A REMARK ON FINE'S ARITHMETIC FUNCTIONS

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Abstract

In this note, we consider some arithmetic identities of Fine associated with divisor functions. We connect these functions with indefinite quadratic forms using a result due to Andrews. As a consequence, arithmetic theorems are extracted.

– Dedicated to Weston Sutherland McCoy

1. Introduction and Main Theorems

In Fine's monograph [3], several infinite products are paraphrased in terms of divisor functions. The main product identity we will study herein is [3, Page 12, Equations (10.6)–(10.7)]

$$\prod_{n=1}^{\infty}\frac{(1-q^{pn})^2}{(1-q^{pn-r})(1-q^{pn-p+r})}=\sum_{n=0}^{\infty}E_r(2pn+r(p-r);2p)q^n, \tag{1}$$

for all positive integers r and p with $r<p$ and $(r,2p)=1$ and every $q\in\mathbb{C}$ with $|q|<1$. We will also study the square of Equation (1), [3, Page 76, Equation (31.5)]

$$q^r\left(\prod_{n=1}^{\infty}\frac{(1-q^{pn})^2}{(1-q^{pn-r})(1-q^{pn-p+r})}\right)^2=\frac{1}{p}\sum_{n=1}^{\infty}\left(\sum_{\substack{\delta d=pn-r^2\\ d\equiv r \bmod p}}(d+\delta)\right)q^n, \tag{2}$$

for all positive integers r and p with $r<p$ and $(r,p)=1$, where we let

$$E_r(N;m)=\sum_{\substack{d\mid N\\ d\equiv r \bmod m}}1-\sum_{\substack{d\mid N\\ d\equiv -r \bmod m}}1. \tag{3}$$

The corollaries of Equations (1) and (2) that follow in [3] are primarily connections between positive definite quadratic forms and divisor functions, which are

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classical and elegant in their own right. However, the product contained on the left-hand side of Equation (1), which is also the main factor of the left-hand side of Equation (2), is a rather special product, having an indefinite quadratic form expansion. This observation leads us to the following theorems, which we prove in the next section.

Theorem 1. *For all positive integers r and p with $r < p$ and $(r, 2p) = 1$, we have*

$$E_r(2pn + r(p-r); 2p) = \sum_{\substack{k,l=-\infty \\ k \geq |l| \\ n = p(k^2-l^2)/2 + p(k+l)/2 - lr}}^{\infty} (-1)^{k+l}.$$

For all integers k and m with $m \geq 1$, define

$$D_{r,m}(n) := \sum_{\substack{d|n \\ d \equiv r \bmod m}} 1,$$

so that, by Equation (3), $E_r(n; m) = D_{r,m}(n) - D_{-r,m}(n)$. We now state a corollary that follows from our first theorem.

Corollary 1. *Let r and p be positive integers with $r < p$ and $(r, 2p) = 1$. We have that $D_{r,2p}(2pn + r(p-r)) = D_{-r,2p}(2pn + r(p-r))$ if and only if n is not of the form $Q_{r,p}(k, l) := p(k^2 - l^2)/2 + p(k + l)/2 - lr$, $l, k \in \mathbb{Z}$ with $k \geq |l|$ or the number of pairs (k, l) of this form have an equal number which have $k + l \equiv 0 \pmod 2$ as those which have $k+l \equiv 1 \pmod 2$. Moreover, $D_{r,2p}(2pn+r(p-r))$ is greater (resp. less) than $D_{-r,2p}(2pn + r(p-r))$ if the the number of pairs $(k, l) \in \mathbb{Z}^2$ with $k \geq |l|$ and $n = Q_{r,p}(k, l)$ such that $k + l \equiv 0 \bmod 2$ is greater (resp., smaller) than those with $k + l \equiv 1 \pmod 2$.*

Our next theorem follows from Equation (2).

Theorem 2. *For all positive integers r and p with $r < p$ and $(r, p) = 1$, we have*

$$\frac{1}{p} \sum_{\substack{\delta d = pn - r^2 \\ d \equiv r \pmod p}} (d + \delta) = \sum_{\substack{k_1,l_1,k_2,l_2=-\infty \\ k_1 \geq |l_1|, k_2 \geq |l_2| \\ n = p(k_1^2-l_1^2)/2 + p(k_1+l_1)/2 - l_1 r + p(k_2^2-l_2^2)/2 + p(k_2+l_2)/2 - l_2 r + r}}^{\infty} (-1)^{k_1+l_1+k_2+l_2}.$$

Next we have the equivalent of a non-negativity result concerning certain indefinite quaternary quadratic forms.

Corollary 2. *Let r and p be positive integers with $r < p$ and $(r, p) = 1$. For each positive number n, the number of quadruples $(k_1, k_2, l_1, l_2) \in \mathbb{Z}^4$ that satisfy $n = Q_{r,p}(k_1, l_1) + Q_{r,p}(k_2, l_2) + r$, $k_1 \geq |l_1|$, $k_2 \geq |l_2|$, $l_1, l_2, k_1, k_2 \in \mathbb{Z}$, and have $k_1 + k_2 + l_1 + l_2 \equiv 0 \pmod 2$, are greater than or equal to those which have $k_1 + k_2 + l_1 + l_2 \equiv 1 \pmod 2$.*

2. Proofs of Results

For $1 < |z| < |q|^{-1}$, $|q| < 1$, Andrews [1, Lemma 1] discovered the expansion,

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-zq^n)(1-z^{-1}q^{n-1})} = \sum_{\substack{k,l=-\infty \\ k\geq |l|}}^{\infty} (-1)^{k+l} z^l q^{(k^2-l^2)/2+(k+l)/2}. \tag{4}$$

Proof of Theorem 1. This formula follows from replacing q by q^p in Equation (4) and then $z = q^{-r}$, where $0 < r < p$, $(r, 2p) = 1$. Equating coefficients of q^n with Equation (1) now gives the result. □

Proof of Corollary 1. This follows from inspecting the weight in the sum on the right-hand side of Theorem 1.1, and noting that $D_{\pm r,p}(n)$ is a non-negative quantity. □

Proof of Theorem 2. We use the $(z, q) \to (q^{-r}, q^p)$ case of Equation (4) we obtained in the proof of Theorem 1.1 in Equation (2), and then equate coefficients of q^n with Equation (2). □

Proof of Corollary 2. It is enough to show the left side of Theorem 1.2 is non-negative. One way to see this is to use the re-writing found in [2, Proposition 1, Equation (8)],

$$\frac{1}{p} \sum_{\substack{d|np-r^2,d>0 \\ d\equiv r \bmod p}} (d + \frac{np-r^2}{d}).$$

Notice that if the summand were positive then so would $d^2 + np - r^2$. Since $d \equiv r \pmod{p}$, there exist integers $m \geq 0$, such that $d^2 + np - r^2 = (pm + r)^2 + np - r^2 = (pm)^2 + 2pmr + np$. Since $np > 0$, it follows that the sum is non-negative. □

3. Observations

Our results coupled with [2] suggest that indefinite quadratic forms seem to have some natural connection with Klein forms. This seems to follow naturally from the fact that Klein forms have integral weight, and Hecke modular forms associated with indefinite quadratic forms have weight one [4]. The example presented here of this phenomenon being Equation (1), which is a holomorphic modular form of weight one [2, Theorem 2, Equation (1)]. Likewise, it was noted/proved in [2, Theorem 2, Equation (8)] that the square of the left-hand side of Equation (1), which shows up in the left-hand side of Equation (2), is a holomorphic modular form of weight two.

References

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