COPRIME MAPPINGS ON GAUSSIAN LINES

Elise Carlos
Department of Mathematics, Seattle University, Seattle, Washington
carlosel@oregonstate.edu

Nestor Iwanojko
Department of Mathematics, Seattle University, Seattle, Washington
niwanojko@student.ethz.ch

Reece Keller
Department of Mathematics, Seattle University, Seattle, Washington
rdkeller@andrew.cmu.edu

Leanne Robertson
Department of Mathematics, Seattle University, Seattle, Washington
robertle@seattleu.edu

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Abstract

If A and B are sets of integers, then a bijection $f : A \to B$ is called a *coprime mapping* if a and $f(a)$ are coprime for all $a \in A$. The existence of coprime mappings between intervals of positive integers has been well studied. Here we extend this study to coprime mappings between intervals of Gaussian integers on lines in the complex plane. We give simple necessary and sufficient conditions that such a line L must satisfy to guarantee that, for all $n \in \mathbb{N}$, a coprime mapping exists between any two contiguous intervals on L consisting of n Gaussian integers. For lines L where these conditions are not met, we conjecture that there is a bound B_L such that if $n > B_L$, then a coprime mapping exists between any two contiguous intervals of length n on L, and find such bounds for certain infinite families of lines. Finally, we consider coprime mappings on lines over other imaginary quadratic fields of class number one, and generalize the necessary and sufficient conditions for the existence of coprime mappings between contiguous intervals to these lines.

1. Introduction

Around 1960, D. J. Newman conjectured that if n is a positive integer and B is an interval of length n that consists of positive integers, then a bijection f from

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$[1,n]$ to B exists with the property that a and $f(a)$ are coprime for all $a \in [1,n]$ (here the integer interval notation $[a,b]$ denotes the set $\{k \in \mathbb{Z} | \, a \leq k \leq b\}$). Such a bijection is called a *coprime mapping*. In 1963, Daykin and Baines [3] investigated the existence of coprime mappings between contiguous intervals, and proved Newman's conjecture in the special case where $B = [n+1, 2n]$. Pomerance and Selfridge [9] proved Newman's conjecture in its entirety in 1980. More recently, Robertson and Small [10] extended their result and determined when a coprime mapping exists from $A = [1,n]$ or $A = \{1,3,5,\ldots,2n-1\}$ to a set of n integers in arithmetic progression.

Note that in all the results mentioned above, the set A consists of either the first n integers or the first n odd integers, so in particular $1 \in A$ in all cases. If $1 \notin A$, then a coprime mapping may or may not exist from A to an interval B of the same length, even if A and B are contiguous. The simplest example is that no coprime mapping exists from $A = \{2,3,4\}$ to $B = \{5,6,7\}$ since 6 shares a common divisor with every element in A. Very recently, Bohman and Peng [2] proved that if A is an interval of length $n \geq 4$ and $n \in A$, then a coprime mapping always exists from A to the contiguous interval of length n, and thus resolved a conjecture of Larsen et al. [5]. More generally, they consider non-contiguous intervals and show that there is a positive constant $\mathcal{C}$ such that if n is sufficiently large, $m > \exp(\mathcal{C}(\log\log n)^2)$, and $A, B \subset [1,n]$ are intervals of length $2m$, then there is a coprime mapping from A to B. Recently, Pomerance [8] improved their result and showed that there is a positive constant c such that the result holds for $m > c(\log n)^2$. In both cases, the authors gave an application of their coprime mapping theorem to the lonely runner conjecture.

We extend the study of coprime mapping from the rational integers on the real line to the Gaussian integers on other lines in the complex plane. Following Gethner, Wagon, and Wick [4], we call a line in the complex plane a *Gaussian line* if it contains two, and hence infinitely many, Gaussian integers. Since the Gaussian integers have the unique factorization property, we can ask when a coprime mapping exists between two contiguous intervals of Gaussian integers on a Gaussian line. We show that in some ways the situation is nicer for the Gaussian integers than for the rational integers. In particular, there are infinitely many Gaussian lines for which a coprime mapping exists between *any* two contiguous intervals of the same length that consist of consecutive Gaussian integers on the line, so regardless where the first interval starts or its length. We give easy to check necessary and sufficient conditions that a Gaussian line must satisfy in order for this property to hold (Theorem 5), then generalize this result to lines over other imaginary quadratic fields whose integer rings have the unique factorization property (Theorem 11).

An outline of the paper and other results is as follows. In Section 2, we discuss Gaussian lines. Many of the properties of the rational integers on the real line extend to the set of Gaussian integers on a Gaussian line, including the periodicity of divisibility, the Chinese remainder theorem, the existence of arbitrarily long sequences of composites, and the property of being a Pillai sequence (see [4, 6, 7]), but in this section we just focus on the properties and notation used to extend questions about coprime mappings to Gaussian lines. All of the results in Section 2 are proven in [6] and [7], but are included here for the convenience of the reader. Our main theorem in Section 3 (Theorem 5) provides necessary and sufficient conditions that a Gaussian line must satisfy in order for coprime mappings to exist between any two contiguous intervals on L of length n, for all $n \in \mathbb{N}$. For $n \geq 2$, our proof requires that between n and $2n$ there exists integers of the form 2^t and $3^r 7^s p^t$, where $p = 11$ or $p = 17$. The existence of such integers is also proven in Section 3. In Section 4, we show that if the necessary and sufficient conditions are not met for a Gaussian line L, then in certain cases there is a bound B_L such that coprime mappings exists between any two contiguous intervals on L of any length $n > B_L$. We conjecture that such a bound B_L exists for all Gaussian lines L. Along the way, we use the explicit upper and lower bounds given by Bennett et al. [1] for the number of primes in a bounded arithmetic progression, to show that if $n \geq 13$, then there exists a prime $p \equiv 3 \pmod 4$ such that $n < p \leq 1.5n$. Finally, in Section 5, we discuss extending this work to other imaginary quadratic fields whose integer rings are unique factorization domains and provide the analogous necessary and sufficient conditions for a coprime mapping to always exist between contiguous intervals on lines in this more general setting.

2. Gaussian Lines

In this section, we provide the definitions and results involving Gaussian lines that we use to extend questions about coprime mappings to these lines. These results are included for the convenience of the reader and are stated without proof (see [6] and [7] for the proofs).

We begin with notation involving the Gaussian integers $\mathbb{Z}[i]$. The unit group of the Gaussian integers is $\{\pm 1, \pm i\}$, so two Gaussian integers, α and β, are *associates* if and only if $\alpha = \pm\beta$ or $\alpha = \pm i\beta$. The *norm* of the Gaussian integer $\alpha = x + iy$ is defined by $N(\alpha) = \alpha \cdot \overline{\alpha} = x^2 + y^2 \in \mathbb{Z}$, where the "bar" denotes complex conjugation. The ring $\mathbb{Z}[i]$ is a unique factorization domain, and this gives the Gaussian integers a well-defined notion of primality. To avoid confusion, we use the terminology *rational prime* for a prime in the rational integers $\mathbb{Z}$, and *Gaussian prime* for a prime in $\mathbb{Z}[i]$. Recall that the Gaussian primes can be classified in terms of the factorization of the rational primes p as follows:

1. If $p = 2$, then p ramifies in $\mathbb{Z}[i]$. Specifically, $2 = -i(1 + i)^2$, so $1 + i$ is a Gaussian prime of norm 2.

2. If $p \equiv 1 \pmod 4$, then $p = \pi_p \cdot \overline{\pi}_p$ splits as a product of two conjugate Gaussian primes of norm p that are not associates in $\mathbb{Z}[i]$.

3. If $p \equiv 3 \pmod 4$, then p is inert in $\mathbb{Z}[i]$ and has norm p^2.

Every Gaussian prime is an associate of one of the Gaussian primes described above. If π is a Gaussian prime then we say π *lies over* p if π divides the rational prime p, in which case we often write π_p for π.

Turning to Gaussian lines, we first recall from [7] that a Gaussian line L can be uniquely defined by two Gaussian integers, $\alpha_0 = a + bi$ and $\delta = c + di$, as follows. Let α_0 be the Gaussian integer on L of minimum norm, and if there are two such integers, let α_0 be the one with the larger real part. If L is vertical, then take $\delta = i$. Otherwise, let α_1 be the Gaussian integer on L closest to α_0 (so $N(\alpha_1 - \alpha_0)$ is minimal) and with $\mathrm{Re}(\alpha_1) > \mathrm{Re}(\alpha_0)$. In this case, take $\delta = \alpha_1 - \alpha_0 = c + di$. Then, c and d are coprime (by the choice of α_1) and $c \geq 0$. Note that α_0 is on the line L, but δ is not, provided $\alpha_0 \neq 0$. Moreover, the Gaussian integers on L are exactly the Gaussian integers

$$\alpha_k = \alpha_0 + \delta k,$$

where $k \in \mathbb{Z}$ (see [7, Lemma 1]). Although we use the terminology of a line, we are essentially interested in the arithmetic progression $\alpha_0, \alpha_1, \alpha_2, \ldots$ of Gaussian integers with common difference δ.

Gaussian lines provide a way of thinking about contiguous intervals of Gaussian integers and of studying coprime mappings on these intervals. Certainly, if all the Gaussian integers on the line share a common Gaussian prime divisor, then a coprime mapping does not exist between any pair of intervals on the line. We call a Gaussian line *primitive* if it contains two coprime Gaussian integers, which happens if and only if α_0 and δ are coprime in $\mathbb{Z}[i]$. Thus, we are interested in the existence of coprime mappings on contiguous intervals of Gaussian integers on primitive Gaussian lines.

From now on, assume that L is a primitive Gaussian line with α_0 and δ defined as above. We also define a rational integer Δ associated to L by

$$\Delta = ad - bc.$$

Then, $\Delta = 0$ if and only if L is the real line $\mathrm{Im}(z) = 0$ or the imaginary line $\mathrm{Re}(z) = 0$, which holds if and only if $\alpha_0 = 0$ (see [7, Lemma 2]). Note that Δ can be obtained from any Gaussian integer on L, not just from α_0. Namely, if $\alpha = x + yi$ is a Gaussian integer on L, then $\alpha = \alpha_0 + n\delta$ for some $n \in \mathbb{Z}$. Thus, $x = a + nc$ and $y = b + nd$, and so $xd - yc = ad - bc = \Delta$.

Together, Δ and δ enable us to easily determine if a given Gaussian prime divides some Gaussian integer on L. Define the *prime set of* L, denoted $\mathbb{P}(L)$, to be the set of Gaussian primes that divide some Gaussian integer on L. The following theorem provides a simple test for whether a rational or non-rational Gaussian prime occurs in $\mathbb{P}(L)$.

Theorem 1 ([7, Theorems 4 and 5])**.** *Let L be a primitive Gaussian line.*

(a) If $p \in \mathbb{Z}$ is a rational prime, then $p \in \mathbb{P}(L)$ if and only if $p \mid \Delta$.

(b) If $\pi \in \mathbb{Z}[i]$ is a non-rational Gaussian prime, then $\pi \in \mathbb{P}(L)$ if and only if $\pi \nmid \delta$.

Notice that if L is not the real or imaginary line (so $\Delta \neq 0$), then it follows from Theorem 1 that there are only finitely many rational primes p that divide some Gaussian integer on L. It also follows that the prime set $\mathbb{P}(L)$ contains at least one Gaussian prime lying over p for every rational prime $p \equiv 1 \pmod 4$ since no rational integer divides $\delta = c + di$ (otherwise c and d would not be coprime as required). Specifically, for every rational prime $p \equiv 1 \pmod 4$, if p does not divide $N(\delta)$, then $\mathbb{P}(L)$ contains exactly two non-associate Gaussian primes lying over p, and if p divides $N(\delta)$ then it contains exactly one such prime. Both of these facts are important in our later work and serve to distinguish the set of Gaussian integers on a Gaussian line from set of rational integers on the real line.

The set of Gaussian integers on a primitive Gaussian line shares several properties with the set of rational integers on the real line (see [4, 6, 7]). The two main shared properties important here are the periodicity of divisibility and the Chinese remainder theorem. The Gaussian line analogies to these two theorems are stated below. Note, however, that the theorems below are really special cases of the theorems in [7] since here they are stated in terms of Gaussian prime divisors rather than in terms of all Gaussian integer divisors.

Theorem 2 ([7, Theorem 3])**.** *Let L be a primitive Gaussian line and π_p be a Gaussian prime that lies over the rational prime p. Suppose π_p divides some Gaussian integer α_t on L. Then π_p also divides the Gaussian integer α_k on L if and only if $k \equiv t \pmod p$.*

Theorem 3 ([7, Theorem 8])**.** *Let L be a primitive Gaussian line. Also, let $b_1, b_2, \ldots, b_k$ be rational integers (not necessarily distinct) and $\pi_{p_1}, \pi_{p_2}, \ldots, \pi_{p_k}$ be Gaussian primes in the prime set $\mathbb{P}(L)$ of L that lie over distinct rational primes $p_1, p_2, \ldots, p_k$, respectively. Then there is a unique rational integer t modulo the product $p_1 p_2 \cdots p_k$ such that*

$$\pi_{p_1} \mid \alpha_{t+b_1},\ \pi_{p_2} \mid \alpha_{t+b_2},\ \ldots,\ \pi_{p_k} \mid \alpha_{t+b_k}.$$

Another useful theorem says that if you want a Gaussian line such that certain Gaussian primes divide specified elements on the line and certain Gaussian primes do not divide any elements on the line, then you are in luck — there are infinitely many such lines as long as your desired requirements don't violate the periodicity of divisibility described in Theorem 2.

Theorem 4 ([6, Theorem 4]). *Let $k, m \in \mathbb{Z}$, with $1 \leq k \leq m$. Also, let $b_1, b_2, \ldots, b_k$ be rational integers (not necessarily distinct) and $\pi_{p_1}, \pi_{p_2}, \ldots, \pi_{p_m}$ be Gaussian primes that lie over distinct rational primes $p_1, p_2, \ldots, p_m$, respectively. Then there are infinitely many primitive Gaussian lines L that satisfy both of the following properties:*

(a) π_j divides α_{b_j} on L for $1 \leq j \leq k$;

(b) $\pi_j \notin \mathbb{P}(L)$ for $k < j \leq m$.

We will use this theorem, along with the definitions and other results in this section, in our study of coprime mappings on Gaussian lines.

3. Coprime Mappings between Contiguous Intervals

The main theorem in this section (Theorem 5) provides necessary and sufficient conditions that a Gaussian line must satisfy in order for a coprime mapping to exist between any two contiguous intervals of the same length that consist of Gaussian integers on the line. The idea is that a primitive Gaussian line will have this property if certain small Gaussian prime divisors are not in the prime set of the line.

We begin with three preliminary lemmas about the existence of powers of small primes between n and $2n$ for $n \geq 2$. These lemmas are used to construct the coprime mappings used to prove Theorem 5.

Lemma 1. *Let $n \geq 2$ be an integer. Then there exists a positive integer x such that $n \leq 2^x < 2n$.*

Proof. Let $n \geq 2$. Then $2^m < n \leq 2^{m+1}$ for some $m \in \mathbb{Z}$. Multiplication by 2 gives $n \leq 2^{m+1} < 2n$, so we may take $x = m + 1$. □

Lemma 2. *Let $n \geq 2$, $n \neq 10$, be an integer. Then there exists non-negative integers x and y such that $n \leq 3^x 7^y < 2n$.*

Proof. The lemma does not hold for $n = 10$, but one can easily verify that it holds for $2 \leq n \leq 27$, $n \neq 10$. Thus, let $n \geq 28$ and $m = 3^s 7^t$ be the largest integer of this form that is strictly smaller than n. If $t = 0$, then $m = 3^s$ and $s \geq 3$ since $n \geq 28$. Thus, $m < 3^{s-3} 7^2 < 2m$ since $1 < 49/27 < 2$, so we may take $x = s - 3$ and $y = 0$ in this case. Similarly, if $t > 0$, then $m < 3^{s+2} 7^{t-1} < 2m$ since $1 < 9/7 < 2$, so we may take $x = s + 2$ and $y = t - 1$ in this case. Then we have $m < 3^x 7^y < 2m$ in both cases. In addition, we have $n \leq 3^x 7^y$ since $m < 3^x 7^y$ and $m = 3^s 7^t$ is the largest integer of this form that is strictly less than n. Thus,

$$n \leq 3^x 7^y < 2m < 2n,$$

as needed. □

Notice that when $n = 10$, we have $n \leq p < 2n$ for $p = 11, 19$. Thus, the next lemma is immediate.

Lemma 3. *Let $n \geq 2$ be an integer and $p = 11$ or 19. Then there exist non-negative integers x, y, and z such that $n \leq 3^x 7^y p^z < 2n$.*

We are now ready to give necessary and sufficient conditions for a coprime mapping to exist between any two contiguous intervals of Gaussian integers on a primitive Gaussian line L. The main idea is that such a mapping will always exist if there are some small Gaussian primes that are not in the prime set of the line. By Theorem 1, this condition can be stated in terms of the divisors of δ and Δ. We give both formulations below. For $k, s \in \mathbb{Z}$, $k < s$, we extend the rational integer interval notation and use the notation $[\alpha_k, \alpha_s]$ to denote the interval $\{\alpha_k, \alpha_{k+1}, \ldots, \alpha_s\}$ of Gaussian integers on L, and simply call it an *interval on L*. In particular, for $n \in \mathbb{N}$, the intervals $[\alpha_k, \alpha_{k+n-1}]$ and $[\alpha_{k+n}, \alpha_{k+2n-1}]$ both have length n and are contiguous.

Theorem 5. *Let L be a primitive Gaussian line and $\mathbb{P}(L)$ be its prime set. Then a coprime mapping*

$$f : [\alpha_k, \alpha_{k+n-1}] \to [\alpha_{k+n}, \alpha_{k+2n-1}] \tag{1}$$

exists for all $k \in \mathbb{Z}$ and all $n \in \mathbb{N}$ if and only if at least one of the following three conditions holds:

1. *the Gaussian prime $1 + i$ is not in $\mathbb{P}(L)$ (i.e., $1 + i$ divides δ);*
2. *none of 3, 7, or 11 are in $\mathbb{P}(L)$ (i.e., Δ is not divisible by $3, 7$, or 11);*
3. *none of 3, 7, or 19 are in $\mathbb{P}(L)$ (i.e., Δ is not divisible by $3, 7$, or 19).*

Proof. Let L be a primitive Gaussian line and $k \in \mathbb{Z}$. We first suppose that one of the three conditions in the theorem holds and use strong induction to show that a coprime mapping f as in (1) exists for all $n \in \mathbb{N}$. The key is that if α_s and α_t are Gaussian integers on L, then by Theorem 2 the only possible common Gaussian prime divisors of α_s and α_t lie over rational primes that divide $t - s$. Thus, we construct f such that if $f(\alpha_s) = \alpha_t$, then $t - s$ is only divisible by Gaussian primes not in $\mathbb{P}(L)$.

First suppose $1 + i \notin \mathbb{P}(L)$. Then $\mathbb{P}(L)$ does not contain a prime of norm 2, so α_s and α_t are coprime over $\mathbb{Z}[i]$ whenever $t - s$ is a power of 2. Let $n \in \mathbb{N}$. If $n = 1$, then a coprime mapping f as in (1) exists since any two consecutive Gaussian integers on L are coprime by Theorem 2. Suppose such a coprime mapping f exists for all $1 \leq n < m$, for some integer $m \geq 2$. Consider $n = m$. By Lemma 1, there is an integer x such that $m \leq 2^x < 2m$. Define f on the interval $[\alpha_k, \alpha_{k+2m-2^x-1}]$ to $[\alpha_{k+2^x}, \alpha_{k+2m-1}]$ by

$$f(\alpha_j) = \alpha_{j+2^x}, \ k \leq j \leq k + 2m - 2^x - 1.$$

If $m = 2^x$ then we have defined a coprime mapping f as in (1) as needed. Otherwise, it remains to show we can define f from $[\alpha_{k+2m-2^x}, \alpha_{k+m-1}]$ to $[\alpha_{k+m}, \alpha_{k+2^x-1}]$. Such a coprime mapping exists by our induction hypothesis since these intervals are contiguous and have length $2^x - m$, which is smaller than m. Thus, a coprime mapping f as in (1) exists for $n = m$, and by induction for all $n \in \mathbb{N}$.

Now suppose $3, 7, p \notin \mathbb{P}(L)$, where $p = 11$ or 19. Follow the strong induction proof above with Lemma 3 instead of Lemma 1 to show a coprime mapping f as in (1) exists for all $n \in \mathbb{N}$. The only difference here is that for $n = m$ we we get non-negative integers x, y, and z such that $m \leq 3^x 7^y p^z < 2m$. Then f is defined from $[\alpha_k, \alpha_{k+2m-3^x7^yp^z-1}]$ to $[\alpha_{k+3^x7^yp^z}, \alpha_{k+2m-1}]$ by

$$f(\alpha_j) = \alpha_{j+3^x7^yp^z},\ k \leq j \leq k + 2m - 3^x7^yp^z - 1,$$

and on $[\alpha_{k+2m-3^x7^yp^z-1}, \alpha_{k+m-1}]$ to the contiguous interval $[\alpha_{k+m-1}, \alpha_{k+3^x7^yp^z-1}]$ by the induction hypothesis. Again this is a coprime mapping since for all j, the only possible common Gaussian prime divisors of α_j and $\alpha_{j+3^x7^yp^z}$ are 3, 7, or p, none of which are in $\mathbb{P}(L)$. Thus, a comprime mapping f as in (1) exists when $n = m$, and by induction it exists for all $n \in \mathbb{N}$.

For the converse, we must show that if none of the three conditions in the theorem is satisfied, then there is a $k \in \mathbb{Z}$ and an $n \in \mathbb{N}$ such that no coprime mapping f as in (1) exists. This is divided into three cases. In each case, we give one value of n and infinitely many $k \in \mathbb{Z}$ for which a coprime mapping f as in (1) does not exist.

`Case 1`: Suppose $1 + i, 3 \in \mathbb{P}(L)$. By Theorem 3, there is a $j \in \mathbb{Z}$ such that $1 + i$ and 3 both divide α_j. Thus, let $n = 2$ and take any $k \equiv j \pmod 6$. No coprime mapping exists between $[\alpha_k, \alpha_{k+1}]$ and $[\alpha_{k+2}, \alpha_{k+3}]$ since, by Theorem 2, $1 + i$ is a common divisor of α_k and α_{k+2}, and 3 is a common divisor of α_k and α_{k+3}.

`Case 2`: Suppose $1 + i, 7 \in \mathbb{P}(L)$. Since 5 is split in $\mathbb{Z}[i]$, we also have $\pi_5 \in \mathbb{P}(L)$, for some Gaussian prime π_5 lying over 5 (see the remark following Theorem 1). By Theorem 3, there is a $j \in \mathbb{Z}$ such that $1 + i$, π_5, and 7 all divide α_j. Thus, let $n = 4$ and take any $k \equiv j \pmod{70}$. No coprime mapping exists between $[\alpha_k, \alpha_{k+3}]$ and $[\alpha_{k+4}, \alpha_{k+7}]$ since again α_k shares a common Gaussian prime divisor with every element of $[\alpha_{k+4}, \alpha_{k+7}]$. Indeed, by Theorem 2, $1 + i$ divides α_k, α_{k+4}, and α_{k+6}; π_5 divides α_k and α_{k+5}; and 7 divides α_k and α_{k+7}.

`Case 3`: Suppose $1 + i, 11, 19 \in \mathbb{P}(L)$. We also have $\pi_5, \pi_{13}, \pi_{17} \in \mathbb{P}(L)$, for some Gaussian primes lying over 5, 13, and 17, respectively. By Theorem 3, there is a $j \in \mathbb{Z}$ such that $1 + i$, π_5, 11, π_{13}, π_{17}, and 19 all divide α_j. Thus, let $n = 10$ and take any $k \equiv j \pmod{461{,}890}$, since $461{,}890 = 2 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. Then no coprime mapping exists from $[\alpha_k, \alpha_{k+9}]$ to $[\alpha_{k+10}, \alpha_{k+19}]$ since α_k shares a common Gaussian prime divisor with every element of $[\alpha_{k+10}, \alpha_{k+19}]$ by the same argument as in Case 2. $\square$

It follows from Theorem 5, for example, that if L is a primitive Gaussian line with $\Delta = 1$ or such that Δ is only divisible by primes $p \equiv 1 \pmod 4$, then a coprime mapping exists between any two contiguous intervals on L of the same length. In both cases the prime set $\mathbb{P}(L)$ of L does not contain any primes $p \equiv 3 \pmod 4$, so the last two conditions of Theorem 5 are both satisfied. By contrast, the next theorem shows that if $\mathbb{P}(L)$ contains $1+i$ and every prime $p \equiv 3 \pmod 4$, $p \leq m$, for some $m \in \mathbb{N}$, then for all $1 < n < m$, there are infinitely many contiguous intervals on L of length n such that no coprime mapping exists between the two intervals.

Theorem 6. *Let m be a positive integer. There are infinitely many Gaussian lines L with the property that for every integer n with $1 < n < m$, there are infinitely many integers k such that no coprime mapping*

$$f : [\alpha_k, \alpha_{k+n-1}] \to [\alpha_{k+n}, \alpha_{k+2n-1}] \tag{2}$$

exists. In particular, no such coprime mapping exists for infinitely many values of k if $\mathbb{P}(L)$ contains $1+i$ and every prime $p \equiv 3 \pmod 4$, $p \leq m$.

Proof. Let m be a positive integer. By Theorem 4, there are infinitely many primitive Gaussian lines whose prime set $\mathbb{P}(L)$ contains $1+i$ and every prime $p \equiv 3 \pmod 4$, $p \leq m$. Let L be one of these lines. Then, by the remark following Theorem 1, $\mathbb{P}(L)$ contains a Gaussian prime π_p lying over p for every rational prime $p \leq m$. By Theorem 3, there are infinitely many integers t such that π_p divides the Gaussian integer α_t on L for all $p \leq m$.

Let $n \in \mathbb{Z}$ satisfy $1 < n < m$. The interval $A_{t,n} = [\alpha_{t-(n-2)}, \alpha_{t+1}]$ consists of n consecutive Gaussian integers on L, including α_t. The contiguous interval of length n on L is $B_{t,n} = [\alpha_{t+2}, \alpha_{t+n+1}]$. No coprime mapping from $A_{t,n}$ to $B_{t,n}$ exists since α_t shares a common Gaussian prime divisor with every element of $B_{t,n}$. Indeed, for all $2 \leq j \leq n+1$ there is a rational prime $p \leq m$ that divides j, so the Gaussian prime π_p is a common divisor of α_t and $\alpha_t + j$ by Theorem 2. Thus, we may take $k = t - (n-2)$ in (2) for each of the infinitely many possible t. $\square$

4. Coprime Mappings between Long Intervals

In the previous section, we gave necessary and sufficient conditions that a Gaussian line L must satisfy in order for a coprime mapping to exist between any two contiguous intervals on L of the same length. For lines where these conditions are not satisfied (e.g., the lines described in Theorem 6), it is natural to ask if there is a bound B_L such that a coprime mapping exists between any two contiguous intervals on L of length $n \geq B_L$. We investigate this question in this section. We prove that if all the primes $p \equiv 3 \pmod 4$ in the prime set of L are smaller than 100, then such a bound B_L exists and we may take $B_L = 102$. Our proof uses the explicit bounds given by Bennett et al. [1] for the number of primes occurring in bounded arithmetic progressions to show that for every integer $n \geq 13$, there exists a prime $p \equiv 3 \pmod 4$ that satisfies $n < p \leq 1.5n$. It is plausible that such a bound B_L similarly exists for all Gaussian lines.

Let L be a primitive Gaussian line different from the real or imaginary line (so $\Delta \neq 0$). By Theorem 1, there are only finitely rational primes $p \equiv 3 \pmod 4$ in the prime set $\mathbb{P}(L)$ of L. That is, there are only finitely many inert prime in $\mathbb{P}(L)$. We first observe that if p is an inert prime and $p \notin \mathbb{P}(L)$, then for any $k \in \mathbb{Z}$ a coprime mapping

$$f : [\alpha_k, \alpha_{k+n-1}] \to [\alpha_{k+n}, \alpha_{k+2n-1}] \tag{3}$$

exists if $n = p$ or $n = p - 1$. Indeed, if $n = p$, then f defined by $f(\alpha_j) = \alpha_{j+p}$, $k \leq j \leq k + n - 1$, is a coprime mapping since, by Theorem 2, the only possible common prime divisor of α_j and α_{j+p} is p and $p \notin \mathbb{P}(L)$. Similarly, if $n = p-1$, then f defined by $f(\alpha_j) = \alpha_{j+p}$, $k \leq j \leq k + n - 2$ and $f(\alpha_{k+n-1}) = \alpha_{k+n}$ is coprime since any two consecutive Gaussian integers on a Gaussian line are coprime.

To construct a coprime mapping as in (3) for other values of n, we would similarly like to send each α_j, $k \leq j \leq n-1$, to α_{j+p} for some inert prime $p \notin \mathbb{P}(L)$, perhaps with different primes p used for different values of j. Our next lemma clarifies this strategy.

Lemma 4. *Let L be a primitive Gaussian line and n be a positive integer. If there is a bijection*

$$g : \{0, 1, \ldots, n - 1\} \to \{n, n + 1, \ldots, 2n - 1\}$$

such that for all $j \in \{0, 1, \ldots, n - 1\}$, every prime divisor p of $g(j) - j$ satisfies $p \equiv 3 \pmod 4$ and $p \notin \mathbb{P}(L)$, then a coprime mapping exists between any two contiguous intervals on L of length n.

Proof. Let $k \in \mathbb{Z}$. Suppose there is a bijection g as described in the lemma. Then

$$f : [\alpha_k, \alpha_{k+n-1}] \to [\alpha_{k+n}, \alpha_{k+2n-1}]$$

defined by

$$f(\alpha_{k+j}) = \alpha_{k+g(j)},\ 1 \leq j \leq n - 1,$$

is a coprime mapping by Theorem 2. □

To construct coprime mappings between long contiguous intervals on a Gaussian line L, we want a sufficient number of primes $p \equiv 3 \pmod 4$ not in $\mathbb{P}(L)$ so that the bijections as in Lemma 4 exist. It is helpful to use the following theorem, which is a consequence of the work of Bennett et al. [1].

Theorem 7. *If $n \geq 13$, then there exists a prime $p \equiv 3 \pmod 4$ such that*

$$n < p \leq 1.5n.$$

Proof. We wrote a program in Python to verify the theorem for $13 \leq n \leq 800$. So assume $n > 800$. Let $\pi(x; 4, 3)$ denote the number of primes less than or equal to x that are congruent to 3 modulo 4. It immediately follows from Corollary 1.6 in [1] that

$$\pi(1.5n; 4, 3) - \pi(n; 4, 3) > \frac{1.5n}{2\ln(1.5n)} - \frac{n}{2\ln(n)}\left(1 + \frac{5}{2\ln(n)}\right),$$

where φ is Euler's totient function. Note that $\ln(1.5n) = \ln(1.5) + \ln(n) < 1.07\ln(n)$, since $\ln(1.5)/(0.07) < \ln(800) \leq \ln(n)$ for all $n \geq 800$. Thus, for all $n \geq 800$, we have

$$\begin{aligned} \pi(1.5n; 4, 3) - \pi(n; 4, 3) &> \frac{n}{2}\left(\frac{1.5}{1.07\ln(n)} - \frac{1}{\ln(n)}\left(1 + \frac{5}{2\ln(800)}\right)\right) \\ &> \frac{n(1.40 - 1.38)}{2\ln(n)} \\ &> 0. \end{aligned}$$

Since the difference $\pi(1.5n; 4, 3) - \pi(n; 4, 3)$ is an integer, it follows that there must be at least one prime $p \equiv 3 \pmod 4$ with $n < p \leq 1.5n$. □

We now use the strategy for constructing coprime mapping given in Lemma 4, together with Theorem 7 and induction, to prove that for certain families of lines where the conditions of Theorem 5 are not met, coprime mappings still exist between all sufficiently long contiguous intervals on these lines. To fix ideas, we first consider all Gaussian lines L for which all the primes $p \equiv 3 \pmod 4$ in $\mathbb{P}(L)$ are smaller than 10. Then, we extend the analysis to lines for which all the primes $p \equiv 3 \pmod 4$ in $\mathbb{P}(L)$ are smaller than 100. In both cases, we let $B_L = q - 1$, where q is the smallest prime larger than 10 (or 100) that is congruent to 3 modulo 4, and prove that a coprime mapping exists between any two contiguous intervals on L of length $n \geq B_L$.

Theorem 8. *Let L be a primitive Gaussian line such that every prime $p \equiv 3 \pmod 4$ in $\mathbb{P}(L)$ is smaller than 10. If $n \geq 10$, then a coprime mapping exists between any two contiguous intervals on L of length n.*

Proof. Let L be a primitive Gaussian line such that every prime $p \equiv 3 \pmod 4$ in $\mathbb{P}(L)$ is smaller than 10. Let $k \in \mathbb{Z}$. To prove Theorem 8, it follows from Lemma 4 that it is sufficient to show that for all $n \geq 10$, there is a bijection

$$g : \{0, 1, \ldots, n-1\} \to \{n, n+1, \ldots, 2n-1\} \tag{4}$$

such that for all $j \in \{0, 1, \ldots, n-1\}$, every prime divisor p of $g(j) - j$ satisfies $p \equiv 3 \pmod 4$ and $p > 10$. We use complete induction to prove such a bijection g exists for all $n \geq 10$. An example of such a bijection for the base cases $10 \leq n \leq 15$ is given in Table 1.

n	$g : \{0, 1, \ldots, n-1\} \to \{n, n+1, \ldots, 2n-1\}$
10	$g(j) = \begin{cases} j+11, & \text{if } 0 \le j \le 8; \\ 10, & \text{if } j = 9. \end{cases}$
11	$g(j) = j + 11$ for all $0 \le j \le 10$.
12	$g(j) = \begin{cases} 23, & \text{if } j = 0; \\ j+11, & \text{if } 1 \le j \le 11. \end{cases}$
13	$g(j) = \begin{cases} j+23, & \text{if } 0 \le j \le 2; \\ j+11, & \text{if } 3 \le j \le 11; \\ 13, & \text{if } j = 12. \end{cases}$
14	$g(j) = \begin{cases} j+19, & \text{if } j = 0 \text{ or } j = 8; \\ j+23, & \text{if } 1 \le j \le 3; \\ j+11, & \text{if } 4 \le j \le 7 \text{ or } 9 \le j \le 12; \\ 14, & \text{if } j = 13. \end{cases}$
15	$g(j) = \begin{cases} j+19, & \text{if } 0 \le j \le 2 \text{ or } 8 \le j \le 10; \\ 26, & \text{if } j = 3; \\ j+11, & \text{if } 4 \le j \le 7 \text{ or } 11 \le j \le 14. \end{cases}$

Table 1. Bijections for the base cases.

Now, suppose that a bijection g as in (4) exists for all $10 \le n < N$ for some integer $N \ge 16$. This is our induction hypothesis. Consider $n = N$. We first claim that there is a prime $q \equiv 3 \pmod 4$ such that $N + 10 \le q < 2N$. Indeed, for $16 \le N \le 21$, take $q = 31$; for $22 \le N \le 32$, take $q = 43$; and for $N > 32$, take $q \equiv 3 \pmod 4$ such that $N + 10 \le q \le 1.5(N + 10) < 2N$, which is guaranteed to exist by Theorem 7. Since $10 \le q - N < N$, it follows from our induction hypothesis that there is a bijection

$$h : \{0, 1, \ldots, q-N-1\} \to \{q-N, q-N+1, \ldots, 2(q-N)-1\}$$

such that for all $j \in \{0, 1, \ldots, q-N-1\}$, every prime divisor p of $h(j) - j$ satisfies $p \equiv 3 \pmod 4$ and $p > 10$. Define a bijection

$$g : \{0, 1, \ldots, N-1\} \to \{N, N+1, \ldots, 2N-1\}$$

by

$$g(j) = \begin{cases} j+q, & \text{if } \ 0 \le j \le 2N-q-1; \\ h(j-2N+q), & \text{if } \ 2N-q \le j \le N-1. \end{cases}$$

Then, for all $j \in \{0, 1, \ldots, N-1\}$, if p is a prime divisor of $g(j)-j$ then $p \equiv 3 \pmod 4$ and $p > 10$, since h has this property and the prime $q \equiv 3 \pmod 4$ was chosen so that $q > 10$. Thus, the bijection g in (4) exists for $n = N$, and by induction, it exists for all $n \ge 10$. $\square$

One can follow the proof of Theorem 8 to prove that a bound B_L similarly exists for other Gaussian lines L, but it is tedious to explicitly provide the bijections g in the base cases (as in Table 1) if the prime set of L contains large primes $p \equiv 3 \pmod 4$. Instead, for such Gaussian lines we use determinants to simply show the bijections exist in the base cases. Namely, let $n \in \mathbb{N}$ and L be a primitive Gaussian line. Also, let $q \equiv 3 \pmod 4$ be an inert prime that is larger than every inert prime in $\mathbb{P}(L)$. Then a rational prime p is not in the prime set of L if $p \equiv 3 \pmod 4$ and $p \geq q$. Thus, to show a bijection g as in Lemma 4 exists, it is sufficient to show a permutation σ of $\{1, 2, \ldots, n\}$ exists with the property that for all $j \in \{1, 2, \ldots, n\}$, there is a prime $p \equiv 3 \pmod 4$, $p \geq q$, such that $\sigma(j) - j + n = p$. Define an $n \times n$ matrix $M_n = [m_{j\,k}]$ by

$$m_{j\,k} = \begin{cases} 1, & \text{if } k - j + n = p \text{ for some prime } p \equiv 3 \pmod 4,\ p \geq q; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

By the definition of determinant, the determinant of M_n is

$$\det(M_n) = \sum_{\sigma} sgn(\sigma) \prod_{j=1}^{n} m_{j\,\sigma(j)},$$

where the summation is taken over all permutations σ of $\{1, 2, \ldots, n\}$. Recall that the sign $sgn(\sigma)$ is equal to ± 1 depending on whether σ is an even or odd permutation. Thus, if $\det(M_n)$ is nonzero, then there must be a permutation σ such that the product $\prod_{j=1}^{n} m_{j\,\sigma(j)}$ is nonzero, that is, all of the terms $m_{j\,\sigma(j)}$ are nonzero. Therefore, if $\det(M_n)$ is nonzero, then there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that for all $j \in \{1, 2, \ldots, n\}$, there is a prime $p \equiv 3 \pmod 4$, $p \geq q$, such that $\sigma(j) - j + n = p$. Notice this only gives us a sufficient condition for such a permutation σ to exist, not a necessary one. It is possible that there there are a lot of permutations σ with the desired property, but an equal number of them are even as are odd, leaving $\det(M_n) = 0$. Putting this together with Lemma 4 gives the following result.

Lemma 5. *Let L be a primitive Gaussian line and n be a positive integer. If $\det(M_n) \neq 0$, then a coprime mapping exists between any two contiguous intervals on L of length n.*

For a given Gaussian line L, it is computationally easy to construct M_n (choose q larger than all primes $p \equiv 3 \pmod 4$ that divide Δ) and to compute its determinant. Thus, for a positive integer n, Lemma 5 provides a simple method for establishing the existence of a coprime mapping between any two contiguous intervals on L of length n. We used this method to prove that we may take $B_L = 102$ in the special case where every prime $p \equiv 3 \pmod 4$ in $\mathbb{P}(L)$ is smaller than 100.

Theorem 9. *Let L be a primitive Gaussian line such that every prime $p \equiv 3 \pmod 4$ in $\mathbb{P}(L)$ is smaller than 100. If $n \geq 102$, then a coprime mapping exists between any two contiguous intervals on L of length n.*

Proof. Let L be a primitive Gaussian line such that every prime $p \equiv 3 \pmod 4$ in $\mathbb{P}(L)$ is smaller than 100. We use strong induction to prove the theorem for every $n \geq 102$. First consider the base cases $102 \leq n \leq 112$. For each n, take $q = 103$ and let M_n be the $n \times n$ matrix defined in (5). We used Python to compute $\det(M_n)$ for $102 \leq n \leq 112$, and saw that it was nonzero in every case. It follows from Lemma 5 that a coprime mapping exists between any two contiguous intervals on L of length n for $102 \leq n \leq 112$.

Now, suppose that the desired coprime mapping exists whenever $102 \leq n < N$ for some $N \geq 112$. Consider $n = N$. We first claim that there is a prime $q \equiv 3 \pmod 4$ such that

$$N + 102 \leq q < 2N.$$

Indeed, if $112 \leq N \leq 306$, then we may take q as follows:

$$q = \begin{cases} 223, & \text{if } 112 \leq N \leq 121; \\ 239, & \text{if } 122 \leq N \leq 137; \\ 271, & \text{if } 138 \leq N \leq 169; \\ 331, & \text{if } 170 \leq N \leq 229; \\ 443, & \text{if } 230 \leq N \leq 306. \end{cases}$$

If $N > 306$, then take $q \equiv 3 \pmod 4$ such that $N + 102 \leq q \leq 1.5(N + 102) < 2N$, which is guaranteed to exist by Theorem 7.

Now, let $[\alpha_k, \alpha_{k+N-1}]$ and $[\alpha_{k+N}, \alpha_{k+2N-1}]$ be two arbitrary contiguous intervals of length N on L. Since $102 \leq q - N < N$, it follows from the induction hypothesis that there is a coprime mapping h from the interval $[\alpha_{k+2N-q}, \alpha_{k+N-1}]$ of length $q - N$ to the interval $[\alpha_{k+N}, \alpha_{k+q-1}]$. Thus, we can define a coprime mapping

$$f : [\alpha_k, \alpha_{k+N-1}] \to [\alpha_{k+N}, \alpha_{k+2N-1}]$$

by

$$f(\alpha_j) = \begin{cases} j + q, & \text{if } k \leq j \leq k + 2N - q - 1; \\ h(\alpha_j), & \text{if } k + 2N - q \leq j \leq k + N - 1. \end{cases}$$

Therefore, a coprime mapping exists between any two contiguous intervals on L of length N, and by induction, one exists between any two contiguous intervals on L of length $n \geq 102$. $\square$

The bound of 102 in Theorem 9 is best possible in some cases, but certainly not all. For instance, by Theorem 5, there are infinitely many lines for which such a coprime mapping exists for all $n \geq 1$. To see that the bound of 102 can be best possible, consider a Gaussian line L with the property that $1+i$ and every prime $p \equiv 3 \pmod 4$, $p < 100$, is in the prime set of L. There are infinitely many such lines by Theorem 4. Recall that the prime set of L will also contain at least one Gaussian prime lying over q for every rational prime $q \equiv 1 \pmod 4$. It follows from Theorem 3 that there is a $k \in \mathbb{Z}$ such that the Gaussian integer α_k on L is divisible by $1+i$, every rational prime $p \equiv 3 \pmod 4$ with $p < 100$, and one Gaussian prime π_q for every rational prime $q \equiv 1 \pmod 4$ with $q < 102$. Then, for every $1 < n < 102$, a coprime mapping does not exist from $[\alpha_k, \alpha_{k+n-1}]$ to the contiguous interval $[\alpha_{k+n}, \alpha_{k+2n-1}]$ since α_k shares a common divisor with every Gaussian integer in $[\alpha_{k+n}, \alpha_{k+2n-1}]$ by Theorem 2.

Notice that using induction to extend Theorem 9 to cover lines L where larger inert primes $p \equiv 3 \pmod 4$ are in $\mathbb{P}(L)$ would only require consideration of more base cases. Thus, we are led to make the following conjecture.

Conjecture 1. Let L be a primitive Gaussian line and $q \equiv 3 \pmod 4$ be an inert prime that is larger than every inert prime in $\mathbb{P}(L)$. If $n \geq q-1$, then a coprime mapping

$$f : [\alpha_k, \alpha_{k+n-1}] \to [\alpha_{k+n}, \alpha_{k+2n-1}]$$

exists for all $k \in \mathbb{Z}$.

We end this section with an example of our main theorems. Consider the infinite family of Gaussian lines that are parallel to the real line. A line L is in this family if and only if $\delta = 1$ and $\alpha_0 = bi$ for some $b \in \mathbb{Z}$, in which case we have $1+i \in \mathbb{P}(L)$ and $\Delta = b$. It follows from Theorem 5, that a coprime mapping exists between any two contiguous intervals on L of the same length unless either 3 or 7 divides b, or both 11 and 19 divide b. If this divisibility condition on b is satisfied, but all primes $p \equiv 3 \pmod 4$ that divide b are smaller than 100, then it follows from Theorem 9 that a coprime mapping exists between any two contiguous intervals on L of length $n \geq 102$. If the divisibility condition is satisfied and b also has a prime divisor $p \equiv 3 \pmod 4$ that is larger than 100, then we conjecture that if $q \equiv 3 \pmod 4$ is a prime that is larger than b, then for all $n \geq q-1$, a coprime mapping exists between any two contiguous intervals on L of length n.

5. Coprime Mappings on Lines in Imaginary Quadratic Fields

In this section, we consider coprime mappings on lines in the other imaginary quadratic fields with class number one. Many of the theorems about Gaussian lines easily extend to these fields. Let K be one of the nine imaginary quadratic fields with class number one and L be a line in the complex plane that contains infinitely many algebraic integers in K. We discuss the prime set of L and provide necessary and sufficient conditions that L must satisfy to guarantee that a coprime mapping exists between any two contiguous intervals on L of the same length. Thus, Theorem 5 generalizes to these fields. The details follow the Gaussian case and are largely left to the reader.

Let K be an imaginary quadratic field with class number one, and let $\mathcal{O}_K$ be its ring of integers. We call a line L in the complex plane an $\mathcal{O}_K$-line if it contains two, and hence infinitely many, elements in $\mathcal{O}_K$. We can extend the definitions of α_0, δ, α_k ($k \in \mathbb{Z}$), Δ, *primitive*, and *prime set* from Gaussian lines to $\mathcal{O}_K$-lines since $\mathcal{O}_K$ is a unique factorization domain. Similarly, the theorems and proofs for $\mathcal{O}_K$-lines follow those given for Gaussian lines with the primes congruent to 3 modulo 4 replaced by inert primes, the primes congruent to 1 modulo 4 replaced by split primes, and the prime lying over 2 replaced by ramified primes. In particular, we have the following theorem.

Theorem 10. *Let L be a primitive $\mathcal{O}_K$-line with prime set $\mathbb{P}(L)$. Let p be a rational prime.*

(a) If p is inert in K, then $p \in \mathbb{P}(L)$ if and only if p divides Δ.

(b) If p is split or ramified in K and $\pi_p \in \mathcal{O}_K$ is a prime that lies over p, then $\pi_p \in \mathbb{P}(L)$ if and only if π_p does not divide δ.

Let L be a primitive $\mathcal{O}_K$-line. Then considering split, inert, and ramified primes in K, the proof of Theorem 5 generalizes to give necessary and sufficient conditions for a coprime mapping to exist between any two contiguous intervals on L of the same length. If p splits in K, then the prime set $\mathbb{P}(L)$ of L contains at least one prime lying over p. The existence of a coprime mapping between any two contiguous intervals on L of the same length is determined by whether or not certain small non-split primes are in $\mathbb{P}(L)$. The proof requires a lemma that is a companion to those given in Section 3 since different small primes can be inert in this more general setting.

Lemma 6. *Let $n \geq 2$ be an integer. Then there exist non-negative integers x and y such that*

$$n \leq 3^x 5^y < 2n.$$

Proof. Let $n \geq 2$ and $m = 3^s 5^t$ be the largest integer of this form that is strictly less than n. First we show that there are integers x and y such that $m < 3^x 5^y < 2m$. Indeed, if $t = 0$ then $m = 3^s$ for some $s \geq 1$. Thus, $m < 3^{s-1}5 < 2m$ since $1 < 5/3 < 2$. Similarly, if $t > 0$, then $m < 3^{s+2}5^{t-1} < 2m$ since $1 < 9/5 < 2$. Moreover, we must have $n \leq 3^x 5^y$ since $m < 3^x 5^y$ and $m = 3^s 5^t$ is the largest integer of this form that is strictly less than n. Thus, we have

$$n \leq 3^x 5^y < 2m < 2n,$$

as needed. $\square$

Now, together with Lemma 6, the proof of Theorem 5 generalizes and gives the following theorem.

Theorem 11. *Let K be an imaginary quadratic field with class number one and L be a primitive $\mathcal{O}_K$-line. For a rational prime p, let π_p be a prime in $\mathcal{O}_K$ that divides p. Then a coprime mapping exists between any two contiguous intervals on L of the same length if and only if at least one of the following three conditions is satisfied:*

1. *the prime 2 is not split in K and $\pi_2 \notin \mathbb{P}(L)$;*
2. *the primes 3 and 5 are not split in K and $\pi_3, \pi_5 \notin \mathbb{P}(L)$;*
3. *for some $p \in \{11, 13, 17, 19\}$, the primes 3, 7, and p are not split in K and $\pi_3, \pi_7, \pi_p \notin \mathbb{P}(L)$.*

For each of the nine imaginary quadratic fields K with class number one, at least one of the conditions about specific rational primes not splitting in K is met. Thus, for each K there are infinitely many primitive $\mathcal{O}_K$-lines L with the property that a coprime mapping exists between any two contiguous intervals on L of the same length.

Suppose L is a primitive $\mathcal{O}_K$-line that does not satisfy any of the conditions in Theorem 11. Let q be the smallest inert prime in K that is larger than all the inert primes in $\mathbb{P}(L)$. Such a prime q exists since the prime set of L contains only finitely many inert primes by Theorem 10. In the special case where $K = \mathbb{Q}(i)$, we predict in Conjecture 1 that if $B_L = q - 1$, then a coprime mapping exists between any two contiguous intervals on L of length $n > B_L$. If $K \neq \mathbb{Q}(i)$, then we predict that a bound B_L will similarly exist, but that in general it will be larger than q.

For example, consider $K = \mathbb{Q}(\sqrt{-3})$. Then $\mathcal{O}_K$ is the ring of Eisenstein integers $\mathbb{Z}[(1 + \sqrt{-3})/2]$, and the inert primes are 2 and the rational primes $p \equiv 5 \pmod 6$. It follows from our strategy used for Gaussian lines that to show a coprime mapping

$$f : [\alpha_k, \alpha_{k+n-1}] \to [\alpha_{k+n}, \alpha_{k+2n-1}]$$

exists on any two contiguous intervals of length n on an Eisenstein line L, it is sufficient to show that a bijection

$$g : \{0, 1, \ldots, n-1\} \to \{n, n+1, \ldots, 2n-1\}$$

exists such that for all $j \in \{0, 1, \ldots, n-1\}$ there is an inert prime $p_j \equiv 5 \pmod 6$, $p_j \notin \mathbb{P}(L)$, with $g(j) = j + p_j$. But if $n \equiv 2$ or $3 \pmod 6$, then no such bijection g can exist since otherwise $g(j)$ would be divisible by 6 if and only if $j \equiv 1 \pmod 6$, and there is one few integer in the set $\{n, n+1, \ldots, 2n-1\}$ that is divisible by 6 than there are integers $j \in \{0, 1, \ldots, n-1\}$ with $j \equiv 1 \pmod 6$. Here every inert prime p satisfies $p^2 \equiv 1 \pmod 6$, so we can patch this up and get our desired coprime mapping f by taking $f(\alpha_j) = \alpha_{j+p_j^2}$ for some values of j, but this requires longer intervals than in the Gaussian case. To better see this, suppose L is an Eisenstein line with $2, 5 \in \mathbb{P}(L)$ and $p \notin \mathbb{P}(L)$ for all rational primes $p \equiv 5 \pmod 6$, $p \geq 11$ (so $q = 11$). If $n \equiv 2$ or $3 \pmod 6$ and $n \leq 57$, then a coprime mapping f that satisfies $f(\alpha_j) = \alpha_{j+p_j}$ for some inert prime $p_j \equiv 5 \pmod 6$ does not exist. If $n \geq 58$, then $n = 62$ is the smallest value of $n \equiv 2$ or $3 \pmod 6$, and in this case we have a long enough interval to translate by $121 = 11^2$ and use that $11 \notin \mathbb{P}(L)$. Namely, define f on $[\alpha_k, \alpha_{k+n-1}]$ by

$$f(\alpha_j) = \begin{cases} \alpha_{j+121}, & \text{if } k \leq j \leq k+2; \\ \alpha_{j+59}, & \text{if } k+3 \leq j \leq k+61. \end{cases}$$

This is a coprime mapping onto $[\alpha_{k+n}, \alpha_{k+2n-1}]$ since 11 and 59 are inert primes that are not in $\mathbb{P}(L)$. A similar coprime mapping f exists when $n = 63$. Thus, if L is an Eisenstein line such that every inert prime in $\mathbb{P}(L)$ is smaller than $q = 11$, it appears that we must go up to $B_L = 58$ for a coprime mapping to exist between all contiguous intervals on L of length $n \geq B_L$. Note that $q = 11$ is also inert in $\mathbb{Z}[i]$, so if L were a Gaussian line with this property, then by contrast we could take $B_L = 10$ by Theorem 8.

It seems plausible that if K is one of the nine imaginary quadratic fields with class number one and L is an $\mathcal{O}_K$-line, then a coprime mapping exists between any two sufficiently long contiguous intervals on L. We leave the reader to further investigate this problem.

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