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DISCRETE MEASURES AND THE RIEMANN HYPOTHESIS OF SOME L-FUNCTION IN THE SELBERG CLASS

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Abstract

Given a function F from the Selberg class, we show that if F has a polynomial Euler product, then the generalized Riemann hypothesis is equivalent to a problem on the rate of convergence of certain discrete measures defined on the positive real numbers.

1. Introduction and Statement of the Main Problems

The Riemann zeta function plays an important role in mathematical research, and it constitutes a first link between arithmetic and analysis. It was used by Euler and Riemann to study the distribution of prime numbers. The distribution of the zeros of the Riemann zeta function ζ (as well as of other zeta functions and L-functions) is related to important questions in number theory. Of special interest is the famous yet unproved Riemann hypothesis, which claims that all nontrivial (non-real) zeros of ζ lie on the critical line $1/2 + i\mathbb{R}$. One of the particular charms of the study of the Riemann hypothesis is the great diversity of its equivalent formulations, which we believe may be extended to a large class of L-functions (the Selberg class) [8, 12, 13, 15], the class of automorphic L-functions [6, 10, 11] and the zeta function associated with the function field K of an arbitrary genus over a finite field of constants [1, 2, 9].

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In this paper, while extending a previous result due to Verjovsky [16] for the Riemann zeta function ζ, we consider the general case of an L-function F from the Selberg class $\mathcal{S}$ having a polynomial Euler product representation. Further, we use recent results of Kaczorowski [4], which are associated with the Euler totient function. This simple approach suggests future applications, mainly for a zeta function. Such a zeta function is associated with the function field of an arbitrary genus over a finite field of constants, when the analog of the Riemann hypothesis holds true [17] and yields interesting results on function fields.

The Selberg class $\mathcal{S}$ was introduced by A. Selberg [14], which consists of the Dirichlet series

$$F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}, \qquad \mathrm{Re}\,(s) > 1,$$

satisfying the following properties:

- *Ramanujan hypothesis*: $a(n) = O(n^{\epsilon})$.
- *Euler product*: For s with sufficiently large real part, we have

  $$F(s) = \prod_p \exp\left(\sum_{k=1}^{+\infty} \frac{b(p^k)}{p^{ks}}\right),$$

  where $b(p^k) = O(p^{k\theta})$, $\theta < \frac{1}{2}$.
- *Analytic continuation*: For a non-negative integer m the entire function defined by $(s-1)^m F(s)$ is of finite order. The smallest such number is denoted by m_F and called the polar order of F.
- *Functional equation*: For $1 \le j \le r$, there exist positive real numbers (Q_F, λ_j) and complex numbers ($\mu_j,\ \omega$) with $Re(\mu_j) \ge 0$ and $|\omega| = 1$ such that

  $$\phi_F(s) = \omega\overline{\phi_F(1-\overline{s})} = F(s) Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j).$$

The degree of $F \in \mathcal{S}$ is defined by $d_F = 2\sum_{j=1}^{r} \lambda_j$. One of the most important conjectures about the Selberg class is the generalized Riemann hypothesis which states: for all $F \in \mathcal{S}$, the nontrivial (non-real) zeros of F are located on the critical line $\mathrm{Re}\,(s) = \frac{1}{2}$. The logarithmic derivative of F has a Dirichlet series expansion as given below:

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \Lambda_F(n) n^{-s}, \qquad \mathrm{Re}\, s > 1.$$

where $\Lambda_F(n) = b(n)\log n$ is the generalized von Mangoldt function (supported on the prime powers). The Euler product for $F \in \mathcal{S}$ can be written as

$$F(s) = \prod_p F_p(s), \quad \text{where} \quad F_p(s) = 1 + \sum_{m=1}^{\infty} \frac{a_F(p^m)}{p^{ms}}.$$

It was conjectured in [5] that the p-factors F_p are of polynomial type, and there exist ν_F a positive integer and complex numbers $\alpha_i(p)$ such that

$$F_p(s) = \prod_{i=1}^{\nu_F} \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1}, \qquad |\alpha_i(p)| \le 1.$$

The subsemigroup of the function $F \in \mathcal{S}$ with polynomial Euler product is denoted by $\mathcal{S}^{poly}$, and it is conjectured that $\mathcal{S}^{poly} = \mathcal{S}$.

From now on and until the end of the present paper, we consider $F \in \mathcal{S}^{poly}$ with $\nu_F = d$, where d is as small as possible and there exists at least one prime number p_0 such that $\prod_{j=1}^{d} \alpha_i(p_0) \neq 0$; d is called the *Euler degree* of F. Kaczorowski and Perelli [5] conjecture that $d = d_F$ for every $F \in \mathcal{S}^{poly}$, and this conjecture holds for all classical L-functions in $\mathcal{S}^{poly}$ known to us. Then, the Euler product of F has the form

$$F(s) = \prod_p F_p(s) = \prod_p \prod_{i=1}^{d} \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1};$$

we will say that F has a polynomial Euler product representation. Put

$$C(F) = \frac{1}{2} \prod_p \left(1 - \frac{\gamma(p)}{p^2}\right),$$

where

$$\gamma(p) = p\left(1 - \frac{1}{F_p(1)}\right).$$

Let $\mathbb{R}_+^*$ denote the multiplicative group of positive real numbers and $C_c^r(\mathbb{R}_+^*)$ the space of complex-valued functions $f : \mathbb{R}_+^* \to \mathbb{C}$ of class C^r with compact support.

Our main results are stated in the following theorems.

Theorem 1. *Let F be an L-function from the Selberg class $\mathcal{S}$ which has a polynomial Euler product representation and does not vanish at $s = \frac{1}{2}$. Then, for every $f \in C_c^0(\mathbb{R}_+^*)$, we have*

$$m_y(f) = 2C(F) \int_0^{+\infty} u f(u) du + O\left(y^{\frac{1}{2}} \log^d y\right), \qquad \textit{as } y \to 0.$$

Upon considering, $m_0(f) = 2C(F) \int_0^{+\infty} u f(u) du$, we have the following theorem.

Theorem 2. *Let F be an L-function from the Selberg class $\mathcal{S}$ which has a polynomial Euler product representation and does not vanish at $s = \frac{1}{2}$ and $s = 1$. Then we have the following:*

1) *The generalized Riemann hypothesis for F is true if and only if for every function $f \in C_c^r(\mathbb{R}_+^*)$ with $d_F + 1 \leq r \leq \infty$, one has*

$$m_y(f) = m_0(f) + O(y^{\frac{3}{4}-\epsilon}),$$

as $y \to 0$ and for all $\epsilon > 0$.

2) *If $\alpha \in]\frac{1}{4}, \frac{3}{4}[$, for all functions $f \in C_c^r(\mathbb{R}_+^*)$ with $d_F + 1 \leq r \leq \infty$, we have*

$$m_y(f) = m_0(f) + O(y^{\alpha-\epsilon}), \tag{1.1}$$

as $y \to 0$, for all $\epsilon > 0$, then F has no zeros in the half-plane $\mathrm{Re}\,(s) > 2(1-\alpha)$. In other words, if F has no zeros in the half-plane $\mathrm{Re}\,(s) > 2(1-\alpha)$, then (1.1) holds for all functions $f \in C_c^r(\mathbb{R}_+^*)$ with $d_F + 1 \leq r \leq +\infty$.

3) *If f is the characteristic function of an interval and $\alpha > \frac{1}{2}$, then*

$$\limsup_{y\to 0} y^{-\alpha} |m_y(f) - m_0(f)| = +\infty.$$

4) *Define a function K with domain in the positive reals as*

$$K(x) = \begin{cases} (1-x)^{d_F} & \text{for } x \leq 1 \\ 0 & \text{for } x > 1. \end{cases} \tag{1.2}$$

Then

$$m_y(K) = m_0(K) + o(y^{\frac{1}{2}}), \quad \text{as } y \to 0$$

and

$$\limsup_{y\to 0} y^{-\alpha} |m_y(K) - m_0(K)| = +\infty$$

for all $\alpha > \frac{1}{2} + \tau$ with $0 \leq \tau < \frac{1}{4}$, if and only if, the generalized Riemann hypothesis for $F(s)$ is false in the strongest possible sense. Hence, there exist zeroes of F which are arbitrarily close to the line $\mathrm{Re}\,(s) = 1 - 2\tau$.

Remark. If $F(1/2) = 0$, we consider a function E defined by $E(s) = \frac{F(s)}{(s-1/2)^\nu}$, where ν is the multiplicity of the eventual zero of F at $s = 1/2$. For an example, define $G_m(s) = (1 - m^{1/2-s})^2 \in \mathcal{S}$, which satisfies the generalized Riemann hypothesis, as its zeros are $\frac{1}{2} + \frac{2k\pi i}{\log m}$, where $k \in \mathbb{Z}$ and each zero has multiplicity two.

2. Preliminary Results

The purpose of this section is to give some results and lemmas which will be useful for the proofs of our main theorems.

In view of our investigation, we observe that the functional equation is of special interest and can be written as

$$F(s) = \Delta_F(s)\overline{F(1-\overline{s})}, \qquad \Delta_F(s) = \omega Q^{1-2s} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1-s)+\overline{\mu_j})}{\Gamma(\lambda_j s + \mu_j)}.$$

By Stirling's formula, we have

$$\Delta_F(\sigma+it) = \left(\lambda Q^2 t^{d_F}\right)^{\frac{1}{2}-\sigma-it} \exp\left(itd_F + \frac{i\pi(\mu-d_F)}{4}\right)\{\omega + O(1/t)\}$$

and

$$-\frac{\Delta_F'}{\Delta_F}(\sigma+it) = \log\left(\lambda Q^2 t^{d_F}\right) + O\left(\frac{1}{t}\right),$$

where $\mu = 2\sum_{j=1}^{r}(1-2\mu_j)$ and $\lambda = \prod_{j=1}^{r}\lambda_j^{2\lambda_j}$. Furthermore, we have

$$\mu_F(\sigma) = \limsup_{t\to\pm\infty} \frac{\log|F(\sigma+it)|}{\log|t|} = \begin{cases} 0 & \text{for} \quad \sigma > 1 \\ (\frac{1}{2}-\sigma)d_F & \text{for} \quad \sigma < 0, \end{cases} \tag{2.1}$$

$$\mu_F(\sigma) \le \frac{1}{2}d_F(1-\sigma) \quad \text{for } 0 \le \sigma \le 1 \tag{2.2}$$

and

$$F(\sigma+it) \ll_\epsilon t^{\mu_F(\sigma)+\epsilon}. \tag{2.3}$$

(see Lemma 2.1 in [7]). The Lindelöf hypothesis for F states that, for any $\epsilon > 0$, we have $F\left(\frac{1}{2}+it\right) = O\left(t^\epsilon\right)$ as $|t| \to +\infty$. Conrey and Ghosh in [3] assert that for $F \in \mathcal{S}^{poly}$ the generalized Riemann hypothesis implies the Lindelöf hypothesis. So we have, $F(s) = O\left(t^\epsilon\right)$ and $F(s)^{-1} = O\left(t^\epsilon\right)$ for $s = \sigma + it$, $\sigma > 1/2$ as $|t| \to +\infty$ and $\epsilon > 0$. Kaczorowski and Perelli in [5] showed that the Prime Number Theorem is equivalent to the nonvanishing of F on the 1-line; their proof is based on a weak zero-density estimate near the 1-line and on a simple almost periodicity argument. Smajlović in [15] showed the following equivalence holds true for all positive integers l:

$$F(1+it) \neq 0 \iff \psi_F(x) = m_F x + o\left(\frac{x}{\log^l x}\right).$$

Define the associated Euler totient function[1] of F as follows ($n \in \mathbb{N}$);

$$\varphi(n, F) = n \prod_{p|n} F_p(1)^{-1}.$$

Kaczorowski [4] gave an asymptotic formula for the sum of $\varphi(n, F)$ over positive integers $n \le x$, which is stated as

$$\sum_{n \le x} \varphi(n, F) = C(F)x^2 + O(x(\log(2x))^d), \tag{2.4}$$

where we recall for the case of the Riemann Zeta function, we have $C(\zeta) = \frac{1}{2\zeta(2)} = \frac{3}{\pi^2}$, that

$$C(F) = \frac{1}{2} \prod_p \left(1 - \frac{\gamma(p)}{p^2}\right)$$

and

$$\gamma(p) = p\left(1 - \frac{1}{F_p(1)}\right).$$

Moreover, Kaczorowski [4] proved that the series $\sum_{n=1}^{\infty} \frac{\varphi(n,F)}{n^s}$ converges absolutely for $\mathrm{Re}\,(s) > 2$ and in this half-plane one has

$$\sum_{n=1}^{\infty} \frac{\varphi(n, F)}{n^s} = \zeta(s-1) \sum_{n=1}^{+\infty} \frac{\alpha(n)}{n^s}, \tag{2.5}$$

where

$$\alpha(n) = \mu(n) \prod_{p|n} \gamma(p).$$

Furthermore, for $\mathrm{Re}\,(s) > 1$ we have

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} = \frac{H(s)}{F(s)}, \tag{2.6}$$

where $H(s) = \sum_{n=1}^{+\infty} \frac{h(n)}{n^s}$ converges absolutely for $\mathrm{Re}\,(s) > \frac{1}{2}$. Moreover, as n runs over square-free positive integers we have

$$h(n) \ll \frac{1}{n} \exp\left(c \frac{\log n}{\log\log(n+2)}\right).$$

[1]For the case of the Riemann Zeta function, we have $\varphi(n, \zeta) = \varphi(n)$, where $\varphi(n)$ is the classical Euler φ-function defined by

$$\varphi(n) = \sum_{1 \le m \le n, (m,n)=1} 1 = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Lemma 1. *We have*

$$2C(F) = \frac{H(2)}{F(2)}.$$

Proof. Combining Equations (2.5) and (2.6), we have

$$\frac{H(s)}{F(s)} = \frac{1}{\zeta(s-1)} \sum_{n=1}^{+\infty} \frac{\varphi(n,F)}{n^s}, \qquad \mathrm{Re}\,(s) > 2.$$

Further, by using properties of the Euler product representation theorem we obtain

$$\begin{aligned} \frac{H(s)}{F(s)} &= \prod_p \left(1 - \frac{1}{p^{s-1}}\right) \prod_p \left(\sum_{k=0}^{+\infty} \frac{\varphi(p,F)}{p^{ks}}\right) \\ &= \prod_p \left(1 - \frac{1}{p^{s-1}}\right) \prod_p \left(1 + \sum_{k=1}^{+\infty} \frac{\varphi(p,F)}{p^{ks}}\right), \end{aligned}$$

where $\varphi(n,F)$ is multiplicative. Using that $\varphi(p,F) = \frac{p}{F_p(1)}$ we obtain

$$\begin{aligned} \frac{H(s)}{F(s)} &= \prod_p \left(1 - \frac{1}{p^{s-1}}\right) \prod_p \left(1 + \frac{1}{F_p(1)} \sum_{k=1}^{+\infty} \frac{1}{p^{k(s-1)}}\right) \\ &= \prod_p \left(1 - \frac{1}{p^{s-1}}\right) \prod_p \left(1 + \frac{1}{F_p(1)} \frac{1}{p^{s-1}-1}\right) \\ &= \prod_p \left(1 - \frac{1}{p^{s-1}}\right) \prod_p \left(1 - \frac{1}{p^{s-1}}\right)^{-1} \left(1 - \frac{\gamma(p)}{p^s}\right) \\ &= \prod_p \left(1 - \frac{\gamma(p)}{p^s}\right). \end{aligned}$$

Hence, the proof of Lemma 1 is complete. □

For each $y \in \mathbb{R}_+^*$, consider the infinite measure (m_y) defined on smooth functions with compact support in $\mathbb{R}_+^*$ given by the formula

$$m_y(f) = \sum_{n \in \mathbb{N}} y\varphi(n,F) f(y^{\frac{1}{2}} n), \tag{2.7}$$

where $\mathbb{N} = \{1, 2, ...\}$ is the set of natural numbers. Let $r \geq 0$ be an integer or infinity. For each $f \in C_c^r(\mathbb{R}_+^*)$ consider the Mellin transform of $m_y(f)y^{-1}$ as follows

$$\mathcal{M}_f(s) = \int_0^{+\infty} m_y(f) y^{s-2} dy, \qquad \mathrm{Re}\,(s) > 1. \tag{2.8}$$

Proposition 1. *The integral defining* $\mathcal{M}_f(s)$ *converges absolutely in the half-plane* $\mathrm{Re}\,(s) > 1$*, and uniformly in* $\mathrm{Re}\,(s) > 1+\epsilon$ *for all* $\epsilon > 0$*. Hence, it defines a holomorphic function in the half-plane* $\mathrm{Re}\,(s) > 1$*.*

Proof. Let $f \in C_c^r(\mathbb{R}_+^*)$ and $\| f \|_\infty = \sup_{y\in\mathbb{R}_+^*} |f(y)|$. We have

$$\begin{aligned} \mid m_y(f) \mid &= \left| \sum_{n\in\mathbb{N}} y\varphi(n,F) f(y^{\frac{1}{2}}n) \right| \\ &\leq \sum_{a\leq ny^{\frac{1}{2}}\leq b} \mid y\varphi(n,F) f(y^{\frac{1}{2}}n) \mid \\ &\leq \left(\sum_{ay^{-\frac{1}{2}}\leq n\leq by^{-\frac{1}{2}}} \mid y\varphi(n,F) \mid \right) \| f \|_\infty \\ &\leq C \| f \|_\infty . \end{aligned}$$

For $C > 0$ and $\mathrm{Re}\,(s) > 1$ one has

$$|\mathcal{M}_f(s)| = \left| \int_0^{+\infty} m_y(f) y^{s-2} dy \right| \ll \| f \|_\infty \left(\frac{T^{\sigma-1}}{\sigma-1} \right),$$

where $\sigma = \mathrm{Re}\,(s)$ and the value of T depends on f such that $f(y) = 0$ for $y > T$. Therefore, we obtain the absolute convergence in $\mathrm{Re}\,(s) > 1$ and the uniform convergence in $\mathrm{Re}\,(s) > 1+\epsilon$ for all $\epsilon > 0$. □

Now, by combining Equations (2.7) and (2.8) we get

$$\begin{aligned} \mathcal{M}_f(s) &= \int_0^{+\infty} m_y(f) y^{s-2} dy \\ &= \int_0^{+\infty} \sum_{n\in\mathbb{N}} y\varphi(n,F) f(y^{\frac{1}{2}}n) y^{s-2} dy \\ &= \sum_{n\in\mathbb{N}} \varphi(n,F) \int_0^{+\infty} f(y^{\frac{1}{2}}n) y^{s-1} dy, \end{aligned}$$

where $\mathrm{Re}\,(s) > 1$. With the change of variable $u = y^{\frac{1}{2}}n$ we obtain

$$\mathcal{M}_f(s) = 2\sum_{n\in\mathbb{N}} \frac{\varphi(n,F)}{n^{2s}} \int_0^{+\infty} f(u) u^{2s-1} dy.$$

Using Equations (2.5) and (2.6) one has

$$\mathcal{M}_f(s) = 2\frac{\zeta(2s-1)}{F(2s)} H(2s) \int_0^{+\infty} f(u) u^{2s-1} du, \quad \mathrm{Re}\,(s) > 2. \tag{2.9}$$

The integral in the last expression represents a holomorphic function on the whole complex plane for any continuous function f with compact support. The Mellin transform $\mathcal{M}_f(s)$ has the same properties as of $2\frac{\zeta(2s-1)}{F(2s)}H(2s)$. Therefore, the function $\mathcal{M}_f(s)$ has a meromorphic continuation to the whole complex plane that is regular for $\mathrm{Re}\,(s) > \frac{1}{2}$ except possibly for a simple pole at $s = 1$ with residue

$$Res_{s=1}\mathcal{M}_f(s) = \frac{H(2)}{F(2)} \int_0^{+\infty} uf(u)du = 2C(F) \int_0^{+\infty} uf(u)du. \tag{2.10}$$

All other possible poles of $\mathcal{M}_f$ are the trivial zeros and the zeros in the strip $0 \leq \sigma < \frac{1}{2}$ of the function defined by $F(2s)$. Moreover, the generalized Riemann Hypothesis for F holds if and only if for all $f \in C_c^0(\mathbb{R}_+^*)$ the function $\mathcal{M}_f$ is regular for $\mathrm{Re}\,(s) > \frac{1}{4}$ except possibly for a simple pole at $s = 1$ with residue given as above.

Lemma 2. *Let F be an L-function from the Selberg class $\mathcal{S}$ which has a polynomial Euler product representation and does not vanish at $s = \frac{1}{2}$. Let $f \in C_c^r(\mathbb{R}_+^*)$ with $r \geq 0$. Then, there exists t_0 independent of f such that*

$$| \mathcal{M}_f(\sigma + it) | \ll \frac{|t|^{\frac{d_F}{2}+\epsilon}}{(1+|t|)^r},$$

for $|t| > t_0$, $\epsilon > 0$ and $\frac{1}{2} \leq \sigma \leq 2$.

Proof. From Equation (2.9), we have

$$\mathcal{M}_f(s) \quad = \quad 2\frac{\zeta(2s-1)}{F(2s)}H(2s) \int_0^{+\infty} f(u)u^{2s-1}du.$$

Upon integrating by parts on the right hand side, we obtain

$$\begin{aligned} \mathcal{M}_f(s) &= 2\frac{\zeta(2s-1)H(2s)(-1)^r}{F(2s)2s(2s+1)(2s+2)...(2s+r-1)} \int_0^{+\infty} f^{(r)}(u)u^{2s+r-1}du \\ &= \frac{\zeta(2s-1)H(2s)(-1)^r}{F(2s)s(2s+1)(2s+2)...(2s+r-1)} \int_0^{T} f^{(r)}(u)u^{2s+r-1}du, \end{aligned}$$

where $f(u) = 0$ for $u > T$, which gives

$$\mathcal{M}_f(s) \quad \ll \quad \frac{|\zeta(2s-1)|}{|s(2s+1)(2s+2)...(2s+r-1)|} \left| \frac{H(2s)}{F(2s)} \right| \parallel f^{(r)} \parallel_\infty \frac{T^{2\sigma+r}}{2\sigma+r}.$$

Combining Equations (2.1), (2.2) and (2.3) we obtain

$$\zeta(2s-1)H(2s) \ll t^{\frac{d_F}{2}+\epsilon},$$

uniformly on $\frac{1}{2} \leq \mathrm{Re}\,(s) \leq 2$. In addition, if $\frac{1}{2} \leq \mathrm{Re}\,(s)$ and $\left|\frac{1}{F(2s)}\right| \ll 1$ then there exists t_0 such that

$$| \mathcal{M}_f(\sigma+it) | \ll \frac{|t|^{\frac{d_F}{2}+\epsilon}}{(1+|t|)^r},$$

for $|t| > t_0$, $\epsilon > 0$ and $\frac{1}{2} \leq \sigma \leq 2$. This finishes the proof of Lemma 2. □

Lemma 3. *Let F be an L-function from the Selberg class $\mathcal{S}$ which has a polynomial Euler product representation and does not vanish at $s = \frac{1}{2}$. Assume that the generalized Riemann hypothesis holds for F. Let $f \in C_c^r(\mathbb{R}_+^*)$ with $r \geq d_F + 1$. Then, for every $\epsilon > 0$ there exists $t_0 > 0$ such that*

$$| \mathcal{M}_f(\sigma+it) | = O_{\epsilon,a}\left(\frac{1}{(1+|t|)^r}\right),$$

for $|t| > t_0$ and $\frac{1}{4} + \epsilon \leq \sigma \leq 2$.

Proof. Under the generalized Riemann hypothesis we have $F(2s)^{-1} = O(t^{(2d_F-1)\epsilon})$. Furthermore, one has $\zeta(2s-1)H(2s) = O(t^{d_F(1-2\epsilon)})$ uniformly in $\frac{1}{4} + \epsilon \leq \sigma \leq 2$. Hence

$$\frac{\zeta(2s-1)H(2s)}{F(2s)} = O(t^{d_F-\epsilon}),$$

uniformly in $\frac{1}{4}+\epsilon \leq \sigma \leq 2$. Now, by integration by parts as in Lemma 2 we deduce that for $f \in C_c^r(\mathbb{R}_+^*)$ with $r \geq d_F + 1$ we get

$$| \mathcal{M}_f(\sigma+it) | = O((1+|t|)^{-(1+\epsilon)}), \quad |t| \to \infty,$$

uniformly in $\frac{1}{4} + \epsilon \leq \sigma \leq 2$. This completes the proof of Lemma 3. □

Let $x > 0$ be a real number and $\Phi_F(x) = \sum_{n \leq x} \varphi(n, F)$. Then, we have the following lemma.

Lemma 4. *For all $\alpha > 1$, we have*

$$\limsup_{x\to+\infty} x^\alpha \left|\frac{\Phi_F(x)}{x^2} - C(F)\right| = +\infty.$$

Proof. Let us assume that there exist $\alpha > 1$, $c > 0$ and a function b_α defined on the positive reals depending on α such that $|b_\alpha(x)| < c$ for all $x > 0$. Then, we have

$$\frac{\Phi_F(x)}{x^2} - C(F) = \frac{b_\alpha(x)}{x^\alpha}. \tag{2.11}$$

Moreover, for all $x > 0$ the function Φ_F satisfies

$$\frac{\Phi_F(x+1)}{(x+1)^2} = \frac{\Phi_F(x)}{(x+1)^2} + \frac{\varphi([x+1], F)}{(x+1)^2} \tag{2.12}$$

and
$$L(x) = b_\alpha(x)\frac{x^2}{(x+1)^2} - b_\alpha(x+1)\left(\frac{x}{x+1}\right)^\alpha.$$
Noting that $L(x)$ is a bounded expression. By combining Equations (2.11) and (2.12) we get
$$L(x) = x^\alpha\left(C(F)\frac{2x+1}{(x+1)^2} - \frac{\varphi([x+1],F)}{(x+1)^2}\right).$$
On the other hand, if $x+1$ is a prime we obtain
$$L(x) = x^\alpha\left(C(F)\frac{2x+1}{(x+1)^2} - \frac{x+1}{F_{x+1}(1)(x+1)^2}\right).$$
This implies that $L(x)$ is unbounded and leads to a contradiction. The proof is complete. □

3. Proof of Theorem 1

In this section, we present two proofs of Theorem 1. The first one uses partial summations and the second the Mellin inversion formula.

Proof. Let F be an L-function from the Selberg class $\mathcal{S}$ which has a polynomial Euler product representation and does not vanish at $s = \frac{1}{2}$. Let $f \in C_c^1(\mathbb{R}_+^*)$ with $supp(f) = [a,b] \subset \mathbb{R}_+^*$. We have
$$\begin{aligned} m_y(f) &= \sum_{n\in\mathbb{N}} y\varphi(n,F)f(y^{\frac{1}{2}}n) \\ &= \sum_{a\le ny^{\frac{1}{2}}\le b} y\varphi(n,F)f(y^{\frac{1}{2}}n) \\ &= \sum_{ay^{-\frac{1}{2}}\le n\le by^{-\frac{1}{2}}} y\varphi(n,F)f(y^{\frac{1}{2}}n). \end{aligned}$$

A partial summation and Equation (2.4) yield

$$\begin{aligned}
m_y(f) &= -yC(F)y^{\frac{1}{2}}\int_{ay^{-\frac{1}{2}}}^{by^{-\frac{1}{2}}} x^2 f'(xy^{\frac{1}{2}})dx + O\left(yy^{\frac{1}{2}}\int_{ay^{-\frac{1}{2}}}^{by^{-\frac{1}{2}}} x\log^d 2xf'(xy^{\frac{1}{2}})dx\right)\\
&= -C(F)\int_a^b u^2 f'(u)du + O\left(y^{\frac{1}{2}}\int_a^b u\log^d\left(2uy^{-\frac{1}{2}}\right)f'(u)du\right)\\
&= -C(F)\int_a^b u^2 f'(u)du + O\left(y^{\frac{1}{2}}\log^d(y)\right)\\
&= 2C(F)\int_a^b uf(u)du + O\left(y^{\frac{1}{2}}\log^d(y)\right)\\
&= 2C(F)\int_0^{+\infty} uf(u)du + O\left(y^{\frac{1}{2}}\log^d(y)\right).
\end{aligned}$$

Hence, Theorem 1 follows by recalling that any continuous function with compact support in $\mathbb{R}_+^*$ can be uniformly approximated by C^1 functions with compact support in $\mathbb{R}_+^*$. The error terms depend only on the support of the functions. □

Theorem 1 can be proved *Alternatively* as follows.

Proof. If $f \in C_c^0(\mathbb{R}_+^*)$, the Mellin inversion formula gives

$$m_y(f) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \mathcal{M}_f(s)y^{1-s}ds, \tag{3.1}$$

for an appropriate $a \in \mathbb{R}$. First, we set $a = 2$, then

$$m_y(f) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathcal{M}_f(2+it)y^{-1-it}ds,$$

where the function defined by $\mathcal{M}_f(\frac{1}{2}+it)$ belongs to $\mathcal{L}_1(\mathbb{R},\mathbb{C})$. From Lemma 2, we move the line of integration to $\mathrm{Re}\,(s) = \frac{1}{2}$. The residue at the point $s = 1$ is equal to $m_0(f)$ (see Equation (2.10)). Hence, by Cauchy's theorem we have

$$m_y(f) - m_0(f) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{M}_f\left(\frac{1}{2}+it\right)y^{\frac{1}{2}}y^{-it}dt.$$

Moreover, one has

$$\left|\int_{-\infty}^{+\infty} \mathcal{M}_f\left(\frac{1}{2}+it\right)y^{\frac{1}{2}}y^{-it}dt\right| = y^{\frac{1}{2}}\left|\int_{-\infty}^{+\infty} \mathcal{M}_f\left(\frac{1}{2}+it\right)y^{-it}dt\right| = o(y^{\frac{1}{2}}).$$

By the Riemann-Lebesgue theorem we get

$$\lim_{y\to 0}\left|\int_{-\infty}^{+\infty} \mathcal{M}_f\left(\frac{1}{2}+it\right)y^{-it}dt\right| = 0.$$

Hence

$$m_y(f) = 2C(F)\int_0^{+\infty} uf(u)du + O\left(y^{\frac{1}{2}}\right).$$

The proof is complete. □

4. Proof of Theorem 2

Proof. Let F be an L-function from the Selberg class $\mathcal{S}$ which has a polynomial Euler product representation and does not vanish at $s = \frac{1}{2}$ and $s = 1$.

1) Let us assume that the generalized Riemann hypothesis holds for F. Then, we replace in formula (3.1) a by $(\frac{1}{4}+\epsilon)$ for $\epsilon > 0$. Also, Lemma 3 implies that the function defined by $\mathcal{M}_f(\frac{1}{2}+it)$ belongs to $\mathcal{L}_1(\mathbb{R},\mathbb{C})$. Hence, the integral of $\mathcal{M}_f(s)y^{1-s}$ over the boundary of the vertical strip $\frac{1}{4} \leq \sigma \leq 2$ exists. Therefore

$$m_y(f) = Res_{s=1}\mathcal{M}_f(s) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{M}_f\left(\frac{1}{4}+\epsilon+it\right)y^{\frac{3}{4}-\epsilon}y^{-it}dt.$$

Again, by the Riemann-Lebesgue theorem we get

$$m_y(f) = 2C(F)\int_0^{+\infty} uf(u)du + O\left(y^{\frac{3}{4}-\epsilon}\right).$$

Conversely, assume that for all $f \in C_c^{\infty}(\mathbb{R}_+^*)$ we have

$$m_y(f) = m_0(f) + O(y^{\frac{3}{4}-\epsilon}), \qquad y \to 0,$$

for all $\epsilon > 0$ and $m_y(f) = m_0(f) + K(y)$. Let T be sufficiently large such that $m_y(f) = 0$ for $y > T$. Then

$$\begin{aligned}\mathcal{M}_f(s) &= \int_0^T m_y(f)y^{s-2}dy = \int_0^T (m_0(f)+K(y))y^{s-2}dy \\ &= \frac{m_0(f)T^{(s-1)}}{s-1} + \int_0^T K(y)y^{s-2}dy.\end{aligned}$$

By substituting $K(y) = O(y^{\frac{3}{4}-\epsilon})$ as $y \to 0$ in the last integral, we see that it converges absolutely and uniformly in the half-plane $\mathrm{Re}\,(s) > (\frac{1}{4}+\epsilon)$. So it defines a holomorphic function in that half-plane. Hence, $\mathcal{M}_f$ is a holomorphic function in the region $\mathrm{Re}\,(s) > (\frac{1}{4}+\epsilon)$ except possibly for a pole at $s = 1$ with residue $m_0(f)$. Since $\mathcal{M}_f(s) = 2\frac{\zeta(2s-1)}{F(2s)}H(2s)\int_0^{+\infty} f(u)u^{2s-1}du$ is holomorphic in the half-plane $\mathrm{Re}\,(s) > (\frac{1}{4}+\epsilon)$, we choose f such that $\int_0^{+\infty} f(u)u^{2s-1}du$ does not vanish at any given zero of F. Then, the function $\frac{\zeta(2s-1)}{F(2s)}H(2s)$ is holomorphic in the half-plane $\mathrm{Re}\,(s) > \frac{1}{4}$. Therefore, the generalized Riemann hypothesis holds for F.

2) Let $\rho = \beta + i\gamma$ be a nontrivial zero of F with max β satisfies $\beta < 1$. We have $\mathcal{M}_f\left(\frac{\beta}{2} + \epsilon + it\right) \in C_c^r(\mathbb{R}_+^*)$ for all $\epsilon > 0$. Since F has no zeros in $Re(s) \geq 1$ ($F(1+it) \neq 0$ for example under the prime number theorem), by the Mellin inversion formula we have

$$m_y(f) = m_0(f) + o\left(y^{1-\frac{\beta}{2}-\epsilon}\right), \qquad \epsilon > 0. \tag{4.1}$$

Conversely, if Equation (4.1) holds with $\alpha = (1 - \frac{\beta}{2},\ \frac{\beta}{2}) \in [1/4, 1/2]$ then the same argument holds true for (4.1). This yields that F has no zero in the region $Re(s) > 2(1-2\alpha)$.

3) Let $x > 0$ be a real number, $\Phi_F(x) = \sum_{n\leq x} \varphi(n, F)$ and set $\Phi_F(x) = 0$ for $0 < x < 1$. By combining Equation (2.4) and Lemma 4 we conclude that the assertion is true for the characteristic function of the interval $]0, 1]$. So, we show that the proof for an arbitrary closed interval is the same.

4) Assume that K is a function as defined by Equation (1.2), then the Mellin transformation of $\mathcal{M}_K(s)$ is given by the following expression

$$\mathcal{M}_K(s) = \frac{2\zeta(2s-1)H(2s)}{F(2s)} \int_0^1 (1-x)^{d_F} x^{2s-1} dx.$$

Further, using

$$\int_0^1 (1-x)^{d_F} x^{2s-1} dx = \frac{\Gamma(d_F+1)\,\Gamma(2s)}{\Gamma(2s+d_F+1)},$$

we obtain

$$\mathcal{M}_K(s) = \frac{2\zeta(2s-1)H(2s)\Gamma(d_F+1)\,\Gamma(2s)}{F(2s)\Gamma(2s+d_F+1)}.$$

Then, the poles of $\mathcal{M}_K(s)$ in the half-plane $\mathrm{Re}\,(s) > 0$ are located at the zeroes of $F(2s)$ and at $s = 1$ since $\Gamma(2s + d_F + 1)$ does not vanish in that half-plane. So, by Lemma 2 we deduce that the function $\mathcal{M}_K(\frac{1}{2} + it) \in \mathcal{L}_1(\mathbb{R}, \mathbb{C})$. Hence, the integral of $\mathcal{M}_K(s)y^{1-s}$ over the boundary of the vertical strip $\frac{1}{2} \leq \sigma \leq 2$ exists. Moreover, it is equal to $Res_{s=1}\mathcal{M}_f(s)$ and to $m_y(K) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{M}_f\left(\frac{1}{2} + it\right) y^{\frac{1}{2}} y^{-it} dt$. Therefore

$$\left|\int_{-\infty}^{+\infty} \mathcal{M}_K\left(\frac{1}{2} + it\right) y^{\frac{1}{2}} y^{-it} dt\right| = y^{\frac{1}{2}} \left|\int_{-\infty}^{+\infty} \mathcal{M}_K\left(\frac{1}{2} + it\right) y^{-it} dt\right| = o(y^{\frac{1}{2}}).$$

Now, by the Riemann-Lebesgue theorem we get

$$\lim_{y\to 0} \left|\int_{-\infty}^{+\infty} \mathcal{M}_K\left(\frac{1}{2} + it\right) y^{-it} dt\right| = 0.$$

Hence

$$m_y(K) \quad = \quad 2C(F)\int_0^{+\infty} uK(u)du + O\left(y^{\frac{1}{2}}\right) = m_0(K) + O\left(y^{\frac{1}{2}}\right).$$

Let us assume that $\limsup_{y\to 0} y^{-\alpha}|m_y(K)-m_0(K)| = +\infty$ for all $\alpha > (\frac{1}{2}+\tau)$ with $0 \leq \tau < \frac{1}{4}$. Consider $\rho = \beta + i\gamma$ to be a nontrivial zero of F with $\beta < 1 - 2\tau$, where $0 \leq \tau < 1/4$. We have $\frac{\zeta(2s-1)H(2s)}{F(2s)} = O\left(t^{d_F}\right)$ for small $0 < \epsilon < 1/4d_F$ uniformly in the strip $(\frac{1}{2} - \tau - \epsilon) \leq \sigma \leq 2$. Hence, $\mathcal{M}_K = O\left((1+|t|)^{-3/2}\right)$ uniformly in the strip $(\frac{1}{2}-\tau-\epsilon) \leq \sigma \leq 2$ and $0 \leq \epsilon < 1/4-\tau$. Further, if we replace the vertical line of integration in the Mellin inversion formula by the vertical line $Re(s) = \frac{1}{2} - \tau - \epsilon$ for $0 \leq \epsilon < 1/4 - \tau$, again by the Riemann-Lebesgue theorem we obtain that the error term is $o\left(q^{\frac{1}{2}+\tau+\epsilon}\right)$, for $0 \leq \epsilon < 1/4 - \tau$. This contradicts the hypothesis that $(1 - \frac{\beta}{2} - \epsilon) > \frac{1}{2}$ if ϵ is small enough. The proof is complete. □

5. Concluding Remarks

The context of function fields with a finite field of constants is particularly interesting due to the fact that the analog of the Riemann hypothesis holds true in this case [17]. Therefore, one is able to deduce many interesting results by using formulaes equivalent to the Riemann hypothesis. The investigation of Theorem 1 provides further information on the invariants of the field. This is a fundamental research problem, and we shall consider it in a sequel to this article.

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