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ON abc TRIPLES OF THE FORM $(1, c-1, c)$

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Abstract

By an abc triple, we mean a triple (a, b, c) of relatively prime positive integers a, b, and c such that $a+b=c$ and $\operatorname{rad}(abc) < c$, where $\operatorname{rad}(n)$ denotes the product of the distinct prime factors of n. The study of abc triples is motivated by the abc conjecture, which states that for each $\epsilon > 0$, there are finitely many abc triples (a, b, c) such that $\operatorname{rad}(abc)^{1+\epsilon} < c$. The necessity of the ϵ in the abc conjecture is demonstrated by the existence of infinitely many abc triples. For instance, $\left(1, 9^k - 1, 9^k\right)$ is an abc triple for each positive integer k. In this article, we study abc triples of the form $(1, c-1, c)$ and deduce two general results that allow us to recover existing sequences of abc triples having $a = 1$ that are in the literature.

1. Introduction

In 1985, Masser and Oesterlé proposed the abc conjecture [11, 9], which states:

Conjecture 1.1 (The abc conjecture)**.** For every $\epsilon > 0$, there are finitely many relatively prime positive integers a, b, and c with $a + b = c$ such that

$$\operatorname{rad}(abc)^{1+\epsilon} < c,$$

where $\operatorname{rad}(n)$ denotes the product of the distinct prime factors of a positive integer n.

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Due to its profound implications, this simple-to-state conjecture is one of the most important open questions in number theory. For instance, some consequences of the *abc* conjecture include an asymptotic version of Fermat's Last Theorem, Faltings's Theorem, Roth's Theorem, and Szpiro's Conjecture [5, 7, 11]. For further information on the *abc* conjecture, see the excellent survey article [8].

The statement of the *abc* conjecture naturally leads us to ask if the ϵ is necessary. This leads us to the "simplistic *abc* conjecture," which asks if there are finitely many relatively prime positive integers a, b, and c with $a + b = c$ for which $\mathrm{rad}(abc) < c$. We call such triples, *abc triples*. The "simplistic *abc* conjecture" is false, as demonstrated by the triple $\left(1, 3^{2^k} - 1, 3^{2^k}\right)$, which is an *abc* triple for each positive integer k. This infinite sequence of *abc* triples is one of the first documented counterexamples to the simplistic *abc* conjecture and was communicated to Lang [7] by Jastrzebowski and Spielman. A theorem of Stewart [14] leads to similar sequences of *abc* triples such as $\left(1, 8^{7^k} - 1, 8^{7^k}\right)$, where k is a positive integer [8]. Jastrzebowski and Spielman's counterexample can also be recovered from the following result: for each odd prime p and each positive integer k, $\left(1, p^{(p-1)k} - 1, p^{(p-1)k}\right)$ is an *abc* triple [3]. Another construction, due to Granville and Tucker [6], shows that for each odd prime p, $\left(1, 2^{p(p-1)} - 1, 2^{p(p-1)}\right)$ is an *abc* triple.

In this article, we prove that $(1, c-1, c)$ is an *abc* triple if and only if $\mathrm{cosocle}(c-1) > \mathrm{rad}(c)$, where $\mathrm{cosocle}(m) = \frac{m}{\mathrm{rad}(m)}$ for m a positive integer (see Proposition 2.2). We note that the term cosocle is borrowed from module theory, where the *cosocle* of an R-module M is the maximal semisimple quotient of M, or equivalently, $\frac{M}{\mathrm{rad}(M)}$. In our setting, the cosocle plays a crucial role in our results, from which we recover each of the above mentioned sequences of *abc* triples. To provide context for our work, we note that the equivalence above requires us to compute $\mathrm{cosocle}(c-1)$ in order to deduce whether $(1, c-1, c)$ is an *abc* triple. The computation of $\mathrm{cosocle}(c-1)$ requires knowledge of the prime factorization of $c - 1$, which becomes computationally difficult as c gets large. Our main results provide a recipe for constructing infinitely many *abc* triples of the form $(1, c-1, c)$ based on knowledge of a divisor of $c - 1$ or c. Our first theorem illustrates this.

Theorem 1. *Let c and m be positive integers with $c > 1$. If m divides $c - 1$ and $\mathrm{cosocle}(m) > \mathrm{rad}(c)$, then $\left(1, c^k - 1, c^k\right)$ is an abc triple for each positive integer k..*

We prove Theorem 1 in Section 2. While the proof is elementary, the result allows us to recover each of the previously mentioned sequences of *abc* triples. It also leads to new sequences of *abc* triples, such as $\left(1, n^{(n-1)k} - 1, n^{(n-1)k}\right)$ which is an *abc* triple for each positive integer k whenever n is a positive integer that is either odd or even and non-squarefree (see Corollary 3.7). A slight modification of the proof of Theorem 1 leads us to our next result (which is also proven in Section 2).

Theorem 2. *Let b and m be positive integers. If m divides $b+1$ and $\mathrm{cosocle}(m) > \mathrm{rad}(b)$, then $\left(1, b^k, b^k + 1\right)$ is an abc triple for each positive odd integer k.*

A consequence of Theorem 1 is that if $(1, c-1, c)$ is an abc triple, then $(1, c^k-1, c^k)$ is an abc triple for each positive integer k (see Corollary 2.4). Similarly, we obtain from Theorem 2 that if $(1, b, b+1)$ is an abc triple, then $(1, b^k, b^k+1)$ is an abc triple for each odd integer k (see Corollary 2.5). These results lead to the following question: given an integer $c > 1$, for what positive integers k is $(1, c^k-1, c^k)$ an abc triple? We answer this question with Theorem 2.9, which provides necessary and sufficient conditions to determine those integers k which yield an abc triple of the form $(1, c^k-1, c^k)$.

In Section 3, we demonstrate various consequences of Theorems 1 and 2. For example, we prove that if $n > 1$ is an integer and p is an odd prime such that $p > \text{rad}(n)$, then $\left(1, n^{p(p-1)k}-1, n^{p(p-1)k}\right)$ is an abc triple for each positive integer k (see Corollary 3.5). In particular, taking $(n, k) = (2, 1)$ allows us to recover Granville and Tucker's original construction [6]. Another consequence is the following: if $n \geq 3$ is an odd integer and $b = n^j - 1$ for some positive integer j, then $\left(1, b^{nk}, b^{nk}+1\right)$ is an abc triple for each positive odd integer k (see Corollary 3.12). Taking $(n, j) = (3, 1)$ gives us that $\left(1, 8^k, 8^k+1\right)$ is an abc triple for each odd integer k.

We conclude the article with Section 4, which is an analysis of the abc triples found by the ABC@Home Project of the form $(1, c-1, c)$ with $c < 10^{18}$. The ABC@Home Project was a network computing project that was started in 2006 by the Mathematics Department of Leiden University, together with the Dutch Kennislink Science Institute. By 2011, they found that there are exactly 14 482 065 abc triples (a, b, c) with $c < 10^{18}$. By the time the project came to a close in 2015, the ABC@Home Project had found a total of 23 827 716 abc triples (a, b, c) with $c < 2^{63}$. We note that this list is not exhaustive of all abc triples with $c < 2^{63}$. In particular, the ABC@Home project found that there are exactly 45 604 abc triples of the form $(1, c-1, c)$ with $c < 10^{18}$. Further observations about the abc triples found by the ABC@Home Project can be found in [12, Chapter 7].

Motivated by the results in Section 3, we study those abc triples found by the ABC@Home Project that are of the form $(1, n^l-1, n^l)$ or $(1, n^l, n^l+1)$ for some integer $l > 1$. We find that this amounts to 8 413 abc triples. For abc triples $(1, c-1, c)$ of the aforementioned form, we show that approximately 48.7% of the abc triples with $c \leq 10^6$ can be obtained from the results proven in Section 3. We also find that for abc triples of the form $(1, n^l-1, n^l)$, there are only four cases where there does not exists a proper divisor m of n^l-1 for which $\text{cosocle}(m) > \text{rad}(n)$.

2. Main Results

In this section, we establish Theorems 1 and 2. To do so, we recall the following elementary property about the radical of a positive integer.

Lemma 2.1. *Let m and n be relatively prime positive integers. Then* $\text{rad}(mn) =$

$\operatorname{rad}(m)\operatorname{rad}(n)$ *and* $\operatorname{rad}(m) \le m$. *Moreover,* $\operatorname{rad}\bigl(m^k\bigr) = \operatorname{rad}(m)$ *for each positive integer* k.

We will assume Lemma 2.1 implicitly throughout this work. Next, we show an important facet about *abc* triples of the form $(1, c-1, c)$, which showcases the importance of the cosocle in our arguments.

Proposition 2.2. *Let* $c > 1$ *be an integer. Then the following are equivalent:*

(i) $\operatorname{cosocle}(c-1) > \operatorname{rad}(c)$*;*

(ii) $\operatorname{cosocle}(c) > \operatorname{rad}(c-1)$*;*

(iii) $(1, c-1, c)$ *is an abc triple.*

Proof. Suppose that $\operatorname{rad}(c) < \operatorname{cosocle}(c-1)$. From the equalities $\operatorname{rad}(c) = \frac{c}{\operatorname{cosocle}(c)}$ and $\operatorname{cosocle}(c-1) = \frac{c-1}{\operatorname{rad}(c-1)}$, we deduce that

$$\begin{aligned}\operatorname{rad}(c) < \operatorname{cosocle}(c-1) \qquad &\text{if and only if} \qquad \frac{c}{\operatorname{cosocle}(c)} < \frac{c-1}{\operatorname{rad}(c-1)}\\ &\text{if and only if} \qquad \operatorname{rad}(c-1) < \frac{c-1}{c}\operatorname{cosocle}(c)\,.\end{aligned}$$

Since $\frac{c-1}{c} < 1$, we have the desired inequality: $\operatorname{rad}(c-1) < \operatorname{cosocle}(c)$.

Next, suppose that $\frac{\operatorname{rad}(c-1)}{\operatorname{cosocle}(c)} < 1$. Since $\operatorname{rad}(c) = \frac{c}{\operatorname{cosocle}(c)}$, we observe that

$$\operatorname{rad}(c(c-1)) = \operatorname{rad}(c)\operatorname{rad}(c-1) = \frac{\operatorname{rad}(c-1)}{\operatorname{cosocle}(c)}c < c,$$

which shows that $(1, c-1, c)$ is an *abc* triple.

Lastly, if $(1, c-1, c)$ is an *abc* triple, then $\operatorname{rad}(c(c-1)) < c$. Consequently,

$$c > \operatorname{rad}(c(c-1)) = \operatorname{rad}(c)\operatorname{rad}(c-1) = \frac{\operatorname{rad}(c)(c-1)}{\operatorname{cosocle}(c-1)}.$$

This implies that

$$\operatorname{rad}(c) < \operatorname{cosocle}(c-1)\frac{c}{c-1}.$$

Since $\operatorname{rad}(c)$ is an integer and $\frac{c}{c-1} > 1$, we deduce that $\operatorname{rad}(c) \le \left\lfloor \operatorname{cosocle}(c-1)\frac{c}{c-1}\right\rfloor$, where $\lfloor x \rfloor$ denotes the floor function. Since $\frac{\operatorname{cosocle}(c-1)}{c-1} < 1$, we observe that

$$\begin{aligned}\left\lfloor \operatorname{cosocle}(c-1)\frac{c}{c-1}\right\rfloor &= \left\lfloor \operatorname{cosocle}(c-1) + \frac{\operatorname{cosocle}(c-1)}{c-1}\right\rfloor\\ &= \operatorname{cosocle}(c-1).\end{aligned}$$

Lastly, c is relatively prime to $c-1$, and thus $\operatorname{cosocle}(c-1) > \operatorname{rad}(c)$. □

An automatic consequence of Proposition 2.2 is that if c or $c-1$ is squarefree, then $(1, c-1, c)$ is not an abc triple since the cosocle of a squarefree positive integer is 1. Our next result establishes that the radical of a positive integer n is preserved if n is divided by the cosocle of any of its divisors.

Lemma 2.3. *Let m and n be positive integers. If m divides n, then* $\operatorname{rad}(n) = \operatorname{rad}\left(\frac{n}{\operatorname{cosocle}(m)}\right)$.

Proof. If $m = 1$, there is nothing to show. So suppose that $m > 1$ and let $m = \prod_{i=1}^{r} p_i^{e_i}$ be the unique prime factorization of m, with each p_i denoting a distinct prime. Since m divides n, we have that $n = q\prod_{i=1}^{r} p_i^{f_i}$ where $e_i \leq f_i$ for $1 \leq i \leq r$ and q is relatively prime to m. Since $\operatorname{cosocle}(m) = \prod_{i=1}^{r} p_i^{e_i - 1}$, we deduce that

$$\frac{n}{\operatorname{cosocle}(m)} = q\prod_{i=1}^{r} p_i^{f_i - e_i + 1}.$$

For $1 \leq i \leq r$, observe that $f_i - e_i + 1 \geq 1$ and thus $\operatorname{rad}\left(\frac{n}{\operatorname{cosocle}(m)}\right) = \operatorname{rad}(n)$. $\square$

With this lemma, we are now ready to prove Theorem 1.

Proof of Theorem 1. Since $c^k - 1 = (c-1)\sum_{j=0}^{k-1} c^j$, we deduce that m divides $c^k - 1$ for each positive integer k. By Lemma 2.3, $\operatorname{rad}(c^k - 1) = \operatorname{rad}\left(\frac{c^k - 1}{\operatorname{cosocle}(m)}\right)$. By assumption, $\frac{\operatorname{rad}(c)}{\operatorname{cosocle}(m)} < 1$ and thus

$$\operatorname{rad}\left(c^k\left(c^k - 1\right)\right) = \operatorname{rad}(c)\operatorname{rad}\left(\frac{c^k - 1}{\operatorname{cosocle}(m)}\right) \leq \frac{\operatorname{rad}(c)}{\operatorname{cosocle}(m)}\left(c^k - 1\right) < c^k - 1.$$

The result now follows since

$$c^k - \operatorname{rad}\left(c^k\left(c^k - 1\right)\right) > c^k - c^k + 1 = 1. \qquad \square$$

An immediate consequence of Theorem 1 and Proposition 2.2 is the following result.

Corollary 2.4. *If $(1, c-1, c)$ is an abc triple, then $\left(1, c^k - 1, c^k\right)$ is an abc triple for each positive integer k.*

In the next section, we will consider further consequences of Theorem 1 that do not require knowledge of an abc triple at the start. The proof of Theorem 1 relies on the factorization of $c^k - 1$. A similar factorization holds for $b^k + 1$ if k is odd, and our proof of Theorem 2 makes use of this.

Proof of Theorem 2. Observe that for each positive odd integer k, the following equality holds: $b^k + 1 = (b+1)\sum_{j=0}^{k-1}(-1)^j b^j$. It follows that m divides $b^k + 1$

for each positive integer k. By Lemma 2.3, $\operatorname{rad}(b^k+1) = \operatorname{rad}\left(\frac{b^k+1}{\operatorname{cosocle}(m)}\right)$. Since $\frac{\operatorname{rad}(b)}{\operatorname{cosocle}(m)} < 1$, we observe that

$$\operatorname{rad}\left(b^k\left(b^k+1\right)\right) = \operatorname{rad}(b)\operatorname{rad}\left(\frac{b^k+1}{\operatorname{cosocle}(m)}\right) \le \frac{\operatorname{rad}(b)}{\operatorname{cosocle}(m)}\operatorname{rad}(b^k+1) < b^k+1.$$

Consequently,

$$b^k+1-\operatorname{rad}\left(b^k\left(b^k+1\right)\right) > b^k+1-b^k-1 = 0. \qquad \square$$

Similarly to the deduction of Corollary 2.4, we now recover the following result as an immediate consequence of Theorem 2 and Proposition 2.2.

Corollary 2.5. *If $(1,b,b+1)$ is an abc triple, then $\left(1,b^k,b^k+1\right)$ is an abc triple for each positive odd integer k.*

Since $(1,8,9)$ is an *abc* triple, we deduce from Corollary 2.5 that $(1,8^k,8^k+1)$ is an *abc* triple for each positive odd integer k. We will also recover this sequence of *abc* triples as a consequence of Corollary 3.12.

By Corollary 2.4, we have that if $(1,c-1,c)$ is an *abc* triple, then $\left(1,c^k-1,c^k\right)$ is an *abc* triple for each positive integer k. This leads us to ask: given a positive integer $c>1$, for what positive integers k is $\left(1,c^k-1,c^k\right)$ an *abc* triple? To answer this question, we first recall a few number theory facts. Given a prime number p and an integer n, the *p-adic valuation* of n, denoted $v_p(n)$, is the unique integer that satisfies $n = p^{v_p(n)}q$ for some integer q that is relatively prime to p. It is easily verified that the following identities hold for each integer x,y,k: $v_p(xy) = v_p(x)+v_p(y)$, $v_p(\frac{x}{y}) = v_p(x)-v_p(y)$, and $v_p(x^k) = kv_p(x)$. In order to determine the exact power of a prime p that divides c^k-1, we first consider the following lemma about the binomial coefficient $\binom{y}{j}$.

Lemma 2.6. *Let $x,y\ge 2$ be integers and let p be a prime that divides x. Then for each j with $2\le j\le y$, the following inequality holds:*

$$v_p(xy) \le v_p\left(\binom{y}{j}x^j\right).$$

Moreover, equality holds if and only if $(p,j)=(2,2)$ and $x\equiv 2 \mod 4$ with y even.

Proof. In the case when $j=2$, we have that

$$\frac{\binom{y}{2}x^2}{xy} = \frac{x(y-1)}{2}.$$

Since p divides x, we have that

$$v_p\left(\frac{x(y-1)}{2}\right) \ge 0 \qquad \text{if and only if} \qquad v_p(xy) \le v_p\left(\binom{y}{2}x^2\right). \tag{2.1}$$

Moreover, equality in Equation (2.1) holds if and only if $p = 2$ and $v_2(x(y-1)) = 1$. Since x is even, it follows that this is equivalent to $x \equiv 2 \mod 4$ and y is even.

Now suppose that $3 \le j \le y$. It suffices to show that $v_p\left(\frac{\binom{y}{j}x^j}{xy}\right) > 0$. To this end, observe that

$$\frac{\binom{y}{j}x^j}{xy} = \binom{y-1}{j-1}\frac{x^{j-1}}{j}.$$

For each prime p, it is always the case that $v_p(j) < j-1$. Thus $v_p\left(\frac{x^{j-1}}{j}\right) > 0$ since $p|x$. The result now follows since $v_p\left(\binom{y-1}{j-1}\right) \ge 0$ and hence

$$v_p\left(\frac{\binom{y}{j}x^j}{xy}\right) \ge v_p\left(\frac{x^{j-1}}{j}\right) > 0. \qquad \square$$

Now suppose that p is a prime that does not divide an integer n. Then the *order* of n modulo p, denoted $\mathrm{ord}_p(n)$, is the least positive integer for which $n^{\mathrm{ord}_p(n)} \equiv 1 \mod p$. By Fermat's Little Theorem, $\mathrm{ord}_p(n)$ divides $p-1$. More generally, $n^k \equiv 1 \mod p$ if and only if $\mathrm{ord}_p(n)$ divides k. With this terminology, we now deduce the exact power of a prime p that divides $c^k - 1$.

Lemma 2.7. *Let c and k be positive integers with $c > 1$. Then p divides $c^k - 1$ if and only if $\mathrm{ord}_p(c)$ divides k. Moreover, if p divides $c^k - 1$, then $v_p(c^k - 1) = f_p + w_p$, where $w_p = v_p(k)$ and*

$$f_p = \begin{cases} v_2(c^2-1) - 1 & \text{if } p = 2, c \equiv 3 \mod 4, \text{ and } k \text{ is even,} \\ v_p(c^{\mathrm{ord}_p(c)} - 1) & \text{otherwise.} \end{cases} \tag{2.2}$$

Proof. The statement that p divides $c^k - 1$ if and only if $\mathrm{ord}_p(c)$ divides k is a standard number theory result. So suppose that p divides $c^k - 1$. Then $\mathrm{ord}_p(c)$ divides k and $p-1$. The latter is due to Fermat's Little Theorem. In particular, $\mathrm{ord}_p(c)$ is not divisible by p. Note that if $k = 1$, then there is nothing to show. We now proceed by cases and suppose that $k \ge 2$.

`Case 1.` Suppose that $p = 2$, $c \equiv 3 \mod 4$, and k is even. The result holds if $k = 2$, and so we may assume that $k \ge 4$. By the Binomial Theorem,

$$c^k = \left(c^2\right)^{\frac{k}{2}} = \left(c^2 - 1 + 1\right)^{\frac{k}{2}} = 1 + \frac{k}{2}\left(c^2-1\right) + \sum_{j=2}^{\frac{k}{2}} \binom{\frac{k}{2}}{j}\left(c^2-1\right)^j.$$

Since $c^2 - 1 \equiv 0 \mod 8$, it follows from Lemma 2.6 that

$$f_2 + w_2 = v_2\left(\frac{k}{2}\left(c^2-1\right)\right) < v_2\left(\binom{\frac{k}{2}}{j}\left(c^2-1\right)^j\right),$$

for $2 \leq j \leq \frac{k}{2}$. In particular, $c^k \equiv 1 + \frac{k}{2}\left(c^2-1\right) \mod 2^{f_2+w_2+1} \neq 1$ and $c^k \equiv 1 \mod 2^{f_2+w_2}$, from which we conclude that $v_2\big(c^k-1\big) = f_2+w_2$.

`Case 2.` Suppose that p is odd or $p=2$ with $c \equiv 1 \mod 4$ or k odd. Write $k = q_1 p^{w_p} \operatorname{ord}_p(c)$ for some integer q_1 that is not divisible by p. By the Binomial Theorem, we obtain

$$\begin{aligned} c^k = \left(c^{\operatorname{ord}_p(c)}\right)^{q_1 p^{w_p}} &= \left(c^{\operatorname{ord}_p(c)} - 1 + 1\right)^{q_1 p^{w_p}} \\ &= 1 + q_1 p^{w_p}\left(c^{\operatorname{ord}_p(c)}-1\right) + \sum_{j=2}^{q_1 p^{w_p}} \binom{q_1 p^{w_p}}{j}\left(c^{\operatorname{ord}_p(c)}-1\right)^j. \end{aligned}$$

From Lemma 2.6, we deduce that

$$f_p + w_p = v_p\Big(q_1 p^{w_p}\left(c^{\operatorname{ord}_p(c)}-1\right)\Big) < v_p\left(\binom{q_1 p^{w_p}}{j}\left(c^{\operatorname{ord}_p(c)}-1\right)^j\right)$$

for $2 \leq j \leq q_1 p^{w_p}$. Therefore $c^k \equiv 1 + q_1 p^{w_p}\left(c^{\operatorname{ord}_p(c)}-1\right) \mod p^{f_p+w_p+1} \neq 1$ and $c^k \equiv 1 \mod p^{f_p+w_p}$. Hence $v_p\big(c^k-1\big) = f_p+w_p$. □

As an immediate consequence of Lemma 2.7 and the Fundamental Theorem of Arithmetic, we obtain the following factorization for c^k-1.

Corollary 2.8. *Let c and k be positive integers with $c>1$. Then with notation as in Lemma 2.7,*

$$c^k - 1 = \prod_{\operatorname{ord}_p(c)\mid k} p^{f_p+w_p}.$$

As a demonstration of Corollary 2.8, let $c=21$ and $k=12$. With notation as above, we see that $w_2=2$, $w_3=1$, and $w_p=0$ for each prime $p \neq 2,3$. Next, we observe that

$$21^{12} - 1 = 2^4 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 61 \cdot 421 \cdot 463 \cdot 3181. \tag{2.3}$$

By Lemma 2.7, the primes appearing in Equation (2.3) are precisely those primes p for which $\operatorname{ord}_p(21)$ divides 12. With a computer algebra system, such as SageMath [13], it is checked that $f_p = 1$ for each prime $p \neq 2$ appearing in Equation (2.3) and $f_2 = 2$. Thus, $21^{12}-1 = \prod_{\operatorname{ord}_p(21)\mid 12} p^{f_p+w_p}$.

Theorem 2.9. *Let c and k be positive integers with $c>1$. With notation as in Corollary 2.8, write*

$$c^k - 1 = \prod_{\operatorname{ord}_p(c)\mid k} p^{f_p+w_p}.$$

Then $\left(1, c^k-1, c^k\right)$ is an abc triple if and only if one of the following conditions hold:

(*i*) *there exists a prime* $p > \mathrm{rad}(c)$ *such that* $\mathrm{ord}_p(c)$ *divides* k *and either* $f_p \geq 2$ *or* $w_p \geq 1$*;*

(*ii*) *there exists a prime* $p < \mathrm{rad}(c)$ *such that* $\mathrm{ord}_p(c)$ *divides* k *and* $f_p + w_p - 1 \geq m_p$*, where* m_p *denote the least positive integer such that* $p^{m_p} > \mathrm{rad}(c)$*;*

(*iii*) *for each prime* p *such that* $\mathrm{ord}_p(c)$ *divides* k*, there exist a non-negative integer* $a_p \leq f_p + w_p - 1$ *such that* $\prod_{\mathrm{ord}_p(c)|k} p^{a_p} > \mathrm{rad}(c)$*.*

Proof. First suppose that $\left(1, c^k - 1, c^k\right)$ is an *abc* triple. By Proposition 2.2, this is equivalent to

$$\mathrm{rad}(c) < \mathrm{cosocle}\left(c^k - 1\right) = \prod_{\mathrm{ord}_p(c)|k} p^{f_p + w_p - 1}.$$

In particular, taking $a_p = f_p + w_p - 1$ yields (*iii*).

Now suppose there is a prime $p > \mathrm{rad}(c)$ such that $\mathrm{ord}_p(c)$ divides k and either $f_p \geq 2$ or $w_p \geq 1$. Note that $f_p \geq 1$ for each prime p such that $\mathrm{ord}_p(c)$ divides k. Consequently, if $f_p \geq 2$ or $w_p \geq 1$, then $f_p + w_p \geq 2$ and thus p^2 divides $c^k - 1$. Then $\left(1, c^k - 1, c^k\right)$ is an *abc* triple by Theorem 1 since $\mathrm{cosocle}\left(p^2\right) = p > \mathrm{rad}(c)$.

Next, suppose that there is a prime $p < \mathrm{rad}(c)$ such that $\mathrm{ord}_p(c)$ divides k and $f_p + w_p - 1 \geq m_p$. Then p^{m_p+1} divides $c^k - 1$ and

$$\mathrm{cosocle}\left(p^{m_p+1}\right) = p^{m_p} > \mathrm{rad}(c)\,.$$

By Theorem 1, we deduce that $\left(1, c^k - 1, c^k\right)$ is an *abc* triple.

Lastly, suppose that for each prime p such that $\mathrm{ord}_p(c)$ divides k, there exists a positive integer $a_p \leq f_p + w_p - 1$ such that $\prod_{\mathrm{ord}_p(c)|k} p^{a_p} > \mathrm{rad}(c)$. Then $\prod_{\mathrm{ord}_p(c)|k} p^{a_p+1}$ divides $c^k - 1$ and the result now follows by Theorem 1 since

$$\mathrm{cosocle}\left(\prod_{\mathrm{ord}_p(c)|k} p^{a_p+1}\right) = \prod_{\mathrm{ord}_p(c)|k} p^{a_p} > \mathrm{rad}(c)\,. \qquad \square$$

As an illustration, consider $c = 21$ and $k = 12$. In the discussion following Corollary 2.8, we noted that $w_2 = f_2 = 2$. Moreover, for each prime $p \neq 2$ appearing in Equation (2.3) we have that $f_p = 1$ and $w_p = 0$. In particular, we see that statements (*i*) and (*ii*) of Theorem 2.9 are not satisfied for each prime p appearing in Equation (2.3). We also have that statement (*iii*) is not satisfied as the only prime for which $f_p + w_p - 1 > 0$ is $p = 2$ and $2^{f_2 + w_2 - 1} = 8 < \mathrm{rad}(21)$. It follows that $\left(1, 21^{12} - 1, 21^{12}\right)$ is not an *abc* triple. In the next section, we will see that 21 is the first odd integer $n > 1$ for which $\left(1, n^{\varphi(n)} - 1, n^{\varphi(n)}\right)$ is not an *abc* triple, where $\varphi(n)$ denotes the Euler-totient function. We note that $\varphi(21) = 12$.

3. Consequences

In this section, we consider various consequences of Theorems 1 and 2. From these consequences, we deduce the sequences of *abc* triples that were mentioned in the introduction. We note that this article began as an investigation of the following question: for what positive odd integers n is $\left(1, n^{\varphi(n)}-1, n^{\varphi(n)}\right)$ an *abc* triple? Here $\varphi(n)$ denotes the Euler-totient function. The question was motivated by the following observation: if n is an odd integer such that $3 \leq n \leq 99$, then $\left(1, n^{\varphi(n)}-1, n^{\varphi(n)}\right)$ is an *abc* triple for each n except $n = 21, 39, 69$, and 87. The fact that the four exceptions are composites is no surprise, as the answer to the question is true for odd primes n [3]. Our investigation of this phenomenon led to our Theorems 1 and 2, and our first consequence provides necessary conditions for when $\left(1, n^{\varphi(n)}-1, n^{\varphi(n)}\right)$ is an *abc* triple for a positive odd integer n. To prove this result, we first recall the following result from elementary number theory.

Lemma 3.1. *Let n be a positive odd integer. Then $n^{2^k} \equiv 1 \mod 2^{k+2}$ for each positive integer k.*

Proof. Since n is odd, there is an integer m such that $n = 2m+1$. By the Binomial Theorem,

$$n^{2^k} = (2m+1)^{2^k} = \sum_{j=0}^{2^k} \binom{2^k}{j} (2m)^j = 1 + 2^{k+1} m \left(1 + \left(2^k - 1\right) m\right) + \sum_{j=3}^{2^k} \binom{2^k}{j} (2m)^j .$$

Now observe that $m\left(1+\left(2^k-1\right)m\right)$ is always even and $\binom{2^k}{j}(2m)^j$ is divisible by 2^{k+2} for $3 \leq j \leq 2^k$. Consequently, $n^{2^k} \equiv 1 \mod 2^{k+2}$. □

With this result, we obtain our first application of Theorem 1.

Corollary 3.2. *Let $n > 1$ be an odd integer and let φ denote the Euler-totient function. Set $d = \gcd(n-1, \varphi(n))$ and $m = 2^{v_2(4\varphi(n)) - 2v_2(d)} d^2$. If $\operatorname{cosocle}(m) > \operatorname{rad}(n)$, then $\left(1, n^{\varphi(n)k} - 1, n^{\varphi(n)k}\right)$ is an abc triple for each positive integer k.*

Proof. Let $P = \sum_{j=0}^{\varphi(n)-1} n^j$ and observe that $n^{\varphi(n)} - 1 = (n-1)\,P$. Since $d = \gcd(n-1, \varphi(n))$ divides $n-1$, we have that $n \equiv 1 \mod d$ and thus

$$P \equiv \sum_{j=0}^{\varphi(n)-1} 1^j \mod d = \varphi(n) \mod d.$$

In particular, d divides P. Since $n^{\varphi(n)} - 1 = (n-1)\,P$, we deduce that d^2 divides $n^{\varphi(n)} - 1$.

Next, write $\varphi(n) = 2^{v_2(\varphi(n))} r$ for r an odd integer. By Lemma 3.1,

$$n^{\varphi(n)} - 1 = (n^r)^{2^{v_2(\varphi(n))}} - 1 \equiv 0 \mod 2^{v_2(\varphi(n))+2}.$$

Hence $2^{v_2(\varphi(n))+2}$ divides $n^{\varphi(n)}-1$. It follows that

$$2^{v_2(\varphi(n))+2}\frac{d^2}{2^{v_2(d^2)}} = 2^{v_2(4\varphi(n))-2v_2(d)}d^2 = m$$

divides $n^{\varphi(n)}-1$. The result now follows from Theorem 1. $\square$

As an illustration, let $n = 75$. Then with notation as in Corollary 3.2, we observe that $\varphi(75) = 40$, $d = 2$, and $m = 32$. Since $\operatorname{cosocle}(32) = 16 > \operatorname{rad}(75) = 15$, we have that $\left(1, 75^{40k}-1, 75^{40k}\right)$ is an *abc* triple for each positive integer k. We note that the converse to Corollary 3.2 does not hold. In fact, if $3 \le n \le 99$ is an odd integer such that $\left(1, n^{\varphi(n)}-1, n^{\varphi(n)}\right)$ is an *abc* triple, then the corollary fails to show the cases corresponding to $n = 33, 35, 55, 57, 63, 65, 77, 93, 95$, and 99. The following result provides an improvement, but comes at the cost of having to compute $v_p(n^{\varphi(n)}-1)$ for each prime p that divides $\gcd(n^{\varphi(n)}-1, \varphi(n))$.

Corollary 3.3. *Let $n > 1$ be an integer and let φ denote the Euler-totient function. Set $d = \gcd(n^{\varphi(n)}-1, \varphi(n))$ and*

$$m = \prod_{p|d} p^{v_p(n^{\varphi(n)}-1)}.$$

If $\operatorname{cosocle}(m) > \operatorname{rad}(n)$, then $\left(1, n^{\varphi(n)k}-1, n^{\varphi(n)k}\right)$ is an abc triple for each positive integer k.

Proof. The result follows from Theorem 1 since m divides $n^{\varphi(n)}-1$. $\square$

For odd integers n such that $3 \le n \le 99$, Corollary 3.3 allows us to conclude that for $n \ne 21, 39, 55, 57, 69$, and 87, $\left(1, n^{\varphi(n)k}-1, n^{\varphi(n)k}\right)$ is an *abc* triple for each positive integer k. As noted at the start of the section, $n = 21, 39, 69$, and 87 are the only n's in this range for which $\left(1, n^{\varphi(n)}-1, n^{\varphi(n)}\right)$ is not an *abc* triple. In particular, Corollary 3.3 fails to show the cases corresponding to $n = 55, 57$. Indeed, when $n = 55$, we have that $\varphi(55) = 40$ and $\gcd\left(55^{40}-1, 40\right) = 8$. Then $m = 2^{v_2(55^{40}-1)} = 64$, and thus $\operatorname{cosocle}(64) = 32 < \operatorname{rad}(55) = 55$. Consequently, the assumption of Corollary 3.3 is not satisfied in the case when $n = 55$. We note that $\operatorname{cosocle}\left(55^{40}-1\right) = 288$, and hence $\left(1, 55^{40k}-1, 55^{40k}\right)$ is an *abc* triple for each positive integer k by Proposition 2.2. The failure of Corollaries 3.2 and 3.3 in the $n = 55$ case stems from the fact that the primes dividing m must divide $\varphi(n)$. Indeed, $\operatorname{cosocle}\left(55^{40}-1\right) = 32 \cdot 9$ and $3 \nmid \varphi(55)$.

To state our next result, we recall the Carmichael function $\lambda : \mathbb{N} \to \mathbb{N}$, which has the property that $\lambda(m)$ is the least positive integer for which $a^{\lambda(m)} \equiv 1 \mod m$ for each integer a that is relatively prime to m. In particular, $\lambda(m)$ divides $\varphi(m)$.

Corollary 3.4. *Let λ and φ denote the Carmichael function and Euler-totient function, respectively. If m and n are relatively prime positive integers such that* cosocle$(m) > \mathrm{rad}(n) > 1$, *then* $\left(1, n^{\lambda(m)k} - 1, n^{\lambda(m)k}\right)$ *and* $\left(1, n^{\varphi(m)k} - 1, n^{\varphi(m)k}\right)$ *are abc triples for each positive integer* k.

Proof. Since $n^{\lambda(m)} \equiv 1 \mod m$, we have that m divides $n^{\lambda(m)} - 1$. By Theorem 1, we have that $\left(1, n^{\lambda(m)k} - 1, n^{\lambda(m)k}\right)$ is an *abc* triple for each positive integer k. Since $\lambda(m) \mid \varphi(m)$, we also have that $\left(1, n^{\varphi(m)k} - 1, n^{\varphi(m)k}\right)$ is an *abc* triple for each positive integer k. □

As an example, choose $n = 11$ and $m = 32$. Then $\mathrm{cosocle}(32) = 16 > \mathrm{rad}(11)$, and therefore the conditions of Corollary 3.4 are satisfied. As a result, we find that $\left(1, 11^{\lambda(32)k} - 1, 11^{\lambda(32)k}\right) = (1, 11^{8k} - 1, 11^{8k})$ is a sequence of *abc* triples. More generally, we have the following application of Corollary 3.4.

Corollary 3.5. *Let* $n > 1$ *be an integer and let* p *be an odd prime such that* $p > \mathrm{rad}(n)$. *Then for each positive integer* k, $\left(1, n^{p(p-1)k} - 1, n^{p(p-1)k}\right)$ *is an abc triple.*

Proof. By assumption, $\mathrm{cosocle}\left(p^2\right) = p > \mathrm{rad}(n)$. Moreover, $\lambda\left(p^2\right) = p\,(p-1)$ since p is prime. It follows from Corollary 3.4 that $\left(1, n^{\lambda(p^2)k} - 1, n^{\lambda(p^2)k}\right) = \left(1, n^{p(p-1)k} - 1, n^{p(p-1)k}\right)$ is an *abc* triple for each positive integer k. □

Taking $(n, k) = (2, 1)$ in Corollary 3.5 yields that $\left(1, 2^{p(p-1)} - 1, 2^{p(p-1)}\right)$ is an *abc* triple for each odd prime p. This result is originally due to Granville and Tucker [6]. Theorem 2.9 gives the following refinement of Corollary 3.5.

Corollary 3.6. *Let* $n > 1$ *be an integer and let* p *be an odd prime such that* $p > \mathrm{rad}(n)$. *Then for each positive integer* k, $\left(1, n^{p\,\mathrm{ord}_p(n)k} - 1, n^{p\,\mathrm{ord}_p(n)k}\right)$ *is an abc triple. In particular, if* $n \equiv 1 \mod p$ *and* $p > \mathrm{rad}(n)$, *then* $\left(1, n^{pk} - 1, n^{pk}\right)$ *is an abc triple for each positive integer* n.

Proof. In the notation of Theorem 2.9, we have that $w_p = v_p(p\,\mathrm{ord}_p(n)) = 1$. Since $p > \mathrm{rad}(n)$ and $\mathrm{ord}_p(n)$ divides $p\,\mathrm{ord}_p(n)$, Theorem 2.9 (*i*) implies that $\left(1, n^{p\,\mathrm{ord}_p(n)} - 1, n^{p\,\mathrm{ord}_p(n)}\right)$ is an *abc* triple. The result now follows by Corollary 2.4. The second statement is automatic since if $n \equiv 1 \mod p$, then $\mathrm{ord}_p(n) = 1$. □

As a demonstration, let $n = 16$ and $p = 5$. Then Corollary 3.6 asserts that $\left(1, 16^{5k} - 1, 16^{5k}\right)$ is an *abc* triple for each positive integer k.

Corollary 3.7. *Let* $n > 1$ *be an integer that is either odd or even and non-squarefree. Then* $\left(1, n^{(n-1)k} - 1, n^{(n-1)k}\right)$ *is an abc triple for each positive integer* k.

Proof. Let $P = \sum_{j=0}^{n-2} n^j$ and observe that $n^{n-1} - 1 = (n-1)\,P$. Moreover,

$$P \equiv \sum_{j=0}^{n-2} (1)^j \mod(n-1) = 0 \mod (n-1)\,.$$

In particular, $(n-1)^2$ divides $n^{n-1} - 1$ and thus

$$\operatorname{rad}(n^{n-1} - 1) = \operatorname{rad}\left(\frac{n^{n-1} - 1}{n-1}\right).$$

Now suppose that n is odd. We claim that 4 divides P. If $n \equiv 1 \mod 4$, then this follows since P is divisible by $n-1$. So suppose that $n \equiv 3 \mod 4$. Then 4 divides $n+1$, and hence 4 divides P since

$$P \equiv \sum_{j=0}^{n-2} (-1)^j \mod(n+1) = 0 \mod (n+1)\,.$$

Consequently,

$$\operatorname{rad}(n^{n-1} - 1) = \operatorname{rad}\left(\frac{n^{n-1} - 1}{2\,(n-1)}\right) \leq \frac{n^{n-1} - 1}{2\,(n-1)}. \tag{3.1}$$

Now observe that by Equation (3.1),

$$\operatorname{cosocle}(n^{n-1} - 1) = \frac{n^{n-1} - 1}{\operatorname{rad}(n^{n-1} - 1)} \geq 2\,(n-1) > \operatorname{rad}(n).$$

The claim now follows by Theorem 1 with $m = n^{n-1} - 1$.

Lastly, suppose that n is an even non-squarefree positive integer. Then $n = a^2 b$ for some positive integers a and b with $a > 1$ and b squarefree. Then $\operatorname{rad}(n) = \operatorname{rad}(ab) \leq ab < n - 1$. Since $\operatorname{rad}(n^{n-1} - 1) = \operatorname{rad}\left(\frac{n^{n-1}-1}{n-1}\right) \leq \frac{n^{n-1}-1}{n-1}$, we deduce that

$$\operatorname{cosocle}(n^{n-1} - 1) = \frac{n^{n-1} - 1}{\operatorname{rad}(n^{n-1} - 1)} \geq n - 1 > \operatorname{rad}(n)\,.$$

The result follows by Theorem 1 with $m = n^{n-1} - 1$. $\square$

From Corollary 3.7, we recover that $\left(1, 9^k - 1, 9^k\right) = \left(1, 3^{2k} - 1, 3^{2k}\right)$ is a sequence of *abc* triples. In particular, we obtain the smallest *abc* triple $(1, 8, 9)$ as a special case. Taking $n = 8$ in Corollary 3.7 gives us the sequence of *abc* triples $\left(1, 8^{7k} - 1, 8^{7k}\right)$, which generalizes the sequence $\left(1, 8^{7^k} - 1, 8^{7^k}\right)$ that appears in [8].

Corollary 3.8. *Let $n > 1$ be an integer. Then $\left(1, n^{(n+1)k} - 1, n^{(n+1)k}\right)$ is an abc triple whenever $(n+1)\,k$ is a positive even integer.*

Proof. Let l be a positive even integer and let $P = \sum_{j=0}^{l-1} (-1)^{j+1} n^j$. Then $n^l - 1 = (n+1)\, P$. We now proceed by cases.

Case 1. Suppose that n is a positive even integer and let $l = 2\,(n+1)$. Since $n \equiv -1 \mod(n+1)$, we have that $P \equiv \sum_{j=0}^{l-1} (-1)^{j+1} = 0 \mod(n+1)$ and thus

$$\mathrm{rad}(n^l - 1) = \mathrm{rad}\left(\frac{n^l - 1}{n+1}\right) \leq \frac{n^l - 1}{n+1}.$$

The claim now holds by Theorem 1 with $m = n^l - 1$ since

$$\mathrm{cosocle}(n^l - 1) = \frac{n^l - 1}{\mathrm{rad}(n^l - 1)} \geq n + 1 > \mathrm{rad}(n).$$

Case 2. Suppose that n is a positive odd integer. Then $l = n + 1$ is even and $P \equiv 0 \mod(n+1)$. A similar argument to that of Case 1 with $m = n^l - 1$ shows that the result holds by Theorem 1. □

As an example, choose $n = 21$. As a result, $(n+1)k$ is even for every positive integer k, and by Corollary 3.8 we know that $\left(1, 21^{22k} - 1, 21^{22k}\right)$ is a sequence of *abc* triples.

Corollary 3.9. *Let $j \geq 2$ be an integer. Then $\left(1, \left(2^j - 1\right)^{2k} - 1, \left(2^j - 1\right)^{2k}\right)$ is an abc triple for each positive integer k.*

Proof. Observe that $\mathrm{rad}\left(\left(2^j - 1\right)^2\right) = \mathrm{rad}\left(2^j - 1\right) \leq 2^j - 1$. Since $\left(2^j - 1\right)^2 - 1 = 2^{j+1}\left(2^{j-1} - 1\right)$, we deduce that

$$\mathrm{cosocle}\left(\left(2^j - 1\right)^2 - 1\right) = \frac{2^{j+1}\left(2^{j-1} - 1\right)}{2\,\mathrm{rad}(2^{j-1} - 1)} = \frac{2^j\left(2^{j-1} - 1\right)}{\mathrm{rad}(2^{j-1} - 1)} \geq 2^j.$$

The result now follows from Theorem 1, since $\mathrm{cosocle}((2^j - 1)^2 - 1) > \mathrm{rad}((2^j - 1)^2)$. □

The $j = 2$ and $j = 3$ cases in Corollary 3.9 result in the sequences of *abc* triples $\left(1, 9^k - 1, 9^k\right)$ and $\left(1, 49^k - 1, 49^k\right)$, respectively. Of note is that the proof of the corollary is made possible by the lower bound, $\mathrm{cosocle}\left(\left(2^j - 1\right)^2 - 1\right) \geq 2^j$. This leads us to ask, can Corollary 3.9 be generalized to deduce sequences of *abc* triples $(1, c-1, c)$ with $\mathrm{cosocle}(c-1)$ bounded below by n^j for some positive integer of the form n^j? The answer is yes, but we have to take $c = \left(n^j - 1\right)^k$ for some positive even integer k that is divisible by n to allow a similar argument to that of Corollary 3.9 to work. This is shown below.

Corollary 3.10. *Let $n \geq 3$ and $j \geq 1$ be integers. If k is a positive integer such that nk is even, then $\left(1, \left(n^j - 1\right)^{nk} - 1, \left(n^j - 1\right)^{nk}\right)$ is an abc triple.*

Proof. Observe that $\operatorname{rad}\Big(\big(n^j-1\big)^{nk}\Big) \leq n^j - 1$ and

$$\big(n^j-1\big)^{nk} - 1 = -1 + \sum_{l=0}^{nk} \binom{nk}{l} n^{jl}\,(-1)^{nk-l} = -kn^{j+1} + \sum_{l=2}^{nk} \binom{nk}{l} n^{jl}\,(-1)^{nk-l}.$$

Note that in the last expression, each term in the sum is divisible by n^{j+1}. From this, we deduce that $\operatorname{cosocle}\Big(\big(n^j-1\big)^{nk}-1\Big) \geq n^j$. Hence $\operatorname{cosocle}\Big(\big(n^j-1\big)^{nk}-1\Big) > \operatorname{rad}\Big(\big(n^j-1\big)^{nk}\Big)$, and the result now follows by Theorem 1. $\square$

As an illustration, consider $(n,j) = (3,1)$ and $k = 2l$ for some positive integer l. This results in the sequence of *abc* triples $\big(1, 64^l - 1, 64^l\big)$.

Corollary 3.11. *Let n be a positive even integer. Then $\big(1, n^{(n+1)k}, n^{(n+1)k}+1\big)$ is an abc triple for each positive odd integer k.*

Proof. Observe that $n^{n+1} + 1 = (n+1)\sum_{j=0}^{n} (-1)^j\, n^j$. Since $n \equiv -1 \mod (n+1)$, it follows that

$$\sum_{j=0}^{n} (-1)^j\, n^j \equiv \sum_{j=0}^{n} 1 \mod (n+1) = 0 \mod (n+1)\,.$$

Hence, $\operatorname{rad}(n^{n+1}+1) = \operatorname{rad}\Big(\frac{n^{n+1}+1}{n+1}\Big) \leq \frac{n^{n+1}+1}{n+1}$. Consequently,

$$\operatorname{cosocle}(n^{n+1}+1) = \frac{n^{n+1}+1}{\operatorname{rad}(n^{n+1}+1)} \geq n+1 > \operatorname{rad}(n)\,.$$

The result now follows from Theorem 2 by taking $m = n^{n+1}+1$. $\square$

As a demonstration of the corollary, take $n = 22$. Then $\big(1, 22^{23k}, 22^{23k}+1\big)$ is a sequence of *abc* triples for each positive odd integer k.

Corollary 3.12. *Let $n \geq 3$ be an odd integer and let $j \geq 1$ be an integer. Then for each odd integer k, $\Big(1, \big(n^j-1\big)^{nk}, \big(n^j-1\big)^{nk}+1\Big)$ is an abc triple.*

Proof. Observe that $\operatorname{rad}\big(\big(n^j-1\big)^{n}\big) \leq n^j - 1$ and

$$\big(n^j-1\big)^{n} + 1 = 1 + \sum_{l=0}^{n} \binom{n}{l} n^{jl}\,(-1)^{n-l} = n^{j+1} + \sum_{l=2}^{n} \binom{n}{l} n^{jl}\,(-1)^{n-l}.$$

Note that in the last expression, each term in the sum is divisible by n^{j+1}. From this, we conclude that $\operatorname{cosocle}\big(\big(n^j-1\big)^{n}+1\big) \geq n^j$. Hence $\operatorname{cosocle}\big(\big(n^j-1\big)^{n}+1\big) > \operatorname{rad}\big(\big(n^j-1\big)^{n}\big)$, and the result now follows by Theorem 2. $\square$

As an example, let $n = 3$ and $j = 1$. Then we get the sequence of *abc* triples $\big(1, 8^k, 8^k+1\big)$ for each odd integer k. In particular, we recover the *abc* triple $(1,8,9)$ as a special case.

4. *abc* Triples of the Form $(1, c-1, c)$ and the ABC@Home Project

The ABC@Home project found that there are exactly 14 482 065 *abc* triples (a, b, c) with $c < 10^{18}$. The information found by the ABC@Home project is available on Bart de Smit's webpage [4]. Given an *abc* triple (a, b, c), we define its *quality* to be

$$q(a, b, c) = \frac{\log c}{\log \operatorname{rad}(abc)}.$$

By definition, we have that an *abc* triple (a, b, c) satisfies $\operatorname{rad}(abc) < c$, and thus $q(a, b, c) > 1$. This gives us the following restatement of the *abc* conjecture: For each $\epsilon > 0$, there are finitely many *abc* triples (a, b, c) with $q(a, b, c) > 1 + \epsilon$.

The *abc* triple with the largest known quality is $\left(2, 3^{10} \cdot 109, 23^5\right)$, which has a quality of approximately 1.6299. In fact, Baker's [2] *explicit abc conjecture* asserts that there is no *abc* triple (a, b, c) with $q(a, b, c) \geq \frac{7}{4}$. From this statement, Fermat's Last Theorem for exponent $n > 6$ easily follows. We note that the explicit *abc* conjecture and the *abc* conjecture are not equivalent.

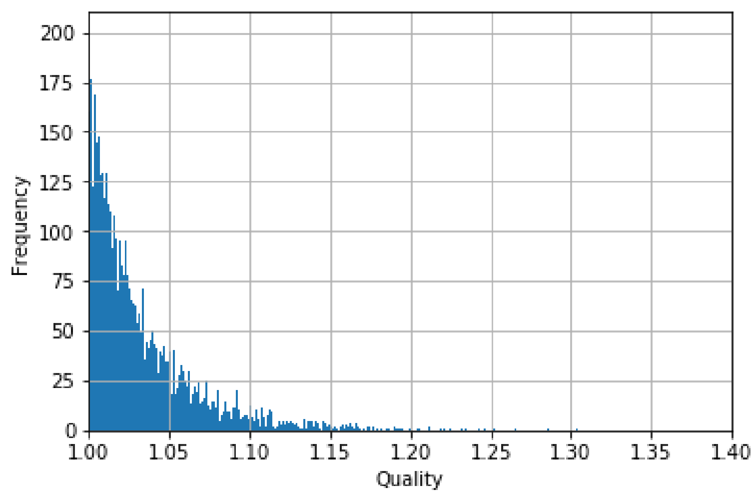

Figure 1: Histogram of the quality of *abc* triples $(1, c-1, c)$ with $c < 10^{18}$

Let S denote the set of *abc* triples of the form $(1, c-1, c)$ with $c < 10^{18}$. From the ABC@Home project, we have that $\#S = 45\,603$. The largest quality occurring in S corresponds to the *abc* triple $(1, 4374, 4375)$, which has quality approximately equal to 1.5679. Figure 1 summarize the distribution of the quality of all *abc* triples in S. The bin size in the histogram is set to 5000. We note that all computations done in this section were done on SageMath [13], and our code is available on GitHub [1].

Table 1 lists the first fifteen *abc* triples of the form $(1, c-1, c)$, their quality, and whether they arise from one of the results proven in Section 3. The only *abc* triple in the table that is not of the form $\left(1, n^l - 1, n^l\right)$ or $\left(1, n^l, n^l + 1\right)$ for some integer $l > 1$ is $(1, 1215, 1216)$. However, most *abc* triples in S are not of the aforementioned form. More precisely, S contains a total of 7376 (resp. 1038) *abc* triples of the form $\left(1, n^l - 1, n^l\right)$ (resp. $\left(1, n^l, n^l + 1\right)$) for some integer $l > 1$. We note that $(1, 8, 9)$ is the only double-counted element since Mihăilescu's Theorem [10] (formerly known as Catalan's conjecture) asserts that 2 and 3 are the only two consecutive perfect powers. Consequently,

$$T = \left\{(1, c-1, c) \in S \mid c = n^l \text{ or } c = n^l + 1 \text{ for some } l > 1\right\}$$

has 8413 elements. The highest quality *abc* triple in T is $(1, 2400, 2401)$, with a quality of approximately 1.4557. Observe that this *abc* triple is obtained from Corollary 3.9 since $(1, 2400, 2401) = \left(1, 7^4 - 1, 7^4\right)$.

$(1, c-1, c)$	$q(1, c-1, c)$	Arises from result in Section 3?
$(1, 8, 9)$	1.2263	Yes; Corollary 3.7 with $(n, k) = (3, 1)$
$(1, 48, 49)$	1.0412	Yes; Corollary 3.9 with $(j, k) = (3, 1)$
$(1, 63, 64)$	1.1127	Yes; Corollary 3.10 with $(n, j, k) = (3, 1, 1)$
$(1, 80, 81)$	1.2920	Yes; Corollary 3.8 with $(n, k) = (3, 1)$
$(1, 224, 225)$	1.0129	Yes; Corollary 3.9 with $(j, k) = (4, 1)$
$(1, 242, 243)$	1.3111	No
$(1, 288, 289)$	1.2252	No
$(1, 512, 513)$	1.3176	Yes; Corollary 3.12 with $(n, j, k) = (3, 1, 2)$
$(1, 624, 625)$	1.0790	Yes; Corollary 3.7 with $(n, k) = (5, 1)$
$(1, 675, 676)$	1.0922	No
$(1, 728, 729)$	1.0459	Yes; Corollary 3.7 with $(n, k) = (3, 3)$
$(1, 960, 961)$	1.0048	Yes; Corollary 3.9 with $(j, k) = (5, 1)$
$(1, 1024, 1025)$	1.1523	Yes; Corollary 3.11 with $(n, k) = (4, 1)$
$(1, 1215, 1216)$	1.1194	No
$(1, 2303, 2304)$	1.0204	No

Table 1: The first fifteen *abc* triples of the form $(1, c-1, c)$

Now suppose that $\left(1, n^l - 1, n^l\right)$ is an *abc* triple for some integer $l > 1$. By Proposition 2.2, we know that $\operatorname{cosocle}\left(n^l - 1\right) > \operatorname{rad}(n)$. However, checking that $\left(1, n^l - 1, n^l\right)$ is an *abc* triple via this criteria gets more difficult as n^l grows. By Theorem 1, we can deduce that $\left(1, n^l - 1, n^l\right)$ is an *abc* triple if there is a divisor m of $n^l - 1$ such that $\operatorname{cosocle}(m) > \operatorname{rad}(n)$. By considering those elements in T

of the form $\big(1, n^l - 1, n^l\big)$ for some integer $l > 1$, we find that m can be taken to be a proper divisor of $n^l - 1$, except for the *abc* triples $(1, c-1, c)$ where $c \in \{9, 676, 11309769, 17380816062160329\}$. Indeed, $\operatorname{rad}(676) = 26$ and $675 = 3^3 5^2$. The only divisor of 675 satisfying $\operatorname{cosocle}(m) > 26$ is $m = 675$.

The above leads us to ask: given $\big(1, n^l - 1, n^l\big) \in T$ with $l > 1$ an integer, what is the least divisor m of $n^l - 1$ for which $\operatorname{cosocle}(m) > \operatorname{rad}(n)$? Using SageMath [13], we answered this question, and our datafile can be accessed in [1, triples_for_thm1.csv]. Table 2 gives the first fifteen elements (a, b, c) in T of the form $\big(1, n^l - 1, n^l\big)$, where n and l are listed, as well as the least divisor m of $n^l - 1$ for which $\operatorname{cosocle}(m) > \operatorname{rad}(n)$ holds. The quality of the *abc* triple is also given.

(a, b, c)	n	l	m	$q(a, b, c)$
$(1, 8, 9)$	3	2	8	1.2263
$(1, 48, 49)$	7	2	16	1.0412
$(1, 63, 64)$	2	6	9	1.1127
$(1, 80, 81)$	3	4	8	1.2920
$(1, 224, 225)$	15	2	32	1.0129
$(1, 242, 243)$	3	5	121	1.3111
$(1, 288, 289)$	17	2	144	1.2252
$(1, 624, 625)$	5	4	16	1.0790
$(1, 675, 676)$	26	2	675	1.0922
$(1, 728, 729)$	3	6	8	1.0459
$(1, 960, 961)$	31	2	64	1.0048
$(1, 2303, 2304)$	48	2	49	1.0204
$(1, 2400, 2401)$	7	4	16	1.4557
$(1, 3024, 3025)$	55	2	432	1.0348
$(1, 3968, 3969)$	63	2	64	1.1554

Table 2: The first fifteen *abc* triples (a, b, c) of the form $\big(1, n^l - 1, n^l\big)$ for $l > 1$, with m the least divisor of $n^l - 1$ satisfying $\operatorname{cosocle}(m) > \operatorname{rad}(n)$

Similarly, we ask the same question in the setting of Theorem 2. That is, given $\big(1, n^l, n^l + 1\big) \in T$ with $l > 1$ an odd integer, what is the least positive divisor m of $n^l + 1$ for which $\operatorname{cosocle}(m) > \operatorname{rad}(n)$? We note that T has 596 elements of the form $\big(1, n^l, n^l + 1\big)$ for some integer $l > 1$. We also answer this question through SageMath, and our datafile is found in [1, triples_for_thm2.csv]. Table 3 gives the first fifteen elements (a, b, c) in T of the form $\big(1, n^l, n^l + 1\big)$, where n and l are listed, as well as the least divisor m of $n^l + 1$ for which $\operatorname{cosocle}(m) > \operatorname{rad}(n)$ holds. In particular, we find that $(1, 8, 9)$ is the only *abc* triple of the form $(1, n^l, n^l + 1)$ in T

with $l > 1$ an odd integer for which there is no proper divisor m of $n^l + 1$ satisfying cosocle$(m) > \mathrm{rad}(n)$.

(a, b, c)	n	l	m	$q(a, b, c)$
$(1, 8, 9)$	2	3	9	1.2263
$(1, 512, 513)$	2	9	9	1.3176
$(1, 6859, 6860)$	19	3	343	1.2281
$(1, 12167, 12168)$	23	3	676	1.2555
$(1, 17576, 17577)$	26	3	81	1.0039
$(1, 29791, 29792)$	31	3	784	1.1424
$(1, 32768, 32769)$	2	15	9	1.0406
$(1, 110592, 110593)$	48	3	49	1.0135
$(1, 250047, 250048)$	63	3	64	1.0351
$(1, 279936, 279937)$	6	7	49	1.0124
$(1, 512000, 512001)$	80	3	81	1.4433
$(1, 1953125, 1953126)$	5	9	27	1.0423
$(1, 2097152, 2097153)$	2	21	9	1.0287
$(1, 3176523, 3176524)$	147	3	676	1.0145
$(1, 7077888, 7077889)$	192	3	169	1.0515

Table 3: The first fifteen *abc* triples (a, b, c) of the form $\left(1, n^l, n^l + 1\right)$ for $l > 1$ an odd integer, with m the least divisor of $n^l + 1$ satisfying cosocle$(m) > \mathrm{rad}(n)$

Next, we investigate how many elements of T arise from the results proven in Section 3. Indeed, each *abc* triple produced by the results of that section are of the form $\left(1, n^l - 1, n^l\right)$ or $\left(1, n^l, n^l + 1\right)$ for some integer $l > 1$. Moreover, for each *abc* triple obtained from one of our corollaries in Section 3, we apply the following result from [15, Section 2.3].

Proposition 4.1. *Let $(1, c-1, c)$ be an abc triple. Then the following are abc triples:*

$$\left(1, (c-1)^3, c\left(c^2 - 3c + 3\right)\right) \qquad \textit{and} \qquad \left(1, c\,(c-2), (c-1)^2\right).$$

As a demonstration, the *abc* triple $(1, 2303, 2304)$ is obtained from the *abc* triple $(1, 48, 49)$ since $2304 = 48^2$. In particular, $(1, 2303, 2304)$ can now be viewed as a consequence of Corollary 3.9 and Proposition 4.1. Proposition 4.1 is part of a more general result in [15, Section 2.3], which provides a way of mapping an *abc* triple (a, b, c) to a new *abc* triple by applying polynomial identities. The more general result arises by splitting the binomial formula $(a + b)^n$ to obtain the following family of identities:

$$a^{n-k}\left(\sum_{j=0}^{k}\binom{n}{j}a^{k-j}b^{j}\right)+b^{k+1}\left(\sum_{j=0}^{n-k-1}\binom{n}{j}a^{j}b^{n-k-1-j}\right)=c^{n}.$$

Taking $k=0$ yields Corollary 2.4. Therefore, the two non-trivial polynomial identities with $a=1$ are those occurring in Proposition 4.1.

Corollaries 3.3 through 3.12 provide us with a recipe for constructing *abc* triples. For each of these corollaries, we consider the set

$$C_i=\{(1,c-1,c)\in T \mid (1,c-1,c) \text{ is obtained from Corollary 3.i}\},$$

where $3\le i\le 12$. By Table 1, we see that $(1,224,225)\in C_9$, but $(1,242,243)\notin C_i$ for each i. Using SageMath, we have the following table:

i	3	4	5	6	7	8	9	10	11	12
$\#C_i$	32	58	12	17	41	29	81	46	18	36

The low number of *abc* triples in T occurring in each C_i is expected. Indeed, for Corollary 3.5 to yield an *abc* triple in T, we require that $n>1$ be an integer, p be an odd prime such that $p>\operatorname{rad}(n)$, and $n^{p(p-1)k}<10^{18}$ for some integer k. For n an odd integer, the only possible (n,p,k) is $(3,5,1)$, which gives the *abc* triple $(1,3486784400,3486784401)$. We also note that since Corollary 3.5 is a special case of Corollary 3.4, we have that $C_5\subseteq C_4$. Now let

$$C=\bigcup_{3\le i\le 12}C_i.$$

We find that $\#C=164$.

Lastly, let D be the set of *abc* triples in T with the property that an element of D is in C or can be obtained from an *abc* triple in C after successive applications of Proposition 4.1 and Corollaries 2.4 and 2.5. As an illustration, the *abc* triple $(1,12214672127,12214672128)$ is not in C, but it is in D. To see this, recall that $(1,2303,2304)$ is obtained from the *abc* triple $(1,48,49)$ via Proposition 4.1. Then,

$$(1,12214672127,12214672128)=\left(1,(c-1)^{3},c\left(c^{2}-3c+3\right)\right),$$

where $c=2304$, which shows that the *abc* triple is in D. In fact, with the exception of the *abc* triple $(1,1215,1216)$, every *abc* triple appearing in Table 1 is in D. Using SageMath, we find that D has 311 elements.

We conclude this article by considering the percentage of *abc* triples $(1,c-1,c)$ in S and T, that are also in D. More precisely, for sets X and Y such that $X\subseteq Y\subseteq S$, we define

$$\delta_{X,Y}(x)=\frac{\#\{(1,c-1,c)\in X \mid c\le x\}}{\#\{(1,c-1,c)\in Y \mid c\le x\}}.$$

In particular, $\delta_{X,Y}(x)$ gives the percentage of *abc* triples $(1, c-1, c)$ of Y with $c \leq x$ that are in X. The table below gives some values of $\delta_{T,S}(x)$, $\delta_{D,S}(x)$, and $\delta_{D,T}(x)$.

x	10^4	10^6	10^8	10^{10}	10^{12}	10^{14}	10^{16}	10^{18}
$\delta_{T,S}(x)$	80%	57.8%	45.2%	35.1%	30.0%	24.6%	20.9%	18.4%
$\delta_{D,S}(x)$	53.3%	28.1%	13.5%	7.03%	3.79%	2.06%	1.14%	0.68%
$\delta_{D,T}(x)$	66.7%	48.7%	29.9%	20.0%	12.6%	8.40%	5.47%	3.70%

In particular, we see that D contains nearly half of the *abc* triples $(1, c-1, c)$ in T with $c \leq 10^6$. This aligns with our earlier observation that with the exception of $(1, 1215, 1216)$, each *abc* triple in Table 1 is in D. However, for $c \geq 10^8$, the percentage of *abc* triples in S that are in T begins to decrease rapidly which leads us to conclude that most *abc* triples in T and S do not fall into families such as those illustrated in Section 3.

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