$

then M contains more than $i + 1$ paths.

Proof. Each odd a_j contributes a vertex of degree one to M, i.e., contributes an endpoint of a path. Thus, if $|\{a_j \mid a_j \text{ is odd}\}| > 2i + 2$, then there must be more than $i + 1$ paths in M. $\square$

Lemma 4. *Let $d > 1$ be odd. If M is the meander of type $\dfrac{d|d}{2|\cdots|2}$, then M consists of a single path.*

Proof. Recall that the vertices of M are labeled from left to right by $v_1, v_2, \ldots, v_{2d}$. Thus, vertex v_{2i-1} is adjacent to vertex v_{2i} by a bottom edge in M for $1 \leq i \leq d$. Removing the bottom edge connecting v_d to v_{d+1} results in the meander M' of type

$$\frac{d|d}{2|\cdots|2|1|1|2|\cdots|2}.$$

Evidently, M' consists of two smaller meanders: one meander M_1 of type $\dfrac{d}{2|\cdots|2|1}$ and one meander M_2 of type $\dfrac{d}{1|2|\cdots|2}$. Considering Lemma 2, we conclude that the meanders M_1 and M_2 each consist of a single path. Since the bottom edge from v_d to v_{d+1} in M connects a vertex of degree one in M_1 to a vertex of degree one in M_2, the result follows. $\square$

Lemma 5. *Let $k > 0$. If M is the meander of type $\dfrac{2|\cdots|2}{2k}$, then M consists of $\left\lceil \frac{k}{2} \right\rceil$ cycles.*

Proof. By induction on k. If $k = 1$ or 2, then it is clear that M consists of a single cycle. Assume the result holds for $k - 1 \geq 2$. Applying $(F_v)(P)(F_v)$ to M results in the meander M' of type $\dfrac{2|\cdots|2|2}{2(k-2)|2}$. Evidently, M' consists of two smaller meanders: one meander M_1 of type $\dfrac{2|\cdots|2}{2(k-2)}$ and one meander M_2 of type $\dfrac{2}{2}$. Applying our inductive hypothesis, M_1 consists of $\lceil \frac{k-2}{2} \rceil$ cycles and M_2 consists of one cycle. The result follows. $\square$

3.1. Proof of Theorem 2

In Propositions 1 and 2 below we establish an exact value and a lower bound, respectively, for the index of certain partitions $\lambda = 1^{f_1}2^{f_2}d^{f_d}$, assuming $w(\lambda)$ is "big enough." Throughout this section, we assume that $d > 1$ is a fixed odd integer.

Proposition 1. *Let* $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ *with* $f_d + f_1 = 2i + 1$ *for* $i \geq 0$. *If* $w(\lambda) \geq 4(i+1)d + 1$, *then* $\operatorname{ind}(\lambda) = i$.

Proof. There are 3 cases.

`Case 1:` $f_d = 0$, $f_1 = 2i + 1$. Let M be the meander of type

$$\frac{\overbrace{2|\cdots|2}^{f_2}\,|\,\overbrace{1|\cdots|1}^{f_1}}{w(\lambda)}.$$

Form a new meander M' by adding $\frac{f_1-1}{2}$ top edges to M connecting the vertices v_{2i-1} and v_{2i} for $\frac{2\cdot f_2+2}{2} \leq i \leq \frac{w(\lambda)-1}{2}$. Note that M' is the meander of type

$$\frac{\overbrace{2|\cdots|2}^{f_2+\frac{f_1-1}{2}}\,|1}{w(\lambda)}.$$

Considering Lemma 2, we conclude that M' consists of a single path. Now, removing each of the $\frac{f_1-1}{2}$ top edges added in forming M' from M breaks one path into two, i.e., each edge removal increases the number of paths by one. Therefore, M consists of

$$1 + \frac{f_1 - 1}{2} = i + 1$$

paths. Applying Theorem 4, the result follows.

`Case 2:` $f_d > 0$ even, f_1 odd. Let M be the meander of type

$$\frac{\overbrace{d|\cdots|d}^{f_d}\,|\,\overbrace{2|\cdots|2}^{f_2}\,|\,\overbrace{1|\cdots|1}^{f_1}}{w(\lambda)}.$$

Form a new meander M' by adding $\frac{f_1-1}{2}$ top edges to M connecting the vertices v_{2i-1} and v_{2i} for $\frac{d\cdot f_d+2\cdot f_2+2}{2} \leq i \leq \frac{w(\lambda)-1}{2}$. Note that M' is the meander of type

$$\frac{a_1|a_2|\dots|a_m}{w(\lambda)} = \frac{\overbrace{d|\cdots|d}^{f_d}|\overbrace{2|\cdots|2}^{f_2+\frac{f_1-1}{2}}|1}{w(\lambda)}.$$

Since

$$f_2 + \frac{f_1-1}{2} = \frac{w(\lambda) - d\cdot f_d - 1}{2} > \frac{4(i+1)d + 1 - (2i+1)d - 1}{2} = \frac{(2i+3)d}{2}$$
$$> \frac{(2i+1)d+1}{2} > \frac{(f_d-1)d+1}{2},$$

it follows that $a_{m-j} = 2$ for $0 < j \leq \frac{(f_d-1)d-1}{2}$. Consequently,

$$\begin{aligned} w(\lambda) - 2\sum_{j=0}^{\frac{(f_d-1)d+1}{2}-2} a_{m-j} &= w(\lambda) - 2\left[2\left(\frac{(f_d-1)d+1}{2} - 2\right)\right] - 2(1) \\ &= w(\lambda) - 4\left(\frac{(f_d-1)d+1}{2} - 2\right) - 2 \\ &\geq 4(i+1)d + 5 - 2(f_d-1)d > 4(i+1)d + 5 - 2(2i)d \\ &= 4d + 5 > 4 = 2a_{m-\left(\frac{(f_d-1)d-1}{2}\right)} \end{aligned}$$

so that $UF\left(\frac{(f_d-1)d+1}{2}, M'\right) = (F_h)(F_v)(P)^{\frac{(f_d-1)d+1}{2}}(F_v)(F_h)$. Thus, since

$$f_2 + \frac{f_1-1}{2} + 1 > \frac{(f_d-1)d+1}{2},$$

applying $UF\left(\frac{(f_d-1)d+1}{2}, M'\right)$ to M' results in the meander M'' of type

$$\frac{\overbrace{d|d|d|\cdots|d|d|d}^{f_d}|\overbrace{2|\cdots|2}^{\frac{w(\lambda)-2d\cdot f_d+d}{2}}}{1|\underbrace{2|\cdots|2}_{\frac{d-1}{2}}|\underbrace{\underbrace{2|\cdots|2}_{d}|\cdots|\underbrace{2|\cdots|2}_{d}}_{\frac{f_d-2}{2}}|w(\lambda)-2d\cdot f_d+2d},$$

where $\frac{w(\lambda)-2d\cdot f_d+d}{2} > 0$. Evidently, M'' consists of $\frac{f_d+2}{2}$ smaller meanders: one meander of type $\frac{d}{1|2|\dots|2}$, $\frac{f_d-2}{2}$ meanders or type $\frac{d|d}{2|\dots|2}$, and one meander of type $\frac{d|2|\dots|2}{w(\lambda)-2d\cdot f_d+2d}$. Applying Lemmas 2 and 4, it follows that M'' consists of $\frac{f_d+2}{2}$ paths, i.e., M' consists of $\frac{f_d+2}{2}$ paths. Now, as in Case 1, removing each of the $\frac{f_1-1}{2}$

top edges added in forming M' from M breaks one path into two. Therefore, M consists of

$$\frac{f_d+2}{2}+\frac{f_1-1}{2}=\frac{f_d+f_1+1}{2}=\frac{2i+2}{2}=i+1$$

paths. Applying Theorem 4, the result follows.

Case 3: f_d odd, f_1 even. Let M be the meander of type

$$\frac{\overbrace{d|\cdots|d}^{f_d}\,|\,\overbrace{2|\cdots|2}^{f_2}\,|\,\overbrace{1|\cdots|1}^{f_1}}{w(\lambda)}.$$

Form a new meander M' by adding $\frac{f_1}{2}$ top edges to M connecting the vertices v_{2i} and v_{2i+1} for $\frac{d\cdot f_d+2\cdot f_2+1}{2}\leq i\leq\frac{w(\lambda)-1}{2}$. Note that M' is the meander of type

$$\frac{a_1|a_2|\ldots|a_m}{w(\lambda)}=\frac{\overbrace{d|\cdots|d}^{f_d}\,|\,\overbrace{2|\cdots|2}^{f_2+\frac{f_1}{2}}}{w(\lambda)}.$$

Since

$$\begin{aligned}f_2+\frac{f_1}{2}&=\frac{w(\lambda)-d\cdot f_d}{2}\geq\frac{4(i+1)d+1-(2i+1)d}{2}=\frac{(2i+3)d+1}{2}\\&>\frac{(2i+1)d+1}{2}>\frac{(f_d-1)d}{2},\end{aligned}$$

it follows that $a_{m-j}=2$ for $0\leq j\leq\frac{(f_d-1)d}{2}-1$. Consequently,

$$\begin{aligned}w(\lambda)-2\sum_{j=0}^{\frac{(f_d-1)d}{2}-2}a_{m-j}&=w(\lambda)-2\left[2\left(\frac{(f_d-1)d}{2}-1\right)\right]\\&=w(\lambda)-4\left(\frac{(f_d-1)d}{2}-1\right)\\&\geq 4(i+1)d+5-2(f_d-1)d\geq 4(i+1)d+5-2(2i)d\\&=4d+5>4=2a_{m-\left(\frac{(f_d-1)d}{2}-1\right)}\end{aligned}$$

so that $UF\left(\frac{(f_d-1)d}{2},M'\right)=(F_h)(F_v)(P)^{\frac{(f_d-1)d}{2}}(F_v)(F_h)$. Thus, since

$$f_2+\frac{f_1}{2}>\frac{(f_d-1)d}{2},$$

applying $UF\left(\frac{(f_d-1)d}{2}, M'\right)$ to M' results in the meander M'' of type

$$\frac{\overbrace{d|d|\cdots|d|d|d}^{f_d}|\quad\overbrace{2|\cdots|2}^{\frac{w(\lambda)-2(f_d-1)d-d}{2}}}{\underbrace{\underbrace{2|\cdots|2}_{d}|\cdots|\underbrace{2|\cdots|2}_{d}}_{\frac{f_d-1}{2}}|w(\lambda)-2(f_d-1)d},$$

where $\frac{w(\lambda)-2(f_d-1)d-d}{2} > 0$. Evidently, M'' consists of $\frac{f_d+1}{2}$ smaller meanders: $\frac{f_d-1}{2}$ meanders of type $\frac{d|d}{2|\ldots|2}$ and one meander of type $\frac{d|2|\ldots|2}{w(\lambda)-2(f_d-1)d}$. Applying Lemmas 2 and 4, it follows that M'' consists of $\frac{f_d+1}{2}$ paths, i.e., M' consists of $\frac{f_d+1}{2}$ paths. Now, as in Case 1, removing each of the $\frac{f_1}{2}$ top edges added in forming M' from M breaks one path into two. Therefore, M consists of

$$\frac{f_d+1}{2}+\frac{f_1}{2}=\frac{f_d+f_1+1}{2}=\frac{2i+2}{2}=i+1$$

paths. Applying Theorem 4, the result follows. □

Proposition 2. *Let* $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ *with* $f_d + f_1 = 2i + 2$ *for* $i \geq 0$*. If* $w(\lambda) = 4(i+1)d + 2k$ *for* $k > 0$*, then* $\text{ind}(\lambda) > i + k$*.*

Proof. There are 5 cases.

`Case 1:` $f_d = 0$, $f_1 = 2i + 2$. Let M be the meander of type

$$\frac{a_1|a_2|\cdots|a_m}{w(\lambda)}=\frac{\overbrace{2|\cdots|2}^{f_2}|\overbrace{1|\cdots|1}^{f_1}}{w(\lambda)}.$$

Since

$$\begin{aligned}w(\lambda)-2\sum_{j=0}^{f_1-2}a_{m-j} &= w(\lambda)-2(f_1-1)=4(i+1)d+2k-4i-2\\ &> 4(i+1)+2k-4i-2=2k+2>2=2a_{m-(f_1-1)},\end{aligned}$$

it follows that $UF(f_1, M) = (F_h)(F_v)(P)^{f_1}(F_v)(F_h)$. Thus, since

$$\begin{aligned}f_2=\frac{w(\lambda)-f_1}{2} &= \frac{4(i+1)d+2k-(2i+2)}{2}>\frac{4(i+1)+2k-(2i+2)}{2}\\ &=\frac{2i+2+2k}{2}>\frac{2i+2}{2}=\frac{f_1}{2},\end{aligned}$$

applying $UF(f_1, M)$ to M results in the meander M' of type

$$\frac{\overbrace{2|\cdots|2}^{\frac{f_1}{2}=i+1}|\overbrace{2|\cdots|2}^{\frac{w(\lambda)-2\cdot f_1}{2}}}{\underbrace{\underbrace{1|1}_{2}|\cdots|\underbrace{1|1}_{2}}_{\frac{f_1}{2}=i+1}|w(\lambda)-2\cdot f_1},$$

where $\frac{w(\lambda)-2\cdot f_1}{2} > 0$. Evidently, M' consists of $i+2$ smaller meanders: $i+1$ meanders of type $\frac{2}{1|1}$ and one meander of type $\frac{2|\cdots|2}{w(\lambda)-2\cdot f_1}$. Clearly, the meander of type $\frac{2}{1|1}$ consists of a single path. Applying Lemma 5, we conclude that the meander of type $\frac{2|\cdots|2}{w(\lambda)-2\cdot f_1}$ consists of $\left\lceil \frac{w(\lambda)-2\cdot f_1}{4} \right\rceil$ cycles. Since

$$\frac{w(\lambda)-2\cdot f_1}{4} = \frac{4(i+1)d+2k-2(2i+2)}{4} > \frac{4(i+1)+2k-2(2i+2)}{4} = \frac{k}{2},$$

it follows that the meander of type $\frac{2|\cdots|2}{w(\lambda)-2\cdot f_1}$ consists of more than $\left\lceil \frac{k}{2} \right\rceil$ cycles. Hence, putting everything together, M consists of $i+1$ paths and more than $\left\lceil \frac{k}{2} \right\rceil$ cycles. Applying Theorem 4, the result follows.

`Case 2:` $f_d = 2i+2$, $f_1 = 0$. Let M be the meander of type

$$\frac{a_1|a_2|\cdots|a_m}{w(\lambda)} = \frac{\overbrace{d|\cdots|d}^{f_d}|\overbrace{2|\cdots|2}^{f_2}}{w(\lambda)}.$$

Since

$$f_2 = \frac{w(\lambda)-d\cdot f_d}{2} = \frac{w(\lambda)-d(2i+2)}{2} = \frac{4(i+1)d+2k-(2i+2)d}{2} = (i+1)d+k,$$

it follows that $a_{m-j} = 2$ for $0 \le j \le (i+1)d-1$. Consequently,

$$\begin{aligned} w(\lambda) - 2\sum_{j=0}^{(i+1)d-2} a_{m-j} &= w(\lambda) - 2[2((i+1)d-1)] = w(\lambda) - 4[(i+1)d-1] \\ &= 4(i+1)d + 2k - 4(i+1)d + 4 = 2k+4 \\ &> 4 = 2a_{m-[(i+1)d-1]} \end{aligned}$$

so that $UF((i+1)d, M) = (F_h)(F_v)(P)^{(i+1)d}(F_v)(F_h)$. Thus, applying

$UF((i+1)d, M)$ to M results in the meander M' of type

$$\frac{\overbrace{d|d|\cdots|d|d|d}^{f_d}|\overbrace{2|\cdots|2}^{\frac{w(\lambda)-4(i+1)d}{2}=k}}{\underbrace{\underbrace{2|\cdots|2}_{d}|\cdots|\underbrace{2|\cdots|2}_{d}}_{\frac{f_d}{2}=i+1}|w(\lambda)-4(i+1)d=2k}.$$

Evidently, M' consists of $i+2$ smaller meanders: $i+1$ meanders of type $\frac{d|d}{2|\ldots|2}$ and one meander of type $\frac{2|\ldots|2}{2k}$. Applying Lemmas 4 and 5, it follows that the meander of type $\frac{d|d}{2|\ldots|2}$ consists of a single path and the meander of type $\frac{2|\ldots|2}{2k}$ consists of $\lceil \frac{k}{2} \rceil$ cycles, respectively. Hence, putting everything together, M must consist of $i+1$ paths and $\lceil \frac{k}{2} \rceil$ cycles. Applying Theorem 4, the result follows.

`Case 3:` $f_1 < d \cdot f_d$ with $0 < f_1, f_d$ even. Let M be the meander of type

$$\frac{\overbrace{d|\cdots|d}^{f_d}|\overbrace{2|\cdots|2}^{f_2}|\overbrace{1|\cdots|1}^{f_1}}{w(\lambda)}.$$

Form a new meander M' by adding $\frac{f_1}{2}$ top edges to M connecting the vertices v_{2i-1} and v_{2i} for $\frac{d \cdot f_d + 2 \cdot f_2 + 2}{2} \leq i \leq \frac{w(\lambda)}{2}$. Note that M' is the meander of type

$$\frac{a_1|a_2|\cdots|a_m}{w(\lambda)} = \frac{\overbrace{d|\cdots|d}^{f_d}|\overbrace{2|\cdots|2}^{f_2+\frac{f_1}{2}}}{w(\lambda)}.$$

Since

$$\begin{aligned} f_2 + \frac{f_1}{2} &= \frac{w(\lambda) - d \cdot f_d}{2} > \frac{4(i+1)d + 2k - d(2i+2)}{2} = \frac{2(i+1)d + 2k}{2} \\ &> \frac{(2i+2)d}{2} > \frac{d \cdot f_d}{2}, \end{aligned}$$

it follows that $a_{m-j} = 2$ for $0 \leq j \leq \frac{d \cdot f_d}{2} - 1$. Consequently,

$$\begin{aligned} w(\lambda) - 2\sum_{j=0}^{\frac{d \cdot f_d}{2}-2} a_{m-j} &= w(\lambda) - 2\left[2\left(\frac{d \cdot f_d}{2} - 1\right)\right] = w(\lambda) - 4\left(\frac{d \cdot f_d}{2} - 1\right) \\ &= 4(i+1)d + 2k - 2d \cdot f_d + 4 \\ &> 4(i+1)d + 2k - 2d(2i+2) + 4 = 2k + 4 \\ &> 4 = 2a_{m-\left(\frac{d \cdot f_d}{2} - 1\right)} \end{aligned}$$

so that $UF\left(\frac{d\cdot f_d}{2}, M'\right) = (F_h)(F_v)(P)^{\frac{d\cdot f_d}{2}}(F_v)(F_h)$. Thus, since $f_2 + \frac{f_1}{2} > \frac{d\cdot f_d}{2}$, it follows that applying $UF\left(\frac{d\cdot f_d}{2}, M'\right)$ to M' results in the meander M'' of type

$$\frac{\overbrace{d|d|\cdots|d|d}^{f_d}|\overbrace{2|\cdots|2}^{\frac{w(\lambda)-2d\cdot f_d}{2}}}{\underbrace{\underbrace{2|\cdots|2}_{d}|\cdots|\underbrace{2|\cdots|2}_{d}}_{\frac{f_d}{2}}|w(\lambda)-2d\cdot f_d},$$

where $\frac{w(\lambda)-2d\cdot f_d}{2} > 0$. Evidently, M'' consists of $\frac{f_d}{2}+1$ smaller meanders: $\frac{f_d}{2}$ meanders of type $\frac{d|d}{2|\cdots|2}$ and one meander of type $\frac{2|\cdots|2}{w(\lambda)-2d\cdot f_d}$. Applying Lemmas 4 and 5, we conclude that the meander of type $\frac{d|d}{2|\cdots|2}$ consists of a single path and the meander of type $\frac{2|\cdots|2}{w(\lambda)-2d\cdot f_d}$ consists of $\left\lceil\frac{w(\lambda)-2d\cdot f_d}{4}\right\rceil$ cycles, respectively. Since

$$\frac{w(\lambda)-2d\cdot f_d}{4} > \frac{4(i+1)d+2k-2d(2i+2)}{4} = \frac{k}{2},$$

it follows that the meander of type $\frac{2|\cdots|2}{w(\lambda)-2d\cdot f_d}$ consists of more than $\left\lceil\frac{k}{2}\right\rceil$ cycles. Hence, putting everything together, M' consists of $\frac{f_d}{2}$ paths and more than $\left\lceil\frac{k}{2}\right\rceil$ cycles. Now, since $d\cdot f_d > f_1$, it follows that each of the $\frac{f_1}{2}$ top edges added in forming M' from M belongs to a path corresponding to one of the meanders of type $\frac{d|d}{2|\cdots|2}$ in M''. Consequently, removing each of the $\frac{f_1}{2}$ top edges added in forming M' from M breaks one path into two. Therefore, M consists of

$$\frac{f_d}{2}+\frac{f_1}{2} = \frac{2i+2}{2} = i+1$$

paths and more than $\left\lceil\frac{k}{2}\right\rceil$ cycles. Applying Theorem 4, the result follows.

`Case 4:` $f_1 < d\cdot f_d$ with $0 < f_1, f_d$ odd. Let M be the meander of type

$$\frac{\overbrace{d|\cdots|d}^{f_d}|\overbrace{2|\cdots|2}^{f_2}|\overbrace{1|\cdots|1}^{f_1}}{w(\lambda)}.$$

Form a new meander M' by adding $\frac{f_1-1}{2}$ top edges to M connecting the vertices v_{2i} and v_{2i+1} for $\frac{d\cdot f_d+2\cdot f_2+1}{2} \le i \le \frac{w(\lambda)-2}{2}$. Note that M' is the meander of type

$$\frac{a_1|a_2|\cdots|a_m}{w(\lambda)} = \frac{\overbrace{d|\cdots|d}^{f_d}|\overbrace{2|\cdots|2}^{f_2+\frac{f_1-1}{2}}|1}{w(\lambda)}.$$

Since

$$f_2 + \frac{f_1 - 1}{2} = \frac{w(\lambda) - d \cdot f_d - 1}{2} \geq \frac{4(i+1)d + 2k - d(2i+1) - 1}{2}$$
$$= \frac{(2i+3)d + 2k - 1}{2} \geq \frac{(2i+3)d + 1}{2} > \frac{d \cdot f_d + 1}{2},$$

it follows that $a_{m-j} = 2$ for $0 < j \leq \frac{d \cdot f_d + 1}{2} - 1$. Consequently,

$$\begin{aligned} w(\lambda) - 2\sum_{j=0}^{\frac{d \cdot f_d+1}{2}-2} a_{m-j} &= w(\lambda) - 2\left[2\left(\frac{d \cdot f_d + 1}{2} - 2\right)\right] - 2(1) \\ &= w(\lambda) - 4\left(\frac{d \cdot f_d + 1}{2} - 2\right) - 2 \\ &= w(\lambda) - 2d \cdot f_d + 4 = 4(i+1)d + 2k - 2d \cdot f_d + 4 \\ &> 4(i+1)d + 2k - 2d(2i+2) + 4 = 2k + 4 \\ &> 4 = 2a_{m-\left(\frac{d(f_d+1)}{2}-1\right)} \end{aligned}$$

so that $UF\left(\frac{d \cdot f_d+1}{2}, M'\right) = (F_h)(F_v)(P)^{\frac{d \cdot f_d+1}{2}}(F_v)(F_h)$. Thus, since

$$f_2 + \frac{f_1 - 1}{2} + 1 > \frac{d \cdot f_d + 1}{2},$$

applying $UF\left(\frac{d \cdot f_d+1}{2}, M'\right)$ to M' results in the meander M'' of type

$$\frac{\overbrace{d|d|d|\cdots|d|d}^{f_d}\,|\,\overbrace{2|\cdots|2}^{\frac{w(\lambda)-2d\cdot f_d}{2}}}{1|2|\cdots|2|\underbrace{\underbrace{2|\cdots|2}_{d}|\cdots|\underbrace{2|\cdots|2}_{d}}_{\frac{f_d-1}{2}}|w(\lambda) - 2d \cdot f_d},$$

where $\frac{w(\lambda)-2d\cdot f_d}{2} > 0$. Evidently, M'' consists of $\frac{f_d+3}{2}$ smaller meanders: one meander of type $\frac{d}{1|2|\cdots|2}$, $\frac{f_d-1}{2}$ meanders of type $\frac{d|d}{2|\cdots|2}$, and one meander of type $\frac{2|\cdots|2}{w(\lambda)-2d\cdot f_d}$. Applying Lemmas 2 and 4, we conclude that the meanders of types $\frac{d}{1|2|\cdots|2}$ and $\frac{d|d}{2|\cdots|2}$, respectively, consist of a single path. Since

$$\frac{w(\lambda) - 2d \cdot f_d}{4} = \frac{4(i+1)d + 2k - 2d \cdot f_d}{4} > \frac{4(i+1)d + 2k - 2d(2i+2)}{4} = \frac{k}{2},$$

applying Lemma 5, it follows that the meander of type $\frac{2|\cdots|2}{w(\lambda)-2d\cdot f_d}$ consists of more than $\lceil \frac{k}{2} \rceil$ cycles. Hence, putting everything together, M' consists of $\frac{f_d-1}{2} + 1 = \frac{f_d+1}{2}$

paths and more than $\left\lceil \frac{k}{2} \right\rceil$ cycles. Now, since $d \cdot f_d > f_1$, it follows that each of the $\frac{f_1-1}{2}$ top edges added in forming M' from M belongs to a path corresponding to one of the meanders of type $\frac{d}{1|2|\cdots|2}$ or $\frac{d|d}{2|\cdots|2}$ in M''. Consequently, removing each of the $\frac{f_1-1}{2}$ top edges added in forming M' from M breaks one path into two. Therefore, M consists of

$$\frac{f_d+1}{2} + \frac{f_1-1}{2} = \frac{f_d+f_1}{2} = \frac{2i+2}{2} = i+1$$

paths and more than $\left\lceil \frac{k}{2} \right\rceil$ cycles. Applying Theorem 4, the result follows.

Case 5: $f_1 \geq d \cdot f_d > 0$. Let M be the meander of type

$$\frac{\overbrace{d|\cdots|d}^{f_d}\,|\,\overbrace{2|\cdots|2}^{f_2}\,|\,\overbrace{1|\cdots|1}^{f_1}}{w(\lambda)}.$$

Since

$$\begin{aligned} w(\lambda) - 2\sum_{j=0}^{f_1-2} a_{m-j} &= w(\lambda) - 2(f_1-1) > 4(i+1)d + 2k - 2(2i+1) \\ &> 4(i+1) + 2k - 2(2i+1) = 2 + 2k > 2 = 2a_{m-(f_1-1)}, \end{aligned}$$

it follows that $UF(f_1, M) = (F_h)(F_v)(P)^{f_1}(F_v)(F_h)$. Thus, since

$$\begin{aligned} w(\lambda) - 2 \cdot f_1 &= 4(i+1)d + 2k - 2 \cdot f_1 > 4(i+1)d + 2k - 2(2i+2) \\ &> 4(i+1) + 2k - 2(2i+2) = 2k > 0 \end{aligned}$$

and $f_1 \geq d \cdot f_d$, applying $UF(f_1, M)$ to M results in the meander M' of type

$$\frac{\overbrace{d|\cdots|d}^{f_d}\,|\,\overbrace{2|\cdots|2}^{f_2}}{\underbrace{\underbrace{1|\cdots|1}_{d}\,|\cdots|\,\underbrace{1|\cdots|1}_{d}}_{f_d}\,|\,\underbrace{\underbrace{1|1}_{2}\,|\cdots|\,\underbrace{1|1}_{2}}_{\frac{f_1-d\cdot f_d}{2}}\,|w(\lambda)-2\cdot f_1},$$

where $\frac{f_1-d\cdot f_d}{2} \geq 0$ and $w(\lambda) - 2 \cdot f_1 > 0$. Evidently, M' consists of $\frac{f_1-f_d(d-2)+2}{2}$ smaller meanders: f_d meanders of type $\frac{d}{1|\cdots|1}$, $\frac{f_1-d\cdot f_d}{2}$ meanders of type $\frac{2}{1|1}$, and one meander of type $\frac{2|\cdots|2}{w(\lambda)-2\cdot f_1}$. Clearly, the meander of type $\frac{2}{1|1}$ consists of one path and the meander of type $\frac{d}{1|\cdots|1}$ consists of $\left\lceil \frac{d}{2} \right\rceil$ paths. Since $\left\lceil \frac{w(\lambda)-2\cdot f_1}{4} \right\rceil > \left\lceil \frac{k}{2} \right\rceil$, applying Lemma 5, it follows that the meander of type $\frac{2|\cdots|2}{w(\lambda)-2\cdot f_1}$ consists of more than $\left\lceil \frac{k}{2} \right\rceil$ cycles. Hence, putting everything together, M consists of

$$f_d\left\lceil \frac{d}{2} \right\rceil + \frac{f_1 - d\cdot f_d}{2} = f_d\frac{d+1}{2} + \frac{f_1-d\cdot f_d}{2} = \frac{f_1+f_d}{2} = \frac{2i+2}{2} = i+1$$

paths and more than $\lceil \frac{k}{2} \rceil$ cycles. Applying Theorem 4, the result follows. $\square$

Using Propositions 1 and 2 above, we now show that the number of partitions $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ of fixed index $i \geq 0$ is periodic assuming that we restrict to those λ with $w(\lambda) \geq 4(i+1)d+1$.

Proposition 3. *Fix $i \geq 0$. If $n \geq 4(i+1)d+1$, then*

$$|\{\lambda \in \mathcal{P}(n,\{1,2,d\}) \mid \operatorname{ind}(\lambda) = i\}| = \begin{cases} 2(i+1), & n \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Combining Theorem 4 and Lemma 3, it follows that any partition λ with $\operatorname{ind}(\lambda) = i$ has at most $2i+2$ odd parts. Take $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ with $2i+2 \geq f_d + f_1$ and $n = w(\lambda) \geq 4(i+1)d+1$. There are 3 cases.

`Case 1:` n odd. In this case, we must have $f_d + f_1$ odd; that is, $f_d + f_1 = 2j+1$ for $0 \leq j \leq i$. Note that

$$4(i+1)d+1 \geq 4(j+1)d+1$$

for $0 \leq j \leq i$. Thus, applying Proposition 1, if $f_d + f_1 = 2j+1$ for $0 \leq j \leq i$, then $\operatorname{ind}(\lambda) = j$. Consequently, if $f_d + f_1$ is odd and $\operatorname{ind}(\lambda) = i$, then $f_d + f_1 = 2i+1$. Evidently, there are $2(i+1)$ partitions $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ of $n = w(\lambda) \geq 4(i+1)d+1$ with $f_d + f_1 = 2i+1$. Therefore,

$$|\{\lambda \in \mathcal{P}(n,\{1,2,d\}) \mid \operatorname{ind}(\lambda) = i\}| = 2(i+1)$$

for $n \geq 4(i+1)d+1$ odd.

`Case 2:` n even, $f_d + f_1 > 0$. Assume that $n = w(\lambda) = 4(i+1)d+2k$ for $k > 0$. In this case, we must have $f_d + f_1$ even; that is, $f_d + f_1 = 2j+2$ for $0 \leq j \leq i$. Note that if $i = j + l$ for $0 \leq l \leq i$, then

$$4(i+1)d+2k = 4(j+l+1)d+2k = 4(j+1)d+2(k+2ld).$$

Applying Proposition 2, it follows that

$$\operatorname{ind}(\lambda) > j+k+2ld = i+k+(2d-1)l > i.$$

Thus, there are no partitions $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ with $w(\lambda) \geq 4(i+1)d+1$, $f_d + f_1 > 0$ even, and $\operatorname{ind}(\lambda) = i$.

`Case 3:` n even, $f_d + f_1 = 0$. Assume that $n = w(\lambda) = 4(i+1)d+2k$ for $k > 0$. In this case, combining Theorem 4 and Lemma 5, it follows that

$$\operatorname{ind}(\lambda) = 2\left\lceil \frac{4(i+1)d+2k}{4} \right\rceil - 1 = 2(i+1)d + 2\left\lceil \frac{k}{2} \right\rceil - 1 > i.$$

Thus, there are no partitions $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ with $w(\lambda) \geq 4(i+1)d+1$ even, $f_d + f_1 = 0$, and $\operatorname{ind}(\lambda) = i$. $\square$

Finally, to finish the proof of Theorem 2, it remains to show that the number of partitions $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ of fixed index $i \geq 0$ is not periodic if we restrict to λ with $w(\lambda) \geq k$ for $k < 4(i+1)d+1$. To do so, considering Proposition 3, in the proposition below we show that $|\{\lambda \in \mathcal{P}(4(i+1)d, \{1,2,d\}) \mid \mathrm{ind}(\lambda) = i\}| = 1$ for $i \geq 0$.

Proposition 4. *If $i \geq 0$, then $|\{\lambda \in \mathcal{P}(4(i+1)d, \{1,2,d\}) \mid \mathrm{ind}(\lambda) = i\}| = 1$.*

Proof. Combining Theorem 4 and Lemma 3, it follows that any partition λ with $\mathrm{ind}(\lambda) = i$ has at most $2i+2$ odd parts. Take $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ with $w(\lambda) = 4(i+1)d$ and $2i+2 \geq f_d + f_1$. Note that since $w(\lambda)$ is even, it follows that $f_d + f_1$ must be even as well. There are 7 cases.

`Case 1:` $f_d = 2i+2$, $f_1 = 0$. Let M be the meander of type

$$\frac{a_1|a_2|\cdots|a_m}{w(\lambda)} = \frac{\overbrace{d|\cdots|d}^{f_d}\,|\,\overbrace{2|\cdots|2}^{f_2}}{w(\lambda)}.$$

Since

$$f_2 = \frac{w(\lambda) - d\cdot f_d}{2} = \frac{w(\lambda) - d(2i+2)}{2} = \frac{4(i+1)d - (2i+2)d}{2} = \frac{2(i+1)d}{2}$$
$$= (i+1)d,$$

it follows that $a_{m-j} = 2$ for $0 \leq j \leq (i+1)d - 1$. Consequently,

$$w(\lambda) - 2\sum_{j=0}^{(i+1)d-2} a_{m-j} = w(\lambda) - 2[2((i+1)d-1)] = w(\lambda) - 4[(i+1)d-1]$$
$$= 4(i+1)d - 4(i+1)d + 4 = 4 = 2a_{m-[(i+1)d-1]}$$

so that $UF((i+1)d, M) = (F_h)(F_v)(P)^{(i+1)d-1}(B)(F_v)(F_h)$. Thus, since $f_2 = (i+1)d$, applying $UF((i+1)d, M)$ to M results in the meander M' of type

$$\frac{\overbrace{d|d|\cdots|d|d|d}^{f_d}}{\underbrace{\underbrace{2|\cdots|2}_{d}\,|\cdots|\,\underbrace{2|\cdots|2}_{d}}_{\frac{f_d}{2}=i+1}}.$$

Evidently, M' consists of $i+1$ smaller meanders: $i+1$ meanders of type $\frac{d|d}{2|\ldots|2}$. Applying Lemma 4, it follows that M consists of $i+1$ paths. Thus, applying Theorem 4, if $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ satisfies $w(\lambda) = 4(i+1)d$, $f_d = 2i+2$, and $f_1 = 0$, then $\mathrm{ind}(\lambda) = i$.

For the remaining 6 cases, no partitions satisfy the given conditions, from which the result follows. As the treatment of these 6 cases is similar to that of those in Proposition 2, we relegate the consideration of the remaining cases to Appendix A. □

4. Parity Periodicity

In this section, for $d > 1$ odd we consider the sequence of differences between the numbers of partitions $\lambda \in \mathcal{P}(n, \{1, 2, d\})$ with $\text{ind}(\lambda)$ even and with $\text{ind}(\lambda)$ odd. In particular, if

$$\mathcal{P}_d^E(n) = |\{\lambda \in \mathcal{P}(n, \{1, 2, d\}) \mid \text{ind}(\lambda) \equiv 0 \ (mod \ 2)\}|$$

and

$$\mathcal{P}_d^O(n) = |\{\lambda \in \mathcal{P}(n, \{1, 2, d\}) \mid \text{ind}(\lambda) \equiv 1 \ (mod \ 2)\}|,$$

then we are interested in the sequence

$$\{\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n)\}_{n \geq 1}.$$

First, given a partition λ, we show that the parity of $\text{ind}(\lambda)$ can be determined by counting the number of parts of λ that are congruent to 2 or 3 modulo 4.

Proposition 5. *Let λ be a partition with $w(\lambda) = n$. If $n \equiv 1, 2 \ (mod \ 4)$, then*

1. $\text{ind}(\lambda) \equiv 1 \ (mod \ 2)$ *if and only if the number of parts of λ that are congruent to 2 or 3 modulo 4 is odd; and*

2. $\text{ind}(\lambda) \equiv 0 \ (mod \ 2)$ *if and only if the number of parts of λ that are congruent to 2 or 3 modulo 4 is even.*

If $n \equiv 0, 3 \ (mod \ 4)$, then

1. $\text{ind}(\lambda) \equiv 1 \ (mod \ 2)$ *if and only if the number of parts of λ that are congruent to 2 or 3 modulo 4 is even; and*

2. $\text{ind}(\lambda) \equiv 0 \ (mod \ 2)$ *if and only if the number of parts of λ that are congruent to 2 or 3 modulo 4 is odd.*

Proof. Let M be the meander of type $\frac{1|\ldots|1}{n}$. Clearly, M consists of $\lceil \frac{n}{2} \rceil$ paths. Thus, by Theorem 4, if $\mathfrak{g}$ is the seaweed algebra of type $\frac{1|\ldots|1}{n}$, then

$$\text{ind}(\mathfrak{g}) = \begin{cases} \frac{n-1}{2}, & n \text{ odd} \\ \frac{n-2}{2}, & n \text{ even,} \end{cases}$$

that is,

$$\mathrm{ind}(\mathfrak{g}) \equiv \begin{cases} 0 \pmod 2, & n \equiv 1,2 \ (mod\ 4) \\ 1 \pmod 2, & n \equiv 0,3 \ (mod\ 4). \end{cases}$$

Now, it remains to consider how replacing blocks of consecutive 1's in the top composition defining the type of M by their sum affects the index of the corresponding seaweed algebra.

Consider the meander N of type

$$\frac{a_1|\cdots|a_n}{\sum_{i=1}^n a_i},$$

where $a_j = 1$ for $j = k, k+1, \ldots, k+p-1$. Form a new meander N' of type

$$\frac{a_1|\cdots|a_{k-1}|p|a_{k+p}|\cdots|a_n}{\sum_{i=1}^n a_i},$$

by adding $\lfloor \frac{p}{2} \rfloor$ top edges joining vertex $v_{i+\sum_{j=1}^{k-1} a_j}$ to vertex $v_{p-i+1+\sum_{j=1}^{k-1} a_j}$ of N for $i = 1, \ldots, \lfloor \frac{p}{2} \rfloor$. Evidently, the addition of each such edge either combines two paths into one, or completes a path into a cycle; that is, by Theorem 4, each of the $\lfloor \frac{p}{2} \rfloor$ edges either increases or decreases the index of the corresponding seaweed algebra by one, changing the parity. Since parts $\equiv 2,3$ $(mod\ 4)$ contribute an odd number of edges and parts $\equiv 0,1$ $(mod\ 4)$ an even number, the result follows. □

Corollary 1. *Let $\lambda = 1^{f_1}2^{f_2} \in \mathcal{P}(n, \{1,2\})$. If $n \equiv 1,2$ $(mod\ 4)$, then*

$$\mathrm{ind}(\lambda) \equiv \begin{cases} 0 \ (mod\ 2), & f_2 \textit{ is even} \\ 1 \ (mod\ 2), & f_2 \textit{ is odd,} \end{cases}$$

and if $n \equiv 0,3$ $(mod\ 4)$, then

$$\mathrm{ind}(\lambda) \equiv \begin{cases} 0 \ (mod\ 2), & f_2 \textit{ is odd} \\ 1 \ (mod\ 2), & f_2 \textit{ is even.} \end{cases}$$

To prove the main result of this section (see Theorem 3 below), we require the following proposition which establishes that the sequence $\{\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n)\}_{n\geq 1}$ is periodic for $d = 1$.

Proposition 6. *If*

$$e_n = |\{\lambda \in \mathcal{P}(n, \{1,2\}) \mid \mathrm{ind}(\lambda) \equiv 0 \ (mod\ 2)\}|$$

and

$$o_n = |\{\lambda \in \mathcal{P}(n, \{1,2\}) \mid \mathrm{ind}(\lambda) \equiv 1 \ (mod\ 2)\}|,$$

then the sequence $\{e_n - o_n\}_{n\geq 1}$ *is periodic with*

$$e_n - o_n = \begin{cases} 1, & n \equiv 1 \ (mod\ 4) \\ 0, & n \equiv 2 \ (mod\ 4) \\ 0, & n \equiv 3 \ (mod\ 4) \\ -1, & n \equiv 0 \ (mod\ 4). \end{cases}$$

Proof. There are 3 cases.

`Case 1:` $n \equiv 1 \pmod 4$. In this case, $n = 4k+1$ for some $k \geq 0$. Note that if $\lambda = 1^{f_1}2^{f_2} \in \mathcal{P}(4k+1, \{1,2\})$, then $0 \leq f_2 \leq 2k$. Thus, there are $k+1$ partitions $\lambda \in \mathcal{P}(4k+1, \{1,2\})$ with f_2 even (including $f_2 = 0$) and k partitions with f_2 odd. Applying Corollary 1, if $n = 4k+1$, then $e_n - o_n = k+1-k = 1$.

`Case 2:` $n \equiv 2, 3 \pmod 4$. Assume that $n \equiv 2 \pmod 4$. In this case, $n = 4k+2$ for some $k \geq 0$. Note that if $\lambda = 1^{f_1}2^{f_2} \in \mathcal{P}(4k+2, \{1,2\})$, then $0 \leq f_2 \leq 2k+1$. Thus, there are $k+1$ partitions $\lambda \in \mathcal{P}(4k+2, \{1,2\})$ with f_2 even (including $f_2 = 0$) and $k+1$ partitions with f_2 odd. Applying Corollary 1, if $n = 4k+2$, then $e_n - o_n = k+1-(k+1) = 0$. The case $n \equiv 3 \pmod 4$ follows via a similar argument.

`Case 3:` $n \equiv 0 \pmod 4$. In this case, $n = 4k$ for some $k \geq 0$. Note that if $\lambda = 1^{f_1}2^{f_2} \in \mathcal{P}(4k, \{1,2\})$, then $0 \leq f_2 \leq 2k$. Thus, there are $k+1$ partitions $\lambda \in \mathcal{P}(4k, \{1,2\})$ with f_2 even (including $f_2 = 0$) and k partitions with f_2 odd. Applying Corollary 1, if $n = 4k$, then $e_n - o_n = k-(k+1) = -1$. ☐

Theorem 3. *If* $d \equiv 3 \ (mod\ 4)$*, then the sequence* $\{\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n)\}_{n\geq 1}$ *is periodic with each period having the following form:*

$$\underbrace{1,0,0,-1,\ldots,1,0,0,-1,1,0}_{d-1}, \underbrace{1,0,1,0,\ldots,1,0,1,0,1}_{d}, \underbrace{1,0,0,1,\ldots,1,0,0,1,1,0,0}_{d}, \underbrace{0,\ldots,0}_{d}, -1.$$

Proof. Set

$$\mathcal{P}_{d,k}(n) = \{1^{f_1}2^{f_2}d^{f_d} \in \mathcal{P}(n, \{1,2,d\}) \mid f_d = k\},$$
$$e_n^k = |\{\lambda \in \mathcal{P}_{d,k}(n) \mid \mathrm{ind}(\lambda) \equiv 0 \pmod 2\}|,$$

and

$$o_n^k = |\{\lambda \in \mathcal{P}_{d,k}(n) \mid \mathrm{ind}(\lambda) \equiv 1 \pmod 2\}|$$

for $k \geq 0$. First, we compute the values $e_n^k - o_n^k$ for $k \geq 0$. To do so, we alter the partitions of $\mathcal{P}_{d,k}(n)$ by removing all parts of size d, resulting in the collection of partitions $\mathcal{P}(n-kd, \{1,2\})$. Note that this defines a bijection $f : \mathcal{P}_{d,k}(n) \to \mathcal{P}(n-kd, \{1,2\})$. Using a combination of Corollary 1 and Propositions 5 and 6, we are then able to compute $e_n^k - o_n^k$.

There are 3 cases. To ease notations, let $(n,k) \equiv (i,j) \pmod m$ denote $n \equiv i \pmod m$ and $k \equiv j \pmod m$.

`Case 1:` $e_n^k - o_n^k$ for $(n,k) \equiv (0,0),(1,3),(2,3)$, and $(3,2) \pmod 4$. In this case, we claim that $e_n^k - o_n^k = -1$. Assume $(n,k) \equiv (0,0) \pmod 4$. Note that $n - kd \equiv 0 \pmod 4$. Applying Corollary 1 and Proposition 6, it follows that

$$|\{1^{f_1}2^{f_2} \in \mathcal{P}(n-kd,\{1,2\}) \mid f_2 \equiv 0 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2} \in \mathcal{P}(n-kd,\{1,2\}) \mid f_2 \equiv 1 \pmod 2\}| + 1.$$

Thus,

$$|\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n) \mid f_2 + k \equiv 0 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n) \mid f_2 + k \equiv 1 \pmod 2\}| + 1.$$

Considering Proposition 5, we conclude that $e_n^k - o_n^k = -1$. The cases $(n,k) \equiv (1,3),(2,3)$, and $(3,2) \pmod 4$ follow via a similar argument.

`Case 2:` $e_n^k - o_n^k$ for $(n,k) \equiv (0,2),(0,3),(1,1),(1,2),(2,0),(2,1),(3,0)$, and $(3,3) \pmod 4$. In this case, we claim that $e_n^k - o_n^k = 0$. Assume $(n,k) \equiv (0,2) \pmod 4$. Note that $n - kd \equiv 2 \pmod 4$. Applying Corollary 1 and Proposition 6, it follows that

$$|\{1^{f_1}2^{f_2} \in \mathcal{P}(n-kd,\{1,2\}) \mid f_2 \equiv 0 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2} \in \mathcal{P}(n-kd,\{1,2\}) \mid f_2 \equiv 1 \pmod 2\}|.$$

Thus,

$$|\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n) \mid f_2 + k \equiv 0 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n) \mid f_2 + k \equiv 1 \pmod 2\}|.$$

Considering Proposition 5, we conclude that $e_n^k - o_n^k = 0$. The cases $(n,k) \equiv (0,3),(1,1),(1,2),(2,0),(2,1),(3,0)$, and $(3,3) \pmod 4$ follow via a similar argument.

`Case 3:` $e_n^k - o_n^k$ for $(n,k) \equiv (0,1),(1,0),(2,2)$, and $(3,1) \pmod 4$. In this case, we claim that $e_n^k - o_n^k = 1$. Assume $(n,k) \equiv (0,1) \pmod 4$. Note that $n - kd \equiv 1 \pmod 4$. Applying Corollary 1 and Proposition 6, it follows that

$$|\{1^{f_1}2^{f_2} \in \mathcal{P}(n-kd,\{1,2\}) \mid f_2 \equiv 0 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2} \in \mathcal{P}(n-kd,\{1,2\}) \mid f_2 \equiv 1 \pmod 2\}| + 1.$$

Thus,

$$|\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n) \mid f_2 + k \equiv 1 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n) \mid f_2 + k \equiv 0 \pmod 2\}| + 1.$$

Considering Proposition 5, we conclude that $e_n^k - o_n^k = 1$. The cases $(n,k) \equiv (1,0),(2,2)$, and $(3,1) \pmod 4$ follow via a similar argument.

Now, if $n = md + i$ for $0 \le i < d$ and $m \ge 0$, then

$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{m} (e_n^j - o_n^j).$$

Note that, if $m \ge 4l-1$ for $l > 0$, then the computations above show that $\sum_{j=0}^{4l-1}(e_n^j - o_n^j) = 0$. Consequently, if $n = md + i$ for $0 \le i < d$ with $m = 4l + k$ for $l \ge 0$ and $0 \le k < 4$, then

$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{m} (e_n^j - o_n^j) = \sum_{j=4l}^{4l+k} (e_n^j - o_n^j) = \sum_{j=0}^{k} (e_n^j - o_n^j).$$

Thus, once again considering the computations above, assuming $n = md + i$ for $0 \le i < d$ with $m = 4l + k$ for $l \ge 0$ and $0 \le k < 4$, it follows that:

- if $n \equiv 1 \pmod 4$ and $m \equiv 0 \pmod 4$, then
$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = e_n^0 - o_n^0 = 1;$$
- if $n \equiv 2 \pmod 4$ and $m \equiv 0 \pmod 4$, then
$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = e_n^0 - o_n^0 = 0;$$
- if $n \equiv 3 \pmod 4$ and $m \equiv 0 \pmod 4$, then
$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = e_n^0 - o_n^0 = 0;$$
- if $n \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 4$, then
$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = e_n^0 - o_n^0 = -1;$$
- if $n \equiv 1 \pmod 4$ and $m \equiv 1 \pmod 4$, then
$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{1} (e_n^j - o_n^j) = 1 + 0 = 1;$$
- if $n \equiv 2 \pmod 4$ and $m \equiv 1 \pmod 4$, then
$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{1} (e_n^j - o_n^j) = 0 + 0 = 0;$$
- if $n \equiv 3 \pmod 4$ and $m \equiv 1 \pmod 4$, then
$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{1} (e_n^j - o_n^j) = 0 + 1 = 1;$$

- if $n \equiv 0 \pmod 4$ and $m \equiv 1 \pmod 4$, then

$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{1}(e_n^j - o_n^j) = -1 + 1 = 0;$$

- if $n \equiv 1 \pmod 4$ and $m \equiv 2 \pmod 4$, then

$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{2}(e_n^j - o_n^j) = 1 + 0 + 0 = 1;$$

- if $n \equiv 2 \pmod 4$ and $m \equiv 2 \pmod 4$, then

$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{2}(e_n^j - o_n^j) = 0 + 0 + 1 = 1;$$

- if $n \equiv 3 \pmod 4$ and $m \equiv 2 \pmod 4$, then

$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{2}(e_n^j - o_n^j) = 0 + 1 - 1 = 0;$$

- if $n \equiv 0 \pmod 4$ and $m \equiv 2 \pmod 4$, then

$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{2}(e_n^j - o_n^j) = -1 + 1 + 0 = 0;$$

- if $n \equiv 1 \pmod 4$ and $m \equiv 3 \pmod 4$, then

$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{3}(e_n^j - o_n^j) = 1 + 0 + 0 - 1 = 0;$$

- if $n \equiv 2 \pmod 4$ and $m \equiv 3 \pmod 4$, then

$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{3}(e_n^j - o_n^j) = 0 + 0 + 1 - 1 = 0;$$

- if $n \equiv 3 \pmod 4$ and $m \equiv 3 \pmod 4$, then

$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{3}(e_n^j - o_n^j) = 0 + 1 - 1 + 0 = 0;$$

- if $n \equiv 0 \pmod 4$ and $m \equiv 3 \pmod 4$, then

$$\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n) = \sum_{j=0}^{3}(e_n^j - o_n^j) = -1 + 1 + 0 + 0 = 0.$$

Therefore, the sequence $\{\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n)\}_{n\geq 1}$ is periodic with each period having the following form:

$$\underbrace{1,0,0,-1,\ldots,1,0,0,-1,1,0}_{d-1}, \underbrace{1,0,1,0,\ldots,1,0,1,0,1}_{d}, \underbrace{1,0,0,1,\ldots,1,0,0,1,1,0,0}_{d}, \underbrace{0,\ldots,0}_{d}, -1.$$

□

Remark 3. Note that if $d = 4m+3$, then the sum of all terms in each period of $\{\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n)\}_{n\geq 1}$ is equal to $1 + (2m+2) + (2m+1) - 1 = 4m+3 = d$.

The following theorem shows that the assumption $d \equiv 3 \pmod 4$ of Theorem 3 is necessary for $d > 1$ odd.

Theorem 5. *If $1 < d \equiv 1 \ (mod\ 4)$, then the sequence $\{\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n)\}_{n\geq 1}$ is not periodic.*

Proof. Let $n(l) = (4l-1)d + 2$ for $l \geq 1$. We show that the sequence

$$\{\mathcal{P}_d^E(n(l)) - \mathcal{P}_d^O(n(l))\}_{l\geq 1}$$

is strictly increasing. Note that $n(l) \equiv 1 \pmod 4$. As in the proof of Theorem 3, set

$$\mathcal{P}_{d,k}(n) = \{1^{f_1}2^{f_2}d^{f_d} \in \mathcal{P}(n,\{1,2,d\}) \mid f_d = k\},$$
$$e_n^k = |\{\lambda \in \mathcal{P}_{d,k}(n) \mid \mathrm{ind}(\lambda) \equiv 0 \pmod 2\}|,$$

and

$$o_n^k = |\{\lambda \in \mathcal{P}_{d,k}(n) \mid \mathrm{ind}(\lambda) \equiv 1 \pmod 2\}|.$$

Mirroring the proof of Theorem 3, we start by calculating the values $e_{n(l)}^k - o_{n(l)}^k$ for $l \geq 1$. As before, this is accomplished by removing all parts of size d from the partitions of $\mathcal{P}_{d,k}(n(l))$ and utilizing Corollary 1 along with Propositions 5 and 6. There are 4 cases.

`Case 1:` $e_{n(l)}^k - o_{n(l)}^k$ for $k \equiv 1 \pmod 4$. In this case, $n(l) - kd \equiv 0 \pmod 4$. Applying Corollary 1 and Proposition 6, it follows that

$$|\{1^{f_1}2^{f_2} \in \mathcal{P}(n(l)-kd,\{1,2\}) \mid f_2 \equiv 0 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2} \in \mathcal{P}(n(l)-kd,\{1,2\}) \mid f_2 \equiv 1 \pmod 2\}| + 1.$$

Thus,

$$|\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n(l)) \mid f_2 \equiv 0 \pmod 2\}|$$

$$= |\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n(l)) \mid f_2 \equiv 1 \pmod 2\}| + 1.$$

Considering Proposition 5, since $d \equiv 1 \pmod 4$, we conclude that $e^k_{n(l)} - o^k_{n(l)} = 1$.

`Case 2:` $e^k_{n(l)} - o^k_{n(l)}$ for $k \equiv 2 \pmod 4$. In this case, $n(l) - kd \equiv 3 \pmod 4$. Applying Corollary 1 and Proposition 6, it follows that

$$|\{1^{f_1}2^{f_2} \in \mathcal{P}(n(l) - kd, \{1,2\}) \mid f_2 \equiv 0 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2} \in \mathcal{P}(n(l) - kd, \{1,2\}) \mid f_2 \equiv 1 \pmod 2\}|.$$

Thus,

$$|\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n(l)) \mid f_2 \equiv 0 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n(l)) \mid f_2 \equiv 1 \pmod 2\}|.$$

Considering Proposition 5, since $d \equiv 1 \pmod 4$, we conclude that $e^k_{n(l)} - o^k_{n(l)} = 0$.

`Case 3:` $e^k_{n(l)} - o^k_{n(l)}$ for $k \equiv 3 \pmod 4$. In this case, $n(l) - kd \equiv 2 \pmod 4$. Applying Corollary 1 and Proposition 6, it follows that

$$|\{1^{f_1}2^{f_2} \in \mathcal{P}(n(l) - kd, \{1,2\}) \mid f_2 \equiv 0 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2} \in \mathcal{P}(n(l) - kd, \{1,2\}) \mid f_2 \equiv 1 \pmod 2\}|.$$

Thus,

$$|\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n(l)) \mid f_2 \equiv 0 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n(l)) \mid f_2 \equiv 1 \pmod 2\}|.$$

Considering Proposition 5, since $d \equiv 1 \pmod 4$, we conclude that $e^k_{n(l)} - o^k_{n(l)} = 0$.

`Case 4:` $e^k_{n(l)} - o^k_{n(l)}$ for $k \equiv 0 \pmod 4$. In this case, $n(l) - kd \equiv 1 \pmod 4$. Applying Corollary 1 and Proposition 6, it follows that

$$|\{1^{f_1}2^{f_2} \in \mathcal{P}(n(l) - kd, \{1,2\}) \mid f_2 \equiv 0 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2} \in \mathcal{P}(n(l) - kd, \{1,2\}) \mid f_2 \equiv 1 \pmod 2\}| + 1.$$

Thus,

$$|\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n(l)) \mid f_2 \equiv 0 \pmod 2\}|$$
$$= |\{1^{f_1}2^{f_2}d^k \in \mathcal{P}_{d,k}(n(l)) \mid f_2 \equiv 1 \pmod 2\}| + 1.$$

Considering Proposition 5, since $d \equiv 1 \pmod 4$, we conclude that $e^k_{n(l)} - o^k_{n(l)} = 1$.

Now, for $l \geq 1$, considering the computations above, we have

$$\mathcal{P}^E_d(n(l)) - \mathcal{P}^O_d(n(l)) = \sum_{j=0}^{4l-1} \left(e^j_{n(l)} - o^j_{n(l)} \right) = l(1 + 1 + 0 + 0) = 2l.$$

$\square$

Remark 4. Theorem 3 does not appear to hold in general when d is even. For example, taking $d = 4$ the first 25 terms in the sequence $\{\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n)\}_{n\geq 1}$ are

$$1, 0, 0, -2, 2, 0, 0, -3, 3, 0, 0, -4, 4, 0, 0, -5, 5, 0, 0, -6, 6, 0, 0, -7, 7,$$

and taking $d = 6$ the first 25 terms are

$$1, 0, 0, -1, 1, -1, 1, -1, 1, -1, 1, -2, 2, -1, 1, -2, 2, -2, 2, -2, 2, -2, 2, -3, 3.$$

5. Epilogue

In this article, we studied two sequences of values defined using integer partitions and the index theory of seaweed subalgebras of $\mathfrak{sl}(n)$. Recall that given a partition λ, we take $\mathrm{ind}(\lambda)$ to be the index of the seaweed subalgebra of $\mathfrak{sl}(n)$ defined by the pair of compositions consisting of λ and its weight $w(\lambda)$. Restricting ind to the partitions of $\mathcal{P}(n, \{1, 2, d\})$ with $d > 1$ odd, we find that (1) the number of partitions $\lambda \in \mathcal{P}(n, \{1, 2, d\})$ with $\mathrm{ind}(\lambda) = i$ is eventually periodic, and (2) when $d \equiv 3 \pmod 4$, the difference between the number of partitions $\lambda \in \mathcal{P}(n, \{1, 2, d\})$ with $\mathrm{ind}(\lambda)$ even and the number with $\mathrm{ind}(\lambda)$ odd is periodic. It was noted in Sections 3 and 4 that experimental evidence suggests that if, instead, $d > 2$ even, then a more complicated version of (1) holds while (2) no longer holds in general. In particular, we are led to pose the following conjecture.

Conjecture 1.

1. For $d > 2$ even and $i \geq 0$, the sequence of values
$$\{|\{\lambda \in \mathcal{P}(n, \{1, 2, d\}) \mid \mathrm{ind}(\lambda) = i\}|\}_{n\geq 1}$$
is eventually periodic with period d.

2. If $d > 2$ even,
$$\mathcal{P}_d^E(n) = |\{\lambda \in \mathcal{P}(n, \{1, 2, d\}) \mid \mathrm{ind}(\lambda) \equiv 0 \pmod 2\}|,$$
and
$$\mathcal{P}_d^O(n) = |\{\lambda \in \mathcal{P}(n, \{1, 2, d\}) \mid \mathrm{ind}(\lambda) \equiv 1 \pmod 2\}|,$$
then the sequence $\{\mathcal{P}_d^E(n) - \mathcal{P}_d^O(n)\}_{n\geq 1}$ is not periodic.

Recall that periodic phenomena among index and integer partitions are not new. In [3], the authors show that the number of partitions $\lambda \in \mathcal{P}(n, \{1, 2, \ldots, k\})$ for $1 \leq k \leq 7$ with $\mathrm{ind}(\lambda) = 0$ is eventually periodic. Thus, given the addition of the new periodicity results established in this paper, we conjecture that a more general theorem exists relating index, restricted classes of integer partitions, and periodic phenomena. In particular, perhaps there exists some Property 1 and Property 2 such that

1. if $S_1 \in \{S \subset \mathbb{Z}_{>0} \mid S$ satisfies Property 1$\}$, then the sequence of values

$$\{|\{\lambda \in \mathcal{P}(n, S_1) \mid \text{ind}(\lambda) = i\}|\}_{n\geq 1}$$

is eventually periodic; and

2. if $S_2 \in \{S \subset \mathbb{Z}_{>0} \mid S$ satisfies Property 2$\}$,

$$\mathcal{P}^E_{S_2}(n) = |\{\lambda \in \mathcal{P}(n, S_2) \mid \text{ind}(\lambda) \equiv 0 \pmod 2\}|,$$

and

$$\mathcal{P}^O_{S_2}(n) = |\{\lambda \in \mathcal{P}(n, S_2) \mid \text{ind}(\lambda) \equiv 1 \pmod 2\}|$$

then $\{\mathcal{P}^E_{S_2}(n) - \mathcal{P}^O_{S_2}(n)\}_{n\geq 1}$ is (eventually) periodic.

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Appendix A: Proof of Proposition 4

Below, the remaining 6 cases in the proof of Proposition 4 are considered.

Proof. `Case 2:` $f_d + f_1 = 0$. Applying Theorem 4 and Lemma 5, it follows that

$$\operatorname{ind}(\lambda) = 2\left\lceil \frac{4(i+1)d}{4} \right\rceil - 1 = 2(i+1)d - 1 > i.$$

Thus, there are no partitions $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ satisfying $w(\lambda) = 4(i+1)d$, $f_d + f_1 = 0$, and $\operatorname{ind}(\lambda) = i$.

`Case 3:` $f_d + f_1 = 2j + 2$ for $0 \le j < i$. Say $i = j + l$ for $0 < l \le i$. Note that

$$4(i+1)d = 4(j+l+1)d = 4(j+1)d + 4ld = 4(j+1)d + 2(2ld)$$

for $2ld > 0$. Applying Proposition 2, it follows that

$$\operatorname{ind}(\lambda) > j + 2ld = (j+l) + (2d-1)l = i + (2d-1)l > i.$$

Thus, there are no partitions $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ satisfying $w(\lambda) = 4(i+1)d$, $f_d + f_1 = 2j + 2$ for $0 \le j < i$, and $\operatorname{ind}(\lambda) = i$.

`Case 4:` $f_1 = 2i + 2$, $f_d = 0$. Let M be the meander of type

$$\frac{a_1|a_2|\cdots|a_m}{w(\lambda)} = \frac{\overbrace{2|\cdots|2}^{f_2}\,|\,\overbrace{1|\cdots|1}^{f_1}}{w(\lambda)}.$$

Since

$$\begin{aligned} w(\lambda) - 2\sum_{j=0}^{f_1-2} a_{m-j} &= w(\lambda) - 2(f_1 - 1) = 4(i+1)d - 2(2i+1) \\ &> 4(i+1) - 2(2i+1) = 2 = 2a_{m-(f_1-1)}, \end{aligned}$$

it follows that $UF(f_1, M) = (F_h)(F_v)(P)^{f_1}(F_v)(F_h)$. Thus, since

$$w(\lambda) - 2\cdot f_1 = 4(i+1)d - 2(2i+2) > 4(i+1) - 2(2i+2) = 0,$$

applying $UF(f_1, M)$ to M results in the meander M' of type

$$\frac{\overbrace{2|\cdots|2}^{\frac{f_1}{2}=i+1}\,|\,\overbrace{2|\cdots|2}^{\frac{w(\lambda)-2\cdot f_1}{2}}}{\underbrace{\underbrace{1|1}_{2}\,|\cdots|\,\underbrace{1|1}_{2}}_{\frac{f_1}{2}=i+1}\,|w(\lambda) - 2\cdot f_1},$$

where $\frac{w(\lambda)-2\cdot f_1}{2} > 0$. Evidently, M' consists of $i+2$ smaller meanders: $i+1$ meanders of type $\frac{2}{1|1}$ and one meander of type $\frac{2|\cdots|2}{w(\lambda)-2\cdot f_1}$. Clearly, the meander of type $\frac{2}{1|1}$ consists of a single path. Applying Lemma 5, we conclude that the meander of type $\frac{2|\cdots|2}{w(\lambda)-2\cdot f_1}$ consists of $\left\lceil \frac{w(\lambda)-2\cdot f_1}{4} \right\rceil$ cycles. Since

$$\frac{w(\lambda)-2\cdot f_1}{4} = \frac{4(i+1)d-2(2i+2)}{4} > \frac{4(i+1)-2(2i+2)}{4} = 0,$$

it follows that the meander of type $\frac{2|\cdots|2}{w(\lambda)-2\cdot f_1}$ consists of at least one cycle. Thus, putting everything together, M consists of $i+1$ paths and at least one cycle. Applying Theorem 4,

$$\text{ind}(\lambda) \geq i+1+2-1 = i+2 > i.$$

Hence, there are no partitions $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ satisfying $w(\lambda) = 4(i+1)d$, $f_d = 0$, $f_1 = 2i+2$, and $\text{ind}(\lambda) = i$.

`Case 5:` $f_d + f_1 = 2i+2$ with $0 < f_1 < d \cdot f_d$ and f_1, f_d even. Let M be the meander of type

$$\frac{\overbrace{d|\cdots|d}^{f_d}\,|\,\overbrace{2|\cdots|2}^{f_2}\,|\,\overbrace{1|\cdots|1}^{f_1}}{w(\lambda)}.$$

Form a new meander M' by adding $\frac{f_1}{2}$ top edges to M connecting the vertices v_{2i-1} and v_{2i} for $\frac{d\cdot f_d+2\cdot f_2+2}{2} \leq i \leq \frac{w(\lambda)}{2}$. Note that M' is the meander of type

$$\frac{a_1|a_2|\cdots|a_m}{w(\lambda)} = \frac{\overbrace{d|\cdots|d}^{f_d}\,|\,\overbrace{2|\cdots|2}^{f_2+\frac{f_1}{2}}}{w(\lambda)}.$$

Since

$$f_2 + \frac{f_1}{2} = \frac{w(\lambda) - d\cdot f_d}{2} > \frac{4(i+1)d - d(2i+2)}{2} = \frac{(2i+2)d}{2} > \frac{d\cdot f_d}{2},$$

it follows that $a_{m-j} = 2$ for $0 \leq j \leq \frac{d\cdot f_d}{2} - 1$. Consequently,

$$\begin{aligned} w(\lambda) - 2\sum_{j=0}^{\frac{d\cdot f_d}{2}-2} a_{m-j} &= w(\lambda) - 2\left[2\left(\frac{d\cdot f_d}{2} - 1\right)\right] = w(\lambda) - 4\left(\frac{d\cdot f_d}{2} - 1\right) \\ &= 4(i+1)d - 2d\cdot f_d + 4 > 4(i+1)d - 2d(2i+2) + 4 \\ &= 4 = 2a_{m-\left(\frac{d\cdot f_d}{2}-1\right)} \end{aligned}$$

so that $UF\left(\frac{d\cdot f_d}{2}, M'\right) = (F_h)(F_v)(P)^{\frac{d\cdot f_d}{2}}(F_v)(F_h)$. Thus, since $f_2 + \frac{f_1}{2} > \frac{d\cdot f_d}{2}$, applying $UF\left(\frac{d\cdot f_d}{2}, M'\right)$ to M' results in the meander M'' of type

$$\frac{\overbrace{d|d|\cdots|d|d}^{f_d}|\overbrace{2|\cdots|2}^{\frac{w(\lambda)-2d\cdot f_d}{2}}}{\underbrace{\underbrace{2|\cdots|2}_{d}|\cdots|\underbrace{2|\cdots|2}_{d}}_{\frac{f_d}{2}}|w(\lambda)-2d\cdot f_d},$$

where $\frac{w(\lambda)-2d\cdot f_d}{2} > 0$. Evidently, M'' consists of $\frac{f_d}{2}+1$ smaller meanders: $\frac{f_d}{2}$ meanders of type $\frac{d|d}{2|\cdots|2}$ and one meander of type $\frac{2|\cdots|2}{w(\lambda)-2d\cdot f_d}$. Applying Lemma 4, we conclude that the meander of type $\frac{d|d}{2|\cdots|2}$ consists of a single path. Since

$$\frac{w(\lambda)-2d\cdot f_d}{4} > \frac{4(i+1)d-2d(2i+2)}{4} = 0,$$

applying Lemma 5, it follows that the meander of type $\frac{2|\cdots|2}{w(\lambda)-2d\cdot f_d}$ consists of at least one cycle. Thus, putting everything together, M' consists of $\frac{f_d}{2}$ paths and at least one cycle. Now, since $d\cdot f_d > f_1$, it follows that each of the $\frac{f_1}{2}$ top edges added in forming M' from M belongs to a path corresponding to one of the meanders of type $\frac{d|d}{2|\cdots|2}$ in M''. Consequently, removing each of the $\frac{f_1}{2}$ top edges added in forming M' from M breaks one path into two. Therefore, M consists of

$$\frac{f_d}{2} + \frac{f_1}{2} = \frac{2i+2}{2} = i+1$$

paths and at least one cycle. Applying Theorem 4,

$$\text{ind}(\lambda) \geq i+1+2-1 = i+2 > i.$$

Hence, there are no partitions $\lambda = 1^{f_1}2^{f_2}d^{f_d}$ satisfying $w(\lambda) = 4(i+1)d$, $f_d + f_1 = 2i+2$, $0 < f_1 < d\cdot f_d$, f_1 and f_d even, and $\text{ind}(\lambda) = i$.

`Case 6:` $f_d + f_1 = 2i+2$ with $0 < f_1 < d\cdot f_d$ and f_1, f_d odd. Let M be the meander of type

$$\frac{\overbrace{d|\cdots|d}^{f_d}|\overbrace{2|\cdots|2}^{f_2}|\overbrace{1|\cdots|1}^{f_1}}{w(\lambda)}.$$

Form a new meander M' by adding $\frac{f_1-1}{2}$ top edges to M connecting the vertices v_{2i-1} and v_{2i} for $\frac{d\cdot f_d+2\cdot f_2+2}{2} \leq i \leq \frac{w(\lambda)-1}{2}$. Note that M' is the meander of type

$$\frac{a_1|a_2|\cdots|a_m}{w(\lambda)} = \frac{\overbrace{d|\cdots|d}^{f_d}|\overbrace{2|\cdots|2}^{f_2+\frac{f_1-1}{2}}|1}{w(\lambda)}.$$

Since

$$f_2 + \frac{f_1 - 1}{2} = \frac{w(\lambda) - d \cdot f_d - 1}{2} \geq \frac{4(i+1)d - d(2i+1) - 1}{2}$$
$$= \frac{(2i+1)d + (2d-1)}{2} > \frac{(2i+1)d + 1}{2} \geq \frac{d \cdot f_d + 1}{2},$$

it follows that $a_{m-j} = 2$ for $0 < j \leq \frac{d \cdot f_d + 1}{2} - 1$. Consequently,

$$\begin{aligned} w(\lambda) - 2 \sum_{j=0}^{\frac{d \cdot f_d + 1}{2} - 2} a_{m-j} &= w(\lambda) - 2\left[2\left(\frac{d \cdot f_d + 1}{2} - 2\right)\right] - 2(1) \\ &= w(\lambda) - 4\left(\frac{d \cdot f_d + 1}{2} - 2\right) - 2 = w(\lambda) - 2d \cdot f_d + 4 \\ &= 4(i+1)d - 2d \cdot f_d + 4 > 4(i+1)d - 2d(2i+2) + 4 \\ &= 4 = 2a_{m - \left(\frac{d(f_d+1)}{2} - 1\right)} \end{aligned}$$

so that $UF\left(\frac{d \cdot f_d + 1}{2}, M'\right) = (F_h)(F_v)(P)^{\frac{d \cdot f_d + 1}{2}}(F_v)(F_h)$. Thus, since

$$f_2 + \frac{f_1 - 1}{2} + 1 > \frac{d \cdot f_d + 1}{2},$$

applying $UF\left(\frac{d \cdot f_d + 1}{2}, M'\right)$ to M' results in the meander M'' of type

$$\frac{\overbrace{d|d|d|\cdots|d|d}^{f_d}\,|\,\overbrace{2|\cdots|2}^{\frac{w(\lambda)-2d\cdot f_d}{2}}}{1|2|\cdots|2\,\underbrace{\underbrace{2|\cdots|2}_{d}|\cdots|\underbrace{2|\cdots|2}_{d}}_{\frac{f_d-1}{2}}|w(\lambda) - 2d \cdot f_d},$$

where $\frac{w(\lambda) - 2d \cdot f_d}{2} > 0$. Evidently, M'' consists of $\frac{f_d + 3}{2}$ smaller meanders: one meander of type $\frac{d}{1|2|\cdots|2}$, $\frac{f_d - 1}{2}$ meanders of type $\frac{d|d}{2|\cdots|2}$, and one meander of type $\frac{2|\cdots|2}{w(\lambda) - 2d \cdot f_d}$. Applying Lemmas 2 and 4, we conclude that the meanders of types $\frac{d}{1|2|\cdots|2}$ and $\frac{d|d}{2|\cdots|2}$ consist of a single path. Since

$$\frac{w(\lambda) - 2d \cdot f_d}{4} = \frac{4(i+1)d - 2d \cdot f_d}{4} > \frac{4(i+1)d - 2d(2i+2)}{4} = 0,$$

applying Lemma 5, it follows that the meander of type $\frac{2|\cdots|2}{w(\lambda) - 2d \cdot f_d}$ consists of at least one cycle. Thus, putting everything together, M' consists of $\frac{f_d - 1}{2} + 1 = \frac{f_d + 1}{2}$ paths and at least one cycle. Now, since $d \cdot f_d > f_1$, it follows that each of the $\frac{f_1 - 1}{2}$

top edges added in forming M' from M belongs to a path corresponding to one of the meanders of type $\frac{d}{1|2|\cdots|2}$ or $\frac{d|d}{2|\cdots|2}$ in M''. Consequently, removing each of the $\frac{f_1-1}{2}$ top edges added in forming M' from M breaks one path into two. Therefore, M consists of

$$\frac{f_d+1}{2}+\frac{f_1-1}{2}=\frac{f_d+f_1}{2}=\frac{2i+2}{2}=i+1$$

paths and at least one cycle. Applying Theorem 4,

$$\operatorname{ind}(\lambda)\geq i+1+2-1=i+2>i.$$

Hence, there are no partitions $\lambda=1^{f_1}2^{f_2}d^{f_d}$ satisfying $w(\lambda)=4(i+1)d$, $f_d+f_1=2i+2$, $0<f_1<d\cdot f_d$, f_1 and f_d odd, and $\operatorname{ind}(\lambda)=i$.

`Case 7:` $f_d+f_1=2i+2$ with $f_1\geq d\cdot f_d>0$. Let M be the meander of type

$$\frac{a_1|a_2|\cdots|a_m}{w(\lambda)}=\frac{\overbrace{d|\cdots|d}^{f_d}\,|\,\overbrace{2|\cdots|2}^{f_2}\,|\,\overbrace{1|\cdots|1}^{f_1}}{w(\lambda)}.$$

Since

$$\begin{aligned}w(\lambda)-2\sum_{j=0}^{f_1-2}a_{m-j}&=w(\lambda)-2(f_1-1)>4(i+1)d-2(2i+1)\\&>4(i+1)-2(2i+1)=2=2a_{m-(f_1-1)},\end{aligned}$$

it follows that $UF(f_1,M)=(F_h)(F_v)(P)^{f_1}(F_v)(F_h)$. Thus, since

$$\begin{aligned}w(\lambda)-2\cdot f_1=4(i+1)d-2\cdot f_1&>4(i+1)d-2(2i+2)\\&>4(i+1)-2(2i+2)=0\end{aligned}$$

and $f_1\geq d\cdot f_d$, applying $UF(f_1,M)$ to M results in the meander M' of type

$$\frac{\overbrace{d|\cdots|d}^{f_d}\,|\,\overbrace{2|\cdots|2}^{f_2}}{\underbrace{\underbrace{1|\cdots|1}_{d}\,|\cdots|\,\underbrace{1|\cdots|1}_{d}}_{f_d}\,|\,\underbrace{\underbrace{1|1}_{2}\,|\cdots|\,\underbrace{1|1}_{2}}_{\frac{f_1-d\cdot f_d}{2}}\,|w(\lambda)-2\cdot f_1},$$

where $\frac{f_1-d\cdot f_d}{2}\geq 0$ and $w(\lambda)-2\cdot f_1>0$. Evidently, M' consists of $\frac{f_1-f_d(d-2)+2}{2}$ smaller meanders: f_d meanders of type $\frac{d}{1|\cdots|1}$, $\frac{f_1-d\cdot f_d}{2}$ meanders of type $\frac{2}{1|1}$, and one meander of type $\frac{2|\cdots|2}{w(\lambda)-2\cdot f_1}$. Clearly, the meander of type $\frac{2}{1|1}$ consists of one path and the meander of type $\frac{d}{1|\cdots|1}$ consists of $\lceil\frac{d}{2}\rceil$ paths. Since

$$\frac{w(\lambda)-2\cdot f_1}{4}>\frac{4(i+1)d-2(2i+2)}{4}>\frac{4(i+1)-2(2i+2)}{4}=0,$$

applying Lemma 5, it follows that the meander of type $\frac{2|\cdots|2}{w(\lambda)-2\cdot f_1}$ consists of at least one cycle. Thus, putting everything together, M consists of

$$f_d\left\lceil\frac{d}{2}\right\rceil+\frac{f_1-d\cdot f_d}{2}=f_d\frac{d+1}{2}+\frac{f_1-d\cdot f_d}{2}=\frac{f_1+f_d}{2}=\frac{2i+2}{2}=i+1$$

paths and at least one cycle. Applying Theorem 4,

$$\mathrm{ind}(\lambda)\geq i+1+2-1=i+2>i.$$

Hence, there are no partitions $\lambda=1^{f_1}2^{f_2}d^{f_d}$ satisfying $w(\lambda)=4(i+1)d$, $f_d+f_1=2i+2$, $f_1\geq d\cdot f_d>0$, and $\mathrm{ind}(\lambda)=i$. $\square$