TRANSCENDENCE OF SOME POWER SERIES IN FUNCTION FIELDS

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Abstract

Let $\mathbb{k}$ be a field of characteristic 0. Let $\mathbb{k}((1/x))$ be the function field with respect to the degree valuation $|\cdot|_\infty$. In this paper, we show that the values of some power series over $\mathbb{k}(x)$ evaluated at certain Liouville numbers in $\mathbb{k}((1/x))$ are either rational over $\mathbb{k}[x]$ or transcendental. Some examples are also included.

1. Introduction

A real number α is considered a Liouville number if, for any positive integer n, there exist integers p_n and $q_n > 1$ satisfying the inequality

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^n}.$$

The question raised by Mahler in 1984 [7] to explore a relationship between Liouville numbers and certain analytic functions. Specifically, he sought for an analytic function $f(T)$ with property that if α is a Liouville number, then $f(\alpha)$ is also a Liouville number. In fact, his inquiry was inspired by the earlier work of Maillet

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[8], who demonstrated that if α is a Liouville number and $f(T)$ is a non-constant rational function with rational coefficients, then $f(\alpha)$ is also a Liouville number. The arithmetic properties of Liouville numbers were also explored in Maillet's work [8].

An extensive body of scholarly research has been dedicated to the exploration of this particular inquiry. Some research efforts have focused on constructing extensive sets of Liouville numbers that are mapped onto themselves by transcendental entire functions; see [9–11]. Others have concentrated on power series with rational coefficients, delving into the transcendence of their values; see [1,4,5]. For instance, Çalişkan [1] demonstrated in 2018 that certain power series over $\mathbb{Q}$ take values of either rational or transcendental numbers for arguments from the set of Liouville numbers, under specific conditions. The analogous results in the p-adic number field were also established in the same work. In 2019, Lelis and Marques [6] introduced weak p-adic numbers, establishing them as p-adic transcendental numbers and demonstrating their inclusion of p-adic Liouville numbers. They also examined the analogous result to Maillet's work in this context.

In 2010, Chaichana and Laohakosol [3] discussed arithmetic properties of Liouville numbers in the function field, particularly in the non-archimedean case. They also introduced a class of Liouville numbers referred to as the Liouville series and established criteria for their algebraic independence. In another study, they extended Erdős's result that every real number can be represented as a sum and a product of two real Liouville numbers to the non-archimedean case. Moreover, they demonstrated that any bilinear rational transformation over $\mathbb{k}(x)$ maps Liouville numbers to Liouville numbers; see [2].

In this work, inspired by the study of Çalişkan [1], we establish an analogous result in the function field case.

2. Main Results

Throughout, let $\mathbb{k}$ be the field of characteristic 0. Let $\mathbb{k}_\infty = \mathbb{k}((1/x))$ be a field of all Laurent series over $\mathbb{k}$ equipped with the degree valuation $|\cdot|_\infty$ defined by $|\alpha|_\infty = e^n$ if

$$\alpha = c_n x^n + \cdots + c_1 x + c_0 + \frac{c_{-1}}{x} + \frac{c_{-2}}{x^2} + \cdots \in \mathbb{k}_\infty \backslash \{0\},$$

where $n \in \mathbb{Z}$, $c_i \in \mathbb{k}$ for all $i \le n$ and $c_n \neq 0$, and $|0|_\infty = 0$.

We first introduce the Liouville numbers in $\mathbb{k}_\infty$.

Definition 1 ([2]). An element $\alpha \in \mathbb{k}_\infty$ is called a $\mathbb{k}_\infty-$*Liouville number* if for any $n \in \mathbb{N}$ there exist $p_n, q_n \in \mathbb{k}[x]\backslash\{0\}$ with $|q_n|_\infty > 1$ such that

$$0 < \left|\alpha - \frac{p_n}{q_n}\right|_\infty < \frac{1}{|q_n|_\infty^n}.$$

The following theorem is the main tool for proving the transcendence of elements in $\mathbb{k}_\infty$.

Theorem 1 (Roth's theorem [12]). *Let $\mathbb{k}$ be a field of characteristic 0. If $\alpha \in \mathbb{k}_\infty \backslash \{0\}$ is an algebraic element over $\mathbb{k}(x)$ of degree ≥ 2, then for each $\varepsilon > 0$, the inequality*

$$\left| \alpha - \frac{P}{Q} \right|_\infty < \frac{1}{|Q|_\infty^{2+\varepsilon}}$$

is satisfied by only a finite number of pairs $P, Q \neq 0$ in $\mathbb{k}[x]$ with $\gcd(P, Q) = 1$.

Our main result states that:

Theorem 2. *Let $f(T)$ be the power series such that*

$$f(T) = \sum_{n=0}^{\infty} c_n T^n \tag{1}$$

where $c_n = b_n / a_n$ with $b_n, a_n \in \mathbb{k}[x] \backslash \{0\}$ and $|a_n|_\infty > 1$ for sufficiently large n satisfying the following two conditions:

$$\sigma := \liminf_{n \to \infty} \frac{\deg(a_{n+1})}{\deg(a_n)} > 1 \tag{2}$$

and

$$\theta := \limsup_{n \to \infty} \frac{\deg(b_n)}{\deg(a_n)} < 1, \tag{3}$$

further let $A_n = lcm[a_0, a_1, \ldots, a_n]$ and derive $u := \limsup_{n \to \infty} \frac{\deg(A_n)}{\deg(a_n)}$.

Let α be a $\mathbb{k}_\infty-$Liouville number for which the following properties hold: there exist sequences $(p_n), (q_n) \subset \mathbb{k}[x] \backslash \{0\}$ with $|q_n|_\infty > 1$ and a sequence of real numbers $(w(n))$ with $\lim_{n \to \infty} w(n) = \infty$ such that

$$\left| \alpha - \frac{p_n}{q_n} \right|_\infty < \frac{1}{|q_n|_\infty^{n w(n)}} \tag{4}$$

and

$$|a_n|_\infty^{\delta_1} \leq |q_n|_\infty^{n} \leq |a_n|_\infty^{\delta_2} \tag{5}$$

hold for sufficiently large n, where $\delta_1, \delta_2 \in \mathbb{R}$ such that $u < \delta_1 \leq \delta_2$.

If $\sigma(1 - \theta) > 4\delta_2$, then $f(\alpha)$ is either in $\mathbb{k}(x)$ or a $\mathbb{k}_\infty-$transcendental number.

We separate the proof of Theorem 2 into two parts.

Part I: We show that $f(T) = \sum_{n=0}^{\infty} c_n T^n$ in (1) converges everywhere on $\mathbb{k}_\infty$ by showing that $\limsup_{n \to \infty} |c_n|_\infty^{1/n} = 0$.

Proof of Part I. From (3), we have $\theta < 1$. Then there exists a sufficiently small $\varepsilon_0 > 0$ such that $\theta + \varepsilon_0 < 1$ and thus $1 - \theta - \varepsilon_0 > 0$. Therefore, for sufficiently large n, we have

$$\frac{\deg(b_n)}{\deg(a_n)} < \theta + \varepsilon_0$$

and we then have, for sufficiently large n,

$$|b_n|_\infty < |a_n|_\infty^{\theta+\varepsilon_0}. \tag{6}$$

Similarly, from (2), we have $\sigma > 1$. Then there is a sufficiently small $\varepsilon_1 > 0$ such that $\sigma_1 := \sigma - \varepsilon_1 > 1$. Since $\sigma_1 < \sigma$, for sufficiently large n, we have

$$\sigma_1 < \frac{\deg(a_{n+1})}{\deg(a_n)}.$$

Since $|a_n|_\infty > 1$ for sufficiently large n, there exists $N_0 \in \mathbb{N}$ such that, for all $n \geq N_0$, (6) holds and

$$\deg(a_{n+1}) > \sigma_1 \deg(a_n) > \deg(a_n) > 1. \tag{7}$$

It is clear that the exponential growth (7) implies

$$\lim_{n\to\infty} \frac{\deg(a_n)}{n} = \infty. \tag{8}$$

Hence, by (6), we have

$$|c_n|_\infty^{1/n} = \left|\frac{b_n}{a_n}\right|_\infty^{1/n} < \frac{1}{|a_n|_\infty^{(1-\theta-\varepsilon_0)/n}} \to 0 \quad (n \to \infty)$$

as desired. □

Part II: Now, we have $f(\alpha) \in \Bbbk_\infty$. We next show that $f(\alpha)$ is either in $\Bbbk(x)$ or a transcendental element by using Roth's theorem.

Proof of Part II. For each $n \in \mathbb{N}$, set

$$f_n(T) = \sum_{\nu=0}^{n} c_\nu T^\nu.$$

Note that α is a $\Bbbk_\infty$−Liouville number and there exist sequences $(p_n), (q_n) \subset \Bbbk[x]\backslash\{0\}$ with $|q_n|_\infty > 1$ such that the inequality (4) holds. Then we have, for each $n \in \mathbb{N}$,

$$\begin{aligned} f_n\left(\frac{p_n}{q_n}\right) &= \sum_{\nu=0}^{n} \frac{b_\nu}{a_\nu}\left(\frac{p_n}{q_n}\right)^\nu = \frac{A_n\left[\frac{b_0}{a_0}q_n^n + \frac{b_1}{a_1}{p_n}^1 q_n^{n-1} + \cdots + \frac{b_n}{a_n}p_n^n\right]}{A_n q_n^n} \\ &:= \frac{h_n}{A_n q_n^n}, \end{aligned}$$

where $A_n = \operatorname{lcm}[a_0, a_1, \ldots, a_n]$. It is clear that h_n and $A_n q_n^n$ are in $\Bbbk[x]$ and thus $f_n\left(\frac{p_n}{q_n}\right) = \frac{h_n}{A_n q_n^n} \in \Bbbk(x)$. Note that, by assumption (5),

$$0 < |a_n|_\infty^{\delta_1} \leq |q_n|_\infty^n \leq |a_n|_\infty^{\delta_2}.$$

Since $\limsup_{n\to\infty} \frac{\deg(A_n)}{\deg(a_n)} = u < \delta_1 \leq \delta_2$, we then have, for sufficiently large $n \geq N_0$, that $\frac{\deg(A_n)}{\deg(a_n)} < \delta_2$, and so

$$0 < |A_n|_\infty < |a_n|_\infty^{\delta_2}.$$

Now we have

$$0 < |A_n|_\infty |q_n^n|_\infty < |a_n|_\infty^{2\delta_2},$$

or equivalently,

$$\frac{1}{|a_n|_\infty} < \frac{1}{|A_n q_n^n|_\infty^{1/2\delta_2}} \tag{9}$$

for sufficiently $n \geq N_0$. Moreover,

$$\begin{aligned}\left|f(\alpha) - \frac{h_n}{A_n q_n^n}\right|_\infty &= \left|f(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_\infty \\ &\leq \max\left\{|f(\alpha) - f_n(\alpha)|_\infty, \left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_\infty\right\}. \end{aligned} \tag{10}$$

By (6) and for all $n \geq N_0$, we obtain

$$\begin{aligned}|f(\alpha) - f_n(\alpha)|_\infty = \left|\sum_{\nu=n+1}^{\infty} c_\nu \alpha^\nu\right|_\infty &\leq \max_{\nu\geq n+1}\left\{\left|\frac{b_\nu}{a_\nu}\right|_\infty \cdot |\alpha|_\infty^\nu\right\} \\ &< \max_{\nu\geq n+1}\left\{\frac{1}{|a_\nu|_\infty^{1-\theta-\varepsilon_0}} \cdot |\alpha|_\infty^\nu\right\}.\end{aligned}$$

By (8), we choose a sufficiently small $\varepsilon_2 > 0$ such that $1 - \theta - \varepsilon_0 - \varepsilon_2 > 0$ and $\lim_{n\to\infty} \frac{\varepsilon_2 \deg(a_n)}{n} = \infty$. Then, for sufficiently large $n \geq N_0$, we obtain

$$\log(|\alpha|_\infty + 1) < \frac{\varepsilon_2 \deg(a_{n+1})}{n+1}$$

and so

$$(|\alpha|_\infty + 1)^{n+1} < |a_{n+1}|_\infty^{\varepsilon_2}. \tag{11}$$

Now we have, for sufficiently large $n \geq N_0$,

$$\begin{aligned}|f(\alpha) - f_n(\alpha)|_\infty &< \max_{\nu\geq n+1}\left\{\frac{1}{|a_\nu|_\infty^{1-\theta-\varepsilon_0-\varepsilon_2}}\right\} = \frac{1}{|a_{n+1}|_\infty^{1-\theta-\varepsilon_0-\varepsilon_2}} \\ &< \frac{1}{|a_n|_\infty^{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_0-\varepsilon_2)}} \\ &< \frac{1}{|A_n q_n^n|_\infty^{\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_0-\varepsilon_2)}{2\delta_2}}}\end{aligned} \tag{12}$$

by (7) and (9).

Next, consider the term $|f_n(\alpha) - f_n(\frac{p_n}{q_n})|_\infty$ in (10). We have, for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_\infty \\
&= \left|\alpha - \frac{p_n}{q_n}\right|_\infty \left|\sum_{\nu=1}^{n} c_\nu \left(\alpha^{\nu-1} + \alpha^{\nu-2}\left(\frac{p_n}{q_n}\right)^1 + \cdots + \left(\frac{p_n}{q_n}\right)^{\nu-1}\right)\right|_\infty \\
&\le \left|\alpha - \frac{p_n}{q_n}\right|_\infty \max_{1\le\nu\le n}\{|c_\nu|_\infty\} \max_{1\le\nu\le n}\left\{\left|\alpha^{\nu-1} + \alpha^{\nu-2}\left(\frac{p_n}{q_n}\right)^1 + \cdots + \left(\frac{p_n}{q_n}\right)^{\nu-1}\right|_\infty\right\}.
\end{aligned}$$

For all sufficiently large $n \ge N_0$, we have

$$\begin{aligned}
\max_{1\le\nu\le n}\{|c_\nu|_\infty\} &= \max\left\{\left|\frac{b_1}{a_1}\right|_\infty, \ldots, \left|\frac{b_{N_0}}{a_{N_0}}\right|_\infty, |b_{N_0+1}|_\infty, \ldots, |b_n|_\infty\right\} \\
&\le \max\left\{M, |b_{N_0+1}|_\infty, \ldots, |b_n|_\infty\right\} \\
&\le \max\left\{M, |a_{N_0+1}|_\infty^{\theta+\varepsilon_0}, \ldots, |a_n|_\infty^{\theta+\varepsilon_0}\right\} \qquad \text{by (6)} \\
&< \max\left\{M, |a_{N_0+1}|_\infty, \ldots, |a_n|_\infty\right\} \qquad \text{since } \theta+\varepsilon_0 < 1 \\
&< \max\left\{M, |a_n|_\infty\right\} \qquad \text{since } \deg(a_n) \text{ is increasing,}
\end{aligned}$$

where $M = \max\left\{\left|\frac{b_1}{a_1}\right|_\infty, \ldots, \left|\frac{b_{N_0}}{a_{N_0}}\right|_\infty\right\}$. Then there exists $N_1 > N_0$ such that, for all $n \ge N_1$,

$$\max_{1\le\nu\le n}\{|c_\nu|_\infty\} < |a_n|_\infty.$$

Moreover, for all $n \ge N_1$, since

$$\left|\frac{p_n}{q_n}\right|_\infty \le \max\left\{|\alpha|_\infty, \left|\alpha - \frac{p_n}{q_n}\right|_\infty\right\} \le \max\{|\alpha|_\infty, 1\} \le |\alpha|_\infty + 1,$$

and $|\alpha|_\infty < |\alpha|_\infty + 1$, we have

$$\begin{aligned}
&\left|\alpha^{\nu-1} + \alpha^{\nu-2}\left(\frac{p_n}{q_n}\right)^1 + \cdots + \left(\frac{p_n}{q_n}\right)^{\nu-1}\right|_\infty \\
&\qquad \le \max\left\{\left|\alpha^{\nu-1}\right|_\infty, \left|\alpha^{\nu-2}\left(\frac{p_n}{q_n}\right)^1\right|_\infty, \ldots, \left|\left(\frac{p_n}{q_n}\right)^{\nu-1}\right|_\infty\right\} \\
&\qquad = (|\alpha|_\infty + 1)^{\nu-1}
\end{aligned}$$

for all $\nu \in \{1, 2, \ldots, n\}$. Now we have, for all $n \ge N_1$,

$$\begin{aligned}
\left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_\infty &\le \left|\alpha - \frac{p_n}{q_n}\right|_\infty |a_n|_\infty (|\alpha|_\infty + 1)^{n-1} \\
&\le \frac{1}{{|q_n|_\infty}^{nw(n)}} |a_n|_\infty^2 \quad \text{(by (11)).}
\end{aligned}$$

Since $u < \delta_1 \leq \delta_2$, there exists $\varepsilon_3 > 0$ such that $0 < u+\varepsilon_3 < \delta_1$. Then $\delta_1 - u - \varepsilon_3 > 0$ and so we obtain $\frac{4}{\delta_1 - u - \varepsilon_3} > 0$. Since $\lim_{n\to\infty} w(n) = \infty$, there exists $N_2 \in \mathbb{N}$ with $N_2 \geq N_1$ such that

$$2 < (\delta_1 - u - \varepsilon_3)\frac{w(n)}{2}$$

for all $n \geq N_2$. Now we have, for all $n \geq N_2$,

$$\begin{aligned}\left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_\infty &< \frac{1}{|q_n|_\infty{}^{nw(n)}}\left(|a_n|_\infty^{\delta_1 - u - \varepsilon_3}\right)^{\frac{w(n)}{2}} \\ &= \frac{1}{|q_n|_\infty{}^{nw(n)}}\left(\frac{|a_n|_\infty^{\delta_1}}{|a_n|_\infty{}^{u+\varepsilon_3}}\right)^{\frac{w(n)}{2}} \\ &\leq \frac{1}{|q_n|_\infty{}^{nw(n)}}\left(\frac{|q_n|_\infty^{n}}{|a_n|_\infty{}^{u+\varepsilon_3}}\right)^{\frac{w(n)}{2}} \\ &= \frac{1}{|q_n|_\infty{}^{\frac{nw(n)}{2}}}\left(\frac{1}{|a_n|_\infty{}^{u+\varepsilon_3}}\right)^{\frac{w(n)}{2}}.\end{aligned}$$

Since $\limsup_{n\to\infty} \frac{\deg(A_n)}{\deg(a_n)} = u < u + \varepsilon_3$, we then have, for sufficiently large $n \geq N_2$, $\frac{\deg(A_n)}{\deg(a_n)} < u + \varepsilon_3$ and so

$$|A_n|_\infty < |a_n|_\infty^{u+\varepsilon_3}.$$

Now we can conclude that

$$\left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_\infty < \frac{1}{|q_n|_\infty{}^{\frac{nw(n)}{2}}}\left(\frac{1}{|A_n|_\infty}\right)^{\frac{w(n)}{2}} = \frac{1}{|A_n q_n^n|_\infty{}^{\frac{w(n)}{2}}} \tag{13}$$

for sufficiently large $n \geq N_2$.

Finally, from (12) and (13), the value of (10) becomes

$$\begin{aligned}\left|f(\alpha) - \frac{h_n}{A_n q_n^n}\right|_\infty &\leq \max\left\{|f(\alpha) - f_n(\alpha)|_\infty, \left|f_n(\alpha) - f_n\left(\frac{p_n}{q_n}\right)\right|_\infty\right\} \\ &< \max\left\{\frac{1}{|A_n q_n^n|_\infty{}^{\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_0-\varepsilon_2)}{2\delta_2}}}, \frac{1}{|A_n q_n^n|_\infty{}^{\frac{w(n)}{2}}}\right\}\end{aligned} \tag{14}$$

for sufficiently large $n \geq N_2$. Since $\lim_{n\to\infty} w(n) = \infty$, there is $N_3 \geq N_2$ such that

$$\frac{w(n)}{2} > \frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_0-\varepsilon_2)}{2\delta_2} \quad (n \geq N_3).$$

Therefore, (14) becomes

$$\left|f(\alpha) - \frac{h_n}{A_n q_n^n}\right|_\infty < \frac{1}{|A_n q_n^n|_\infty{}^{\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_0-\varepsilon_2)}{2\delta_2}}} \tag{15}$$

for all $n \geq N_3$. Since $\frac{\sigma(1-\theta)}{2\delta_2} > 2$ by the assumption and ε_0, ε_1 and ε_2 can be chosen to be sufficiently small and

$$\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_0-\varepsilon_2)}{2\delta_2} > 2,$$

there exists a positive number ε such that

$$\frac{(\sigma-\varepsilon_1)(1-\theta-\varepsilon_0-\varepsilon_2)}{2\delta_2} > 2+\varepsilon. \tag{16}$$

Hence, from (15), (16) and since $|A_n q_n^n|_\infty > 1$ for all $n \in \mathbb{N}$, we conclude that

$$\left| f(\alpha) - \frac{h_n}{A_n q_n^n} \right|_\infty < \frac{1}{{|A_n q_n^n|_\infty}^{2+\varepsilon}}$$

for sufficiently large n (for all $n \geq N_3$), where ε is a suitable positive number dependent on $\varepsilon_0, \varepsilon_1$ and ε_2.
Therefore

$$\left| f(\alpha) - \frac{h_n}{A_n q_n^n} \right|_\infty < \frac{1}{{|A_n q_n^n|_\infty}^{2+\varepsilon}} \longrightarrow 0 \quad (n \to \infty).$$

Hence, it can be concluded that

$$\lim_{n\to\infty} \frac{h_n}{A_n q_n^n} = f(\alpha).$$

Notice that if $\left\{ \frac{h_n}{A_n q_n^n} : n \in \mathbb{N} \right\}$ is a finite set, then there exists a constant subsequence of $\left(\frac{h_n}{A_n q_n^n} \right)$, namely $\left(\frac{P}{Q} \right)$. Therefore,

$$f(\alpha) = \lim_{n\to\infty} \frac{h_n}{A_n q_n^n} = \frac{P}{Q}$$

and we conclude that $f(\alpha) = \frac{P}{Q} \in \mathbb{k}(x)$.

Now, we may assume that there are infinitely many distinct $\frac{h_n}{A_n q_n^n} \in \mathbb{k}(x)$ with $\gcd(h_n, A_n q_n^n) = 1$ and

$$\left| f(\alpha) - \frac{h_n}{A_n q_n^n} \right|_\infty < \frac{1}{{|A_n q_n^n|_\infty}^{2+\varepsilon}} \tag{17}$$

for all $n \geq N_3$. By Roth's theorem, $f(\alpha)$ is either a rational number over $\mathbb{k}(x)$ or a $\mathbb{k}_\infty-$transcendental number. □

We can additionally show that in the latter case of the previous proof we can exclude rationality with details following: suppose $f(\alpha) = \frac{P}{Q} \in \mathbb{k}(x)$. Choose $N = \max\{N_3, \deg(Q)\}$. If $|PA_n q_n^n - h_n Q|_\infty = 0$ for all $n \geq N$, then $PA_n q_n^n - h_n Q = 0$

for all $n \geq N$. That is, $\frac{h_n}{A_n q_n^n} = \frac{P}{Q}$ for all $n \geq N$. Hence, $\left(\frac{h_n}{A_n q_n^n}\right)$ is an eventually constant, a contradiction. Then there is $m \geq N$ such that $|PA_m q_m^m - h_m Q|_\infty \neq 0$. Therefore

$$\left|f(\alpha) - \frac{h_m}{A_m q_m^m}\right|_\infty = \frac{|PA_m q_m^m - h_m Q|_\infty}{|QA_m q_m^m|_\infty} \geq \frac{1}{|QA_m q_m^m|_\infty}.$$

Now, we have

$$\frac{1}{|QA_m q_m^m|_\infty} \leq \left|f(\alpha) - \frac{h_m}{A_m q_m^m}\right|_\infty < \frac{1}{|A_m q_m^m|_\infty{}^{2+\varepsilon}}$$

by (17). Therefore

$$e^m < |A_m q_m^m|_\infty^{1+\varepsilon} < |Q|_\infty < e^{\deg(Q)}$$

which is absurd as we assume $m \geq N = \max\{\deg(Q), N_3\} \geq \deg(Q)$. Hence, $f(\alpha)$ is a $\mathbb{k}_\infty$−transcendental number.

Some interesting examples are constructed as follows.

Example 1. Define an increasing sequence $(\widetilde{p}_n)$ of positive integers by

$$\widetilde{p}_1 = 1, \quad \widetilde{p}_2 = 2 \quad \text{and} \quad \widetilde{p}_{n+1} \geq \widetilde{p}_n \cdot n^2 \quad (n \geq 2).$$

Let $f(T) = \sum_{n=0}^{\infty} \left(\frac{b_n}{a_n}\right) T^n$ where $b_n = x + 1$ $(n \geq 0)$ and

$$a_n = \begin{cases} x^{\widetilde{p}_n \cdot \lfloor \frac{n}{2} \rfloor} & \text{if } n \geq 1, \\ 1 & \text{if } n = 0 \end{cases}$$

be the power series over $\mathbb{k}(x)$. It is easy to see that, for all $n \in \mathbb{N} \cup \{0\}$, b_n and a_n are nonzero polynomials over $\mathbb{k}$ and a_n divides a_{n+1}. Moreover,

$$|b_n|_\infty = e > 1 \quad \text{and} \quad |a_n|_\infty = e^{\widetilde{p}_n \cdot \lfloor \frac{n}{2} \rfloor} > 1 \quad (n \geq 2).$$

Next, we have

$$\sigma = \liminf_{n\to\infty} \frac{\deg(a_{n+1})}{\deg(a_n)} \geq \lim_{n\to\infty} \left(n^2 \cdot \frac{\lfloor \frac{n+1}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor}\right) = \infty > 1$$

and

$$\theta = \limsup_{n\to\infty} \frac{\deg(b_n)}{\deg(a_n)} = \lim_{n\to\infty} \frac{1}{\widetilde{p}_n \cdot \lfloor \frac{n}{2} \rfloor} = 0 < 1.$$

We have $A_n = \mathrm{lcm}[a_0, a_1, \ldots, a_n] = a_n$ and $|A_n|_\infty = |a_n|_\infty$ $(n \geq 2)$. We then obtain

$$u = \limsup_{n\to\infty} \frac{\deg(A_n)}{\deg(a_n)} = 1.$$

Let $\alpha = \sum\limits_{n=1}^{\infty} \frac{1}{x^{\widetilde{p}_n}}$. Observe that the series $\sum\limits_{n=1}^{\infty} \frac{1}{x^{\widetilde{p}_n}}$ converges in $\mathbb{k}_\infty$ because $\lim\limits_{n\to\infty} \frac{1}{|x^{\widetilde{p}_n}|_\infty} = 0$. For each $n \in \mathbb{N}$, set $p_n = x^{\widetilde{p}_n} \sum\limits_{k=1}^{n} \frac{1}{x^{\widetilde{p}_k}}$, $q_n = x^{\widetilde{p}_n}$ and $w(n) = \frac{n}{2}$. Then $p_n, q_n \in \mathbb{k}[x]\backslash\{0\}$ with $|q_n|_\infty = e^{\widetilde{p}_n} > 1$ and $\lim\limits_{n\to\infty} w(n) = \infty$. Moreover,

$$\left|\alpha - \frac{p_n}{q_n}\right|_\infty = \left|\sum_{k=n+1}^{\infty} \frac{1}{x^{\widetilde{p}_k}}\right|_\infty = \frac{1}{|x^{\widetilde{p}_{n+1}}|_\infty} = \frac{1}{e^{\widetilde{p}_{n+1}}} \leq \frac{1}{e^{\widetilde{p}_n \cdot n^2}} < \frac{1}{|q_n|_\infty^{n\cdot w(n)}}.$$

It follows that

$$\left|\alpha - \frac{p_n}{q_n}\right|_\infty < \frac{1}{|q_n|_\infty^{n\cdot w(n)}} \leq \frac{1}{|q_n|_\infty^{n}} \quad (n \geq 2).$$

For $n = 1$, by direct computation, we have

$$\left|\alpha - \frac{p_1}{q_1}\right|_\infty = \left|\sum_{k=2}^{\infty} \frac{1}{x^{\widetilde{p}_k}}\right|_\infty = \frac{1}{|x^{\widetilde{p}_2}|_\infty} = \frac{1}{e^2} < \frac{1}{e} = \frac{1}{|q_1|_\infty}.$$

Therefore, α is $\mathbb{k}_\infty$-Liouville number by Definition 1.

Choose real numbers $\delta_1 = 2$ and $\delta_2 = 3$. Then

$$|a_n|_\infty^{\delta_1} \leq |q_n|_\infty^{n} \leq |a_n|_\infty^{\delta_2}$$

for all $n \in \mathbb{N}$.

Now, we have $u = 1$ and $\sigma(1-\theta) = \infty > 4\delta_2$. By Theorem 2, we can conclude that $f(\alpha)$ is either rational or transcendental. But in fact, $f(\alpha)$ is a transcendental element in $\mathbb{k}_\infty$ because of the following: notice that

$$\begin{aligned} f_n\left(\frac{p_n}{q_n}\right) &= \sum_{\nu=0}^{n} \left(\frac{b_\nu}{a_\nu}\right)\left(\frac{p_n}{q_n}\right)^{\nu} \\ &= (1+x) + \sum_{\nu=1}^{n} \left(\frac{1+x}{x^{\widetilde{p}_\nu \cdot \lfloor \frac{\nu}{2} \rfloor}}\right)\left(\sum_{k=1}^{n} \frac{1}{x^{\widetilde{p}_k}}\right)^{\nu} \in \mathbb{k}(x) \end{aligned}$$

for all $n \in \mathbb{N}$. Taking the x-adic valuation of both sides, it yields

$$\begin{aligned} \left|f_n\left(\frac{p_n}{q_n}\right)\right|_x &= \max_{1\leq\nu\leq n} \left\{1, \left|\frac{1+x}{x^{\widetilde{p}_\nu \cdot \lfloor \frac{\nu}{2} \rfloor}}\right|_x \left|\sum_{k=1}^{n} \frac{1}{x^{\widetilde{p}_k}}\right|_x^{\nu}\right\} \\ &= \max_{1\leq\nu\leq n} \left\{1, e^{\widetilde{p}_\nu \cdot \lfloor \frac{\nu}{2} \rfloor + \widetilde{p}_\nu \cdot \nu}\right\} \\ &= e^{\widetilde{p}_n \cdot \lfloor \frac{n}{2} \rfloor + \widetilde{p}_n \cdot n} \end{aligned}$$

for all $n \in \mathbb{N}$ which is increasing. This guarantees that $f_n\left(\frac{p_n}{q_n}\right)$ $(n \geq 1)$ are all distinct. By the proof of Theorem 2, $f(\alpha)$ is transcendental as required.

Example 2. If the above defining sequences are chosen to be $\widetilde{p}_{n+1} > \widetilde{p}_n \cdot n^2$ and $w(n) = n$ for all $n \in \mathbb{N}$, it can be shown by a similar proof as in Example 1 that $f(\alpha)$ is transcendental.

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