Abstract
In this paper, we introduce a new generalization of Leonardo numbers, which are so-called Leonardo $p$-numbers. We investigate some basic properties of these numbers. We also define incomplete Leonardo $p$-numbers which generalize the incomplete Leonardo numbers.

1. Introduction
There are several generalizations of Fibonacci numbers, one among them is Fibonacci $p$-numbers which are defined by Stakhov and Rozin [14]. For any given integer $p > 0$, the Fibonacci $p$-numbers are defined by the recurrence relation

$$F_{p,n} = F_{p,n-1} + F_{p,n-p-1}, \quad n > p,$$

with initial values $F_{p,0} = 0, F_{p,k} = 1$ for $k = 1, 2, \ldots, p$. The Lucas $p$-numbers also satisfy the same recurrence relation

$$L_{p,n} = L_{p,n-1} + L_{p,n-p-1}, \quad n > p,$$

but begin with initial values $L_{p,0} = p + 1, L_{p,k} = 1$ for $k = 1, 2, \ldots, p$. It is clear to see that when $p = 1$, the Fibonacci $p$-sequence and the Lucas $p$-sequence reduce to the Fibonacci sequence $\{F_n\}_{n=0}^\infty$ and Lucas sequence $\{L_n\}_{n=0}^\infty$, respectively. A connection between Fibonacci $p$-numbers and Lucas $p$-numbers is

$$L_{p,n} = F_{p,n+1} + pF_{p,n-p}. \quad (1)$$

For details related to Fibonacci $p$-numbers and their generalizations, see [1, 10, 14, 15, 16].

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On the other hand, the Leonardo sequence \( \{L_n\}_{n=0}^{\infty} \) is defined by the following non-homogenous recurrence relation:

\[
L_n = L_{n-1} + L_{n-2} + 1, \quad n \geq 2,
\]

with initial values \( L_0 = L_1 = 1 \). In 1981, Dijkstra [7] used these numbers as an integral part of his sorting algorithm. Also the \( n \)th Leonardo number corresponds to the number of nodes in the Fibonacci tree of order \( n \). The properties of Leonardo numbers are studied in papers written by Catarino and Borges [5, 6], Alp and Kocer [2], and Shannon [13]. Several different versions of Leonardo-like sequences were previously studied by various researchers [3, 4, 8, 9, 17]. Some of these are listed in the On-Line Encyclopedia of Integer Sequences (for example, the sequence [A111314] in the On-Line Encyclopedia of Integer Sequences [12]). For the history of the Leonardo sequences, see also [A001595] in the On-Line Encyclopedia of Integer Sequences [12].

Recently, Kuhapatanakul and Chobsorn [11] have introduced a generalization of the Leonardo sequence \( \{L_{k,n}\}_{n=0}^{\infty} \) as:

\[
L_{k,n} = L_{k,n-1} + L_{k,n-2} + k, \quad n \geq 2,
\]

with initial values \( L_{k,0} = L_{k,1} = 1 \). It is clear to see that when \( k = 1 \), it reduces to the Leonardo sequence. When \( k = 2 \), this sequence reduces to the sequence [A111314] in the On-Line Encyclopedia of Integer Sequences [12].

In this article, we consider a new generalization of Leonardo sequence and investigate some basic properties of this sequence.

### 2. Main Results

We start by giving the definition of the Leonardo \( p \)-sequence.

**Definition 1.** For any given integer \( p > 0 \), the Leonardo \( p \)-sequence \( \{L_{p,n}\}_{n=0}^{\infty} \) is defined by the following non-homogenous relation:

\[
L_{p,n} = L_{p,n-1} + L_{p,n-p-1} + p, \quad n > p,
\]

with initial values \( L_{p,0} = L_{p,1} = \cdots = L_{p,p} = 1 \).

Some special cases for the Leonardo \( p \)-sequence can be given as follows. We note that for \( p > 1 \), the Leonardo \( p \)-sequences are new in OEIS.

- For \( p = 1 \), we get the classical Leonardo sequence.
- For \( p = 2 \), the first twenty Leonardo 2-numbers are

\[
1, 1, 1, 4, 7, 10, 16, 25, 37, 55, 82, 121, 178, 262, 385, 565, 829, 1216, 1783, 2614.
\]
For $p = 3$, the first twenty Leonardo 3-numbers are
1, 1, 5, 9, 13, 17, 25, 37, 53, 73, 101, 141, 197, 273, 377, 521, 721, 997, 1377.

For odd $p$, Leonardo $p$-numbers are odd for all $n$.

The non-homogenous recurrence relation of Leonardo $p$-numbers can be converted to the following homogenous recurrence relation:

$$L_{p,n} = L_{p,n-1} + L_{p,n-p} - L_{p,n-2p-1}, \quad n > 2p.$$  

Similar to the relation between the Leonardo numbers and the Fibonacci numbers, now we give a relation between the Leonardo $p$-numbers and the Fibonacci $p$-numbers.

**Theorem 1.** For $n \geq 0$, we have

$$L_{p,n} = (p + 1) F_{p,n+1} - p.$$  

**Proof.** The proof can be done by using induction on $n$. It is clear to check that the relation is true for $n = 0, 1, \ldots, p$. Suppose that the statement is true for all $n > p$. From the induction hypothesis, we get

$$L_{p,n+1} = L_{p,n} + L_{p,n-p} + p$$

$$= (p + 1) F_{p,n+1} - p + (p + 1) F_{p,n-p+1} - p + p$$

$$= (p + 1) (F_{p,n+1} + F_{p,n-p+1}) - p$$

$$= (p + 1) F_{p,n+2} - p,$$

which shows that the relation is true for $n + 1$.

**Remark 1.** If we take $p = 1$ in Theorem 1, we get the identity

$$L_{n} = 2F_{n+1} - 1,$$

which is given in [5, Equation 10].

The following result gives a link between the Leonardo $p$-numbers, the Fibonacci $p$-numbers, and the Lucas $p$-numbers.

**Proposition 1.** For $n \geq 0$, we have

$$L_{p,n} = L_{p,n+p+1} - F_{p,n+p+1} - p.$$  

**Proof.** By using the Equation (1), we get

$$L_{p,n} = (p + 1) F_{p,n+1} - p$$

$$= L_{p,n+p+1} - F_{p,n+p+2} + F_{p,n+1} - p$$

$$= L_{p,n+p+1} - F_{p,n+p+1} - p.$$  

$\square$
Now we state a summation formula for the Leonardo $p$-numbers.

**Proposition 2.** For $n \geq 0$, we have

$$\sum_{k=0}^{n} L_{p,k} = L_{p,n+p+1} - (n+1)p - 1.$$  

**Proof.** By using Theorem 1 and by using the sum formula of Fibonacci $p$-numbers in [16, Property 6], we get

$$\sum_{k=0}^{n} L_{p,k} = \sum_{k=0}^{n} ((p+1)F_{p,k+1} - p)$$

$$= (p+1)\sum_{k=0}^{n} F_{p,k+1} - \sum_{k=0}^{n} p$$

$$= (p+1)\sum_{k=1}^{n+1} F_{p,k} - (n+1)p$$

$$= (p+1)(F_{p,n+p+2} - F_{p,p} - F_{p,0}) - (n+1)p$$

$$= (p+1)(F_{p,n+p+2} - 1) - (n+1)p$$

$$= (p+1)F_{p,n+p+2} - p - 1 - (n+1)p$$

$$= L_{p,n+p+1} - (n+1)p - 1.$$  

$\square$

**Remark 2.** If we take $p = 1$ in Proposition 2, we get the identity

$$\sum_{k=0}^{n} L_k = L_{n+2} - (n+2),$$

which is given in [5, Proposition 3.1 (1)].

The following summation identity is true.

**Theorem 2.** For $n \geq 0$ and $m = 0, 1, \ldots, p-1$, we have

$$\sum_{k=0}^{n} L_{p,(p+1)k+m} = L_{p,(p+1)n+m+1} - pn.$$  

**Proof.** For $n \geq 0$, $k \geq 1$, and any nonnegative integer $m$, we have the following recurrence relation:

$$L_{p,k(p+1)+m} = L_{p,k(p+1)+m+1} - L_{p,(k-1)(p+1)+m+1} - p.$$
Then, we do the following computation:

\[
\sum_{k=0}^{n} \mathcal{L}_{p,k(p+1)+m} = \mathcal{L}_{p,m} + \mathcal{L}_{p,(p+1)+m} + \mathcal{L}_{p,2(p+1)+m} + \cdots + \mathcal{L}_{p,(p+1)n+m} \\
= \mathcal{L}_{p,m} + \left( \mathcal{L}_{p,(p+1)+m+1} - \mathcal{L}_{p,m+1} - p \right) \\
+ \left( \mathcal{L}_{p,2(p+1)+m+1} - \mathcal{L}_{p,(p+1)+m+1} - p \right) \\
+ \cdots + \left( \mathcal{L}_{p,n(p+1)+m+1} - \mathcal{L}_{p,(n-1)(p+1)+m+1} - p \right) \\
= \mathcal{L}_{p,n(p+1)+m+1} + \mathcal{L}_{p,m} - \mathcal{L}_{p,m+1} - np. \quad (2)
\]

We consider the expression \( \mathcal{L}_{p,m} - \mathcal{L}_{p,m+1} \) next.

For \( m = 0, 1, \ldots, p - 1 \),

\[ \mathcal{L}_{p,m} - \mathcal{L}_{p,m+1} = 1 - 1 = 0. \quad (3) \]

**Corollary 1.** For \( n \geq 0 \), we have

\[ \sum_{k=0}^{n} \mathcal{L}_{p,(p+1)k+p} = \mathcal{L}_{p,(p+1)(n+1)} - p(n+1) - 1. \]

**Proof.** For \( m = p \) in Theorem 2, we have

\[ \mathcal{L}_{p,m} - \mathcal{L}_{p,m+1} = 1 - (p + 2) = -p - 1. \quad (4) \]

We get the desired result by substituting (3) and (4) into (2). \( \square \)

**Remark 3.** If we take \( p = 1 \) in Theorem 2 and Corollary 1, we get the identities

\[ \sum_{k=0}^{n} \mathcal{L}_{p,2k} = \mathcal{L}_{p,2n+1} - n, \]
\[ \sum_{k=0}^{n} \mathcal{L}_{p,2k+1} = \mathcal{L}_{p,2n+2} - (n + 2), \]

which are given in [5, Proposition 3.1 (2)-(3)].

To obtain the Binet formula of the Leonardo \( p \)-sequence, we use the Binet formula of the Fibonacci \( p \)-sequence (see [1])

\[ F_{p,n} = \sum_{k=1}^{p+1} \frac{\alpha_k^n}{(p+1)\alpha_k - p}, \]

where \( \alpha_k \) are the distincts roots of the polynomial \( x^{p+1} - x^p - 1 \).
Theorem 3. The Binet formula of the Leonardo \( p \)-sequence is

\[ L_{p,n} = (p + 1) \left( \sum_{k=1}^{p+1} \frac{\alpha_k^{n+1}}{(p+1)\alpha_k - p} \right) - p. \]

Proof. From Theorem 1 and the Binet formula of Fibonacci \( p \)-sequence, we get the desired result. \( \square \)

In the following theorem, we give the Honsberger formula for the Leonardo \( p \)-numbers. We need the following identity which is the Honsberger-like identity for Fibonacci \( p \)-numbers [16]:

\[ F_{p,m+n} = F_{p,m}F_{p,n+1} + \sum_{j=1}^{p} F_{p,m-j}F_{p,n+p-j} \]

where \( m \) and \( n \) are nonnegative integers such that \( m, n \geq p \).

Theorem 4. For nonnegative integers \( m, n \geq p \), we have

\[ \mathcal{L}_{p,m} \mathcal{L}_{p,n+1} + \sum_{j=1}^{p} \mathcal{L}_{p,m-j} \mathcal{L}_{p,n-p+j} \]

\[ = (p + 1) \left( \mathcal{L}_{p,m+n+1} - p \sum_{j=0}^{p} (F_{p,m-j+1} + F_{p,n-p+j+2}) + p(p+1) \right). \]

Proof. By using Theorem 1 and the Honsberger identity for Fibonacci \( p \)-numbers, we get

\[ \mathcal{L}_{p,m} \mathcal{L}_{p,n+1} + \sum_{j=1}^{p} \mathcal{L}_{p,m-j} \mathcal{L}_{p,n-p+j} \]

\[ = (p + 1)^2 F_{p,m+1}F_{p,n+2} - p(p + 1) (F_{p,m+1} + F_{p,n+2}) + p^2 \]

\[ + (p + 1)^2 \sum_{j=1}^{p} F_{p,m-j+1}F_{p,n-j-p+1} - p(p + 1) \sum_{j=1}^{p} (F_{p,m-j+1} + F_{p,n+j-p+1}) + p \sum_{j=1}^{p} p^2 \]

\[ = (p + 1)^2 \left( F_{p,m+n+1} + \sum_{j=1}^{p} F_{p,m-j+1}F_{p,n-p+j+1} \right) \]

\[ - p(p + 1) \left( F_{p,m+1} + F_{p,n+2} + \sum_{j=1}^{p} (F_{p,m-j+1} + F_{p,n-p+j+1}) \right) + p^2(p + 1) \]
\[
\begin{align*}
\text{Remark 4.} & \text{ For } p = 1 \text{ in Theorem 4, we get the following identity for the Leonardo numbers:} \\
& L_{m+1} L_{n+1} + L_m L_{n+1} = 2L_{m+n+1} - L_{m+1} - L_{n+2} + 2.
\end{align*}
\]

3. Incomplete Leonardo \(p\)-numbers

In this section, we define incomplete Leonardo \(p\)-numbers and state some properties of these numbers. For this purpose, first we consider Theorem 1, then we need to use the definition of incomplete Fibonacci \(p\)-numbers [15]:

\[
F_{p,n}(k) = \sum_{i=0}^{k} \binom{n-pi - 1}{i}, \quad 0 \leq k \leq \left\lfloor \frac{n-1}{p+1} \right\rfloor.
\]

**Definition 2.** Let \(n\) be a positive integer and \(k\) be an integer. For \(0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor\), the incomplete Leonardo \(p\)-numbers are defined as

\[
L_{p,n}(k) = (p+1) \sum_{i=0}^{k} (n-pi) - p.
\]

It is clear to see the following special cases:

- \(L_{p,n}(0) = 1\),
- \(L_{p,n}(1) = (p + 1)(n - p) + 1\),
- \(L_{p,n}\left(\left\lfloor \frac{n}{p+1} \right\rfloor\right) = L_{p,n}\).

**Proposition 3.** For \(0 \leq k \leq \frac{n-p-2}{p+1}\), the non-linear recurrence relation of the incomplete Leonardo \(p\)-numbers \(L_{p,n}(k)\) is

\[
L_{p,n}(k + 1) = L_{p,n-1}(k + 1) + L_{p,n-1}(k) + p.
\]
Proof. By using the definition of incomplete Leonardo \( p \)-numbers, we have

\[
\mathcal{L}_{p,n-1}(k+1) + \mathcal{L}_{p,n-p-1}(k)+p
\]
\[
= (p+1) \sum_{i=0}^{k+1} \binom{n-pi-1}{i} - p + (p+1) \sum_{i=0}^{k} \binom{n-p(i+1)-1}{i} - p + p
\]
\[
= (p+1) \sum_{i=0}^{k+1} \binom{n-pi-1}{i} - p + (p+1) \sum_{i=1}^{k+1} \binom{n-pi-1}{i-1}
\]
\[
= (p+1) \sum_{i=0}^{k+1} \left( \binom{n-pi-1}{i} + \binom{n-pi-1}{i-1} \right) - p
\]
\[
= (p+1) \sum_{i=0}^{k+1} \binom{n-pi}{i} - p = \mathcal{L}_{p,n}(k+1).
\]

Proposition 3 can be transformed into the following non-homogeneous recurrence relation:

\[
\mathcal{L}_{p,n}(k) = \mathcal{L}_{p,n-1}(k) + \mathcal{L}_{p,n-p-1}(k-1)+p
\]
\[
= \mathcal{L}_{p,n-1}(k) + \mathcal{L}_{p,n-p-1}(k) + p + (\mathcal{L}_{p,n-p-1}(k-1) - \mathcal{L}_{p,n-p-1}(k))
\]
\[
= \mathcal{L}_{p,n-1}(k) + \mathcal{L}_{p,n-p-1}(k) + p - (p+1) \binom{n-p(k+1)-1}{k}.
\]

(5)

Proposition 4. For \( 0 \leq k \leq \frac{n-p-1}{p+1} \), we have

\[
\sum_{i=0}^{s} \binom{s}{i} \mathcal{L}_{p,n+pi}(k+i) + (2^s - 1) p = \mathcal{L}_{p,n+(p+1)s}(k+s).
\]

Proof. The proof will be done by using induction on \( s \). From Proposition 3, the relation is true for \( s = 0 \) and \( s = 1 \). Assume that the relation is true for all \( j < s+1 \). Now we show that it is true for \( s+1 \).

\[
\sum_{i=0}^{s+1} \binom{s+1}{i} \mathcal{L}_{p,n+pi}(k+i)
\]
\[
= \sum_{i=0}^{s+1} \left[ \binom{s}{i} + \binom{s}{i-1} \right] \mathcal{L}_{p,n+pi}(k+i)
\]
\[
\sum_{i=0}^{s} \binom{s}{i} L_{p,n+p+i}(k + i) + \sum_{i=0}^{s+1} \binom{s+1}{i-1} L_{p,n+p+i}(k + i)
\]

\[
= \sum_{i=0}^{s} \binom{s}{i} L_{p,n+p+i}(k + i) + \sum_{i=0}^{s} \binom{s}{i} L_{p,n+p+i+1}(k + i + 1)
\]

\[
= L_{p,n+(p+1)s}(k + s) - (2^s - 1)p + L_{p,n+(p+1)s+p}(k + s + 1) - (2^s - 1)p
\]

\[
= L_{p,n+(p+1)(s+1)}(k + s + 1) - (2^{s+1} - 1)p.
\]

Note that when \( p = 1 \) in Proposition 4, we obtain
\[
\sum_{i=0}^{s} \binom{s}{i} L_{n+i}(k + i) + 2^s - 1 = L_{n+2s}(k + s),
\]
which is given in [11, Equation (3.4)].

**Proposition 5.** For \( n \geq (p + 1)(k + 1) \) we have
\[
\sum_{i=0}^{s-1} L_{p,n-p+i}(k) + sp = L_{p,n+s}(k + 1) - L_{p,n}(k + 1).
\]

**Proof.** We prove it by using induction on \( s \). It is clear to see that the equality holds for \( s = 1 \). Suppose that it is true for all \( i < s \). Now we prove it for \( s \). From Proposition 3, we have
\[
L_{p,n+s+1}(k + 1) - L_{p,n}(k + 1)
\]

\[
= (L_{p,n+s}(k + 1) + L_{p,n+s-p}(k) + p) - L_{p,n}(k + 1)
\]

\[
= (L_{p,n+s}(k + 1) - L_{p,n}(k + 1)) + L_{p,n+s-p}(k) + p
\]

\[
= \sum_{i=0}^{s-1} L_{p,n-p+i}(k) + sp + L_{p,n+s-p}(k) + p
\]

\[
= \sum_{i=0}^{s} L_{p,n-p+i}(k) + p(s + 1),
\]
which completes the proof. \( \square \)

**Remark 5.** If we take \( p = 1 \) in Proposition 3, Proposition 4, Equation (5), and Proposition 5, then we get the results in [6, Proposition 1-4], respectively.
Finally, we note that the generating function of incomplete Leonardo $p$-numbers can be obtained by using the generating function of incomplete Fibonacci $p$-numbers which is given in [15, Theorem 16] as:

$$R_k^p(t) := \sum_{n=0}^{\infty} F_{p,n}(k) t^n = \frac{t^{k+1}}{(1 - t - t^p + 1)} \times$$

$$\left[ F_{p,k(p+1)+1} + \sum_{i=1}^{p} \left( F_{p,k(p+1)+i} - F_{p,k(p+1)+i} t^i \right) \right]$$

$$= \frac{t^{p+1}}{(1 - t)^{k+1}} \times$$

Since $L_{p,n}(k) = (p + 1) F_{p,n+1}(k) - p$, we get the desired result.

References


[8] G.B. Djordjević, Some properties of the sequences $C_{n,3} = C_{n-1,3} + C_{n-2,3} + r$, *Fibonacci Quart.* 43(3) (2005), 202-207.


