ON LEONARDO p-NUMBERS

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Abstract

In this paper, we introduce a new generalization of Leonardo numbers, which are so-called Leonardo p-numbers. We investigate some basic properties of these numbers. We also define incomplete Leonardo p-numbers which generalize the incomplete Leonardo numbers.

1. Introduction

There are several generalizations of Fibonacci numbers, one among them is Fibonacci p-numbers which are defined by Stakhov and Rozin [14]. For any given integer $p > 0$, the *Fibonacci p-numbers* are defined by the recurrence relation

$$F_{p,n} = F_{p,n-1} + F_{p,n-p-1}, \ n > p,$$

with initial values $F_{p,0} = 0$, $F_{p,k} = 1$ for $k = 1, 2, \ldots, p$. The *Lucas p-numbers* also satisfy the same recurrence relation

$$L_{p,n} = L_{p,n-1} + L_{p,n-p-1}, \ n > p,$$

but begin with initial values $L_{p,0} = p + 1$, $L_{p,k} = 1$ for $k = 1, 2, \ldots, p$. It is clear to see that when $p = 1$, the Fibonacci p-sequence and the Lucas p-sequence reduce to the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ and Lucas sequence $\{L_n\}_{n=0}^{\infty}$, respectively. A connection between Fibonacci p-numbers and Lucas p-numbers is

$$L_{p,n} = F_{p,n+1} + pF_{p,n-p}. \tag{1}$$

For details related to Fibonacci p-numbers and their generalizations, see [1, 10, 14, 15, 16].

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On the other hand, the *Leonardo sequence* $\{\mathcal{L}_n\}_{n=0}^{\infty}$ is defined by the following non-homogenous recurrence relation:

$$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2} + 1, \ n \geq 2,$$

with initial values $\mathcal{L}_0 = \mathcal{L}_1 = 1$. In 1981, Dijkstra [7] used these numbers as an integral part of his sorting algorithm. Also the nth Leonardo number corresponds to the number of nodes in the Fibonacci tree of order n. The properties of Leonardo numbers are studied in papers written by Catarino and Borges [5, 6], Alp and Kocer [2], and Shannon [13]. Several different versions of Leonardo-like sequences were previously studied by various researchers [3, 4, 8, 9, 17]. Some of these are listed in the On-Line Encyclopedia of Integer Sequences (for example, the sequence [A111314] in the On-Line Encyclopedia of Integer Sequences [12]). For the history of the Leonardo sequences, see also [A001595] in the On-Line Encyclopedia of Integer Sequences [12].

Recently, Kuhapatanakul and Chobsorn [11] have introduced a generalization of the Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n=0}^{\infty}$ as:

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k, \ n \geq 2,$$

with initial values $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$. It is clear to see that when $k = 1$, it reduces to the Leonardo sequence. When $k = 2$, this sequence reduces to the sequence [A111314] in the On-Line Encyclopedia of Integer Sequences [12].

In this article, we consider a new generalization of Leonardo sequence and investigate some basic properties of this sequence.

2. Main Results

We start by giving the definition of the Leonardo p-sequence.

Definition 1. *For any given integer $p > 0$, the Leonardo p-sequence $\{\mathcal{L}_{p,n}\}_{n=0}^{\infty}$ is defined by the following non-homogenous relation:*

$$\mathcal{L}_{p,n} = \mathcal{L}_{p,n-1} + \mathcal{L}_{p,n-p-1} + p, \ n > p,$$

with initial values $\mathcal{L}_{p,0} = \mathcal{L}_{p,1} = \cdots = \mathcal{L}_{p,p} = 1$.

Some special cases for the Leonardo p-sequence can be given as follows. We note that for $p > 1$, the Leonardo p-sequences are new in OEIS.

- For $p = 1$, we get the classical Leonardo sequence.
- For $p = 2$, the first twenty Leonardo 2-numbers are

$$1, 1, 1, 4, 7, 10, 16, 25, 37, 55, 82, 121, 178, 262, 385, 565, 829, 1216, 1783, 2614.$$

- For $p = 3$, the first twenty Leonardo 3-numbers are

$$1, 1, 1, 5, 9, 13, 17, 25, 37, 53, 73, 101, 141, 197, 273, 377, 521, 721, 997, 1377.$$

- For odd p, Leonardo p-numbers are odd for all n.

The non-homogenous recurrence relation of Leonardo p-numbers can be converted to the following homogenous recurrence relation:

$$\mathcal{L}_{p,n} = \mathcal{L}_{p,n-1} + \mathcal{L}_{p,n-p} - \mathcal{L}_{p,n-2p-1},\ n > 2p.$$

Similar to the relation between the Leonardo numbers and the Fibonacci numbers, now we give a relation between the Leonardo p-numbers and the Fibonacci p-numbers.

Theorem 1. *For $n \geq 0$, we have*

$$\mathcal{L}_{p,n} = (p+1)\,F_{p,n+1} - p.$$

Proof. The proof can be done by using induction on n. It is clear to check that the relation is true for $n = 0, 1, \ldots, p$. Suppose that the statement is true for all $n > p$. From the induction hypothesis, we get

$$\begin{aligned}
\mathcal{L}_{p,n+1} &= \mathcal{L}_{p,n} + \mathcal{L}_{p,n-p} + p \\
&= (p+1)\,F_{p,n+1} - p + (p+1)\,F_{p,n-p+1} - p + p \\
&= (p+1)\,(F_{p,n+1} + F_{p,n-p+1}) - p \\
&= (p+1)\,F_{p,n+2} - p,
\end{aligned}$$

which shows that the relation is true for $n + 1$. □

Remark 1. If we take $p = 1$ in Theorem 1, we get the identity

$$\mathcal{L}_n = 2F_{n+1} - 1,$$

which is given in [5, Equation 10].

The following result gives a link between the Leonardo p-numbers, the Fibonacci p-numbers, and the Lucas p-numbers.

Proposition 1. *For $n \geq 0$, we have*

$$\mathcal{L}_{p,n} = L_{p,n+p+1} - F_{p,n+p+1} - p.$$

Proof. By using the Equation (1), we get

$$\begin{aligned}
\mathcal{L}_{p,n} &= (p+1)\,F_{p,n+1} - p \\
&= L_{p,n+p+1} - F_{p,n+p+2} + F_{p,n+1} - p \\
&= L_{p,n+p+1} - F_{p,n+p+1} - p.
\end{aligned}$$

□

Now we state a summation formula for the Leonardo p-numbers.

Proposition 2. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n} \mathcal{L}_{p,k} = \mathcal{L}_{p,n+p+1} - (n+1)\,p - 1.$$

Proof. By using Theorem 1 and by using the sum formula of Fibonacci p-numbers in [16, Property 6], we get

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{L}_{p,k} &= \sum_{k=0}^{n} \left((p+1)\,F_{p,k+1} - p\right) \\
&= (p+1)\sum_{k=0}^{n} F_{p,k+1} - \sum_{k=0}^{n} p \\
&= (p+1)\sum_{k=1}^{n+1} F_{p,k} - (n+1)\,p \\
&= (p+1)\left(F_{p,n+p+2} - F_{p,p} - F_{p,0}\right) - (n+1)\,p \\
&= (p+1)\left(F_{p,n+p+2} - 1\right) - (n+1)\,p \\
&= (p+1)\,F_{p,n+p+2} - p - 1 - (n+1)\,p \\
&= \mathcal{L}_{p,n+p+1} - (n+1)\,p - 1.
\end{aligned}$$

□

Remark 2. If we take $p = 1$ in Proposition 2, we get the identity

$$\sum_{k=0}^{n} \mathcal{L}_{k} = \mathcal{L}_{n+2} - (n+2)\,,$$

which is given in [5, Proposition 3.1 (1)].

The following summation identity is true.

Theorem 2. *For $n \geq 0$ and $m = 0, 1, \ldots, p-1$, we have*

$$\sum_{k=0}^{n} \mathcal{L}_{p,(p+1)k+m} = \mathcal{L}_{p,(p+1)n+m+1} - pn.$$

Proof. For $n \geq 0$, $k \geq 1$, and any nonnegative integer m, we have the following recurrence relation:

$$\mathcal{L}_{p,k(p+1)+m} = \mathcal{L}_{p,k(p+1)+m+1} - \mathcal{L}_{p,(k-1)(p+1)+m+1} - p.$$

Then, we do the following computation:

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{L}_{p,k(p+1)+m} &= \mathcal{L}_{p,m}+\mathcal{L}_{p,(p+1)+m}+\mathcal{L}_{p,2(p+1)+m}+\cdots+\mathcal{L}_{p,(p+1)n+m}\\ &= \mathcal{L}_{p,m}+\left(\mathcal{L}_{p,(p+1)+m+1}-\mathcal{L}_{p,m+1}-p\right)\\ &\quad+\left(\mathcal{L}_{p,2(p+1)+m+1}-\mathcal{L}_{p,(p+1)+m+1}-p\right)\\ &\quad+\ldots+\left(\mathcal{L}_{p,n(p+1)+m+1}-\mathcal{L}_{p,(n-1)(p+1)+m+1}-p\right)\\ &= \mathcal{L}_{p,n(p+1)+m+1}+\mathcal{L}_{p,m}-\mathcal{L}_{p,m+1}-np. \qquad (2)\end{aligned}$$

We consider the expression $\mathcal{L}_{p,m}-\mathcal{L}_{p,m+1}$ next.

For $m=0,1,\ldots,p-1$,

$$\mathcal{L}_{p,m}-\mathcal{L}_{p,m+1}=1-1=0. \qquad (3)$$

□

Corollary 1. *For $n\geq 0$, we have*

$$\sum_{k=0}^{n}\mathcal{L}_{p,(p+1)k+p}=\mathcal{L}_{p,(p+1)(n+1)}-p(n+1)-1.$$

Proof. For $m=p$ in Theorem 2, we have

$$\mathcal{L}_{p,m}-\mathcal{L}_{p,m+1}=1-(p+2)=-p-1. \qquad (4)$$

We get the desired result by substituting (3) and (4) into (2). □

Remark 3. If we take $p=1$ in Theorem 2 and Corollary 1, we get the identities

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{L}_{p,2k} &= \mathcal{L}_{p,2n+1}-n,\\ \sum_{k=0}^{n}\mathcal{L}_{p,2k+1} &= \mathcal{L}_{p,2n+2}-(n+2),\end{aligned}$$

which are given in [5, Proposition 3.1 (2)-(3)].

To obtain the Binet formula of the Leonardo p-sequence, we use the Binet formula of the Fibonacci p-sequence (see [1])

$$F_{p,n}=\sum_{k=1}^{p+1}\frac{\alpha_k^n}{(p+1)\alpha_k-p},$$

where α_k are the distincts roots of the polynomial $x^{p+1}-x^p-1$.

Theorem 3. *The Binet formula of the Leonardo p-sequence is*

$$\mathcal{L}_{p,n} = (p+1)\left(\sum_{k=1}^{p+1} \frac{\alpha_k^{n+1}}{(p+1)\alpha_k - p}\right) - p.$$

Proof. From Theorem 1 and the Binet formula of Fibonacci p-sequence, we get the desired result. □

In the following theorem, we give the Honsberger formula for the Leonardo p-numbers. We need the following identity which is the Honsberger-like identity for Fibonacci p-numbers [16]:

$$F_{p,m+n} = F_{p,m}F_{p,n+1} + \sum_{j=1}^{p} F_{p,m-j}F_{p,n-p+j}$$

where m and n are nonnegative integers such that $m, n \geq p$.

Theorem 4. *For nonnegative integers $m, n \geq p$, we have*

$$\begin{aligned} & \mathcal{L}_{p,m}\mathcal{L}_{p,n+1} + \sum_{j=1}^{p} \mathcal{L}_{p,m-j}\mathcal{L}_{p,n-p+j} \\ = \; & (p+1)\left(\mathcal{L}_{p,m+n+1} - p\sum_{j=0}^{p}\left(F_{p,m-j+1} + F_{p,n-p+j+2}\right) + p\,(p+1)\right). \end{aligned}$$

Proof. By using Theorem 1 and the Honsberger identity for Fibonacci p-numbers, we get

$$\begin{aligned} & \mathcal{L}_{p,m}\mathcal{L}_{p,n+1} + \sum_{j=1}^{p} \mathcal{L}_{p,m-j}\mathcal{L}_{p,n-p+j} \\ & = (p+1)^2 F_{p,m+1}F_{p,n+2} - p\,(p+1)\left(F_{p,m+1} + F_{p,n+2}\right) + p^2 \\ & + (p+1)^2 \sum_{j=1}^{p} F_{p,m-j+1}F_{p,n+j-p+1} - p\,(p+1)\sum_{j=1}^{p}\left(F_{p,m-j+1} + F_{p,n+j-p+1}\right) + \sum_{j=1}^{p} p^2 \\ & = (p+1)^2\left(F_{p,m+1}F_{p,n+2} + \sum_{j=1}^{p} F_{p,m-j+1}F_{p,n-p+j+1}\right) \\ & \quad - p\,(p+1)\left(F_{p,m+1} + F_{p,n+2} + \sum_{j=1}^{p}\left(F_{p,m-j+1} + F_{p,n-p+j+1}\right)\right) + p^2\,(p+1) \end{aligned}$$

$$
\begin{aligned}
&= (p+1)^2 F_{p,m+n+2} - p(p+1)\left(\sum_{j=0}^{p} F_{p,m-j+1} + \sum_{j=1}^{p+1} F_{p,n-p+j+1}\right) + p^2(p+1) \\
&= (p+1)\left((p+1)F_{p,m+n+2} - p + p - p\sum_{j=0}^{p}\left(F_{p,m-j+1} + F_{p,n-p+j+2}\right) + p^2\right) \\
&= (p+1)\left(\mathcal{L}_{p,m+n+1} - p\sum_{j=0}^{p}\left(F_{p,m-j+1} + F_{p,n-p+j+2}\right) + p(p+1)\right).
\end{aligned}
$$

□

Remark 4. For $p = 1$ in Theorem 4, we get the following identity for the Leonardo numbers:

$$\mathcal{L}_m\mathcal{L}_{n+1} + \mathcal{L}_{m-1}\mathcal{L}_n = 2\mathcal{L}_{m+n+1} - \mathcal{L}_{m+1} - \mathcal{L}_{n+2} + 2.$$

3. Incomplete Leonardo p-numbers

In this section, we define incomplete Leonardo p-numbers and state some properties of these numbers. For this purpose, first we consider the Theorem 1, then we need to use the definition of incomplete Fibonacci p-numbers [15]:

$$F_{p,n}(k) = \sum_{i=0}^{k} \binom{n-pi-1}{i}, \ 0 \le k \le \left\lfloor \frac{n-1}{p+1} \right\rfloor.$$

Definition 2. *Let n be a positive integer and k be an integer. For $0 \le k \le \left\lfloor \frac{n}{p+1} \right\rfloor$, the incomplete Leonardo p-numbers are defined as*

$$\mathcal{L}_{p,n}(k) = (p+1)\sum_{i=0}^{k} \binom{n-pi}{i} - p.$$

It is clear to see the following special cases:

- $\mathcal{L}_{p,n}(0) = 1$,
- $\mathcal{L}_{p,n}(1) = (p+1)(n-p) + 1$,
- $\mathcal{L}_{p,n}\left(\left\lfloor \frac{n}{p+1} \right\rfloor\right) = \mathcal{L}_{p,n}$.

Proposition 3. *For $0 \le k \le \frac{n-p-2}{p+1}$, the non-linear recurrence relation of the incomplete Leonardo p-numbers $\mathcal{L}_{p,n}(k)$ is*

$$\mathcal{L}_{p,n}(k+1) = \mathcal{L}_{p,n-1}(k+1) + \mathcal{L}_{p,n-p-1}(k) + p.$$

Proof. By using the definition of incomplete Leonardo p-numbers, we have

$$\begin{aligned}
&\mathcal{L}_{p,n-1}(k+1)+\mathcal{L}_{p,n-p-1}(k)+p \\
&\quad= (p+1)\sum_{i=0}^{k+1}\binom{n-pi-1}{i}-p+(p+1)\sum_{i=0}^{k}\binom{n-p(i+1)-1}{i}-p+p \\
&\quad= (p+1)\sum_{i=0}^{k+1}\binom{n-pi-1}{i}-p+(p+1)\sum_{i=1}^{k+1}\binom{n-pi-1}{i-1} \\
&\quad= (p+1)\sum_{i=0}^{k+1}\left(\binom{n-pi-1}{i}+\binom{n-pi-1}{i-1}\right)-p \\
&\quad= (p+1)\sum_{i=0}^{k+1}\binom{n-pi}{i}-p=\mathcal{L}_{p,n}(k+1).
\end{aligned}$$

□

Proposition 3 can be transformed into the following non-homogeneous recurrence relation:

$$\begin{aligned}
\mathcal{L}_{p,n}(k) &= \mathcal{L}_{p,n-1}(k)+\mathcal{L}_{p,n-p-1}(k-1)+p \\
&= \mathcal{L}_{p,n-1}(k)+\mathcal{L}_{p,n-p-1}(k)+p+\left(\mathcal{L}_{p,n-p-1}(k-1)-\mathcal{L}_{p,n-p-1}(k)\right) \\
&= \mathcal{L}_{p,n-1}(k)+\mathcal{L}_{p,n-p-1}(k)+p-(p+1)\binom{n-p(k+1)-1}{k}. \qquad (5)
\end{aligned}$$

Proposition 4. *For* $0 \leq k \leq \frac{n-p-s}{p+1}$, *we have*

$$\sum_{i=0}^{s}\binom{s}{i}\mathcal{L}_{p,n+pi}(k+i)+(2^s-1)p=\mathcal{L}_{p,n+(p+1)s}(k+s).$$

Proof. The proof will be done by using induction on s. From Proposition 3, the relation is true for $s=0$ and $s=1$. Assume that the relation is true for all $j<s+1$. Now we show that it is true for $s+1$.

$$\begin{aligned}
&\sum_{i=0}^{s+1}\binom{s+1}{i}\mathcal{L}_{p,n+pi}(k+i) \\
&\quad= \sum_{i=0}^{s+1}\left[\binom{s}{i}+\binom{s}{i-1}\right]\mathcal{L}_{p,n+pi}(k+i)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{s} \binom{s}{i} \mathcal{L}_{p,n+pi}(k+i) + \sum_{i=0}^{s+1} \binom{s}{i-1} \mathcal{L}_{p,n+pi}(k+i) \\
&= \sum_{i=0}^{s} \binom{s}{i} \mathcal{L}_{p,n+pi}(k+i) + \sum_{i=0}^{s} \binom{s}{i} \mathcal{L}_{p,n+p(i+1)}(k+i+1) \\
&= \mathcal{L}_{p,n+(p+1)s}(k+s) - (2^s-1)p + \mathcal{L}_{p,n+(p+1)s+p}(k+s+1) - (2^s-1)p \\
&= \mathcal{L}_{p,n+(p+1)(s+1)}(k+s+1) - \left(2^{s+1}-1\right)p.
\end{aligned}
$$

□

Note that when $p = 1$ in Proposition 4, we obtain

$$
\sum_{i=0}^{s} \binom{s}{i} \mathcal{L}_{n+i}(k+i) + 2^s - 1 = \mathcal{L}_{n+2s}(k+s),
$$

which is given in [11, Equation (3.4)].

Proposition 5. *For $n \geq (p+1)(k+1)$ we have*

$$
\sum_{i=0}^{s-1} \mathcal{L}_{p,n-p+i}(k) + sp = \mathcal{L}_{p,n+s}(k+1) - \mathcal{L}_{p,n}(k+1).
$$

Proof. We prove it by using induction on s. It is clear to see that the equality holds for $s = 1$. Suppose that it is true for all $i < s$. Now we prove it for s. From Proposition 3, we have

$$
\begin{aligned}
&\mathcal{L}_{p,n+s+1}(k+1) - \mathcal{L}_{p,n}(k+1) \\
&\quad = (\mathcal{L}_{p,n+s}(k+1) + \mathcal{L}_{p,n+s-p}(k) + p) - \mathcal{L}_{p,n}(k+1) \\
&\quad = (\mathcal{L}_{p,n+s}(k+1) - \mathcal{L}_{p,n}(k+1)) + \mathcal{L}_{p,n+s-p}(k) + p \\
&\quad = \sum_{i=0}^{s-1} \mathcal{L}_{p,n-p+i}(k) + sp + \mathcal{L}_{p,n+s-p}(k) + p \\
&\quad = \sum_{i=0}^{s} \mathcal{L}_{p,n-p+i}(k) + p(s+1),
\end{aligned}
$$

which completes the proof. □

Remark 5. If we take $p = 1$ in Proposition 3, Proposition 4, Equation (5), and Proposition 5, then we get the results in [6, Proposition 1-4], respectively.

Finally, we note that the generating function of incomplete Leonardo p-numbers can be obtained by using the generating function of incomplete Fibonacci p-numbers which is given in [15, Theorem 16] as:

$$R_p^k(t) := \sum_{n=0}^{\infty} F_{p,n}(k)\, t^n = \frac{t^{k(p+1)+1}}{(1-t-t^{p+1})} \times \left[F_{p,k(p+1)+1} + \sum_{i=1}^{p}\left(F_{p,k(p+1)+i+1} - F_{p,k(p+1)+i}\right)t^i - \frac{t^{p+1}}{(1-t)^{k+1}}\right].$$

Since $\mathcal{L}_{p,n}(k) = (p+1)\, F_{p,n+1}(k) - p$, we get the desired result.

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