LOCAL DIFFERENCES DETERMINED BY CONVEX SETS

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Abstract

This paper introduces a new problem concerning additive properties of convex sets. Let $S = \{s_1 < \cdots < s_n\}$ be a set of real numbers and let $D_i(S) = \{s_x - s_y : 1 \leq x - y \leq i\}$. We expect that $D_i(S)$ is large, with respect to the size of S and the parameter i, for any convex set S. We give a construction to show that $D_3(S)$ can be as small as $n + 2$, and show that this is the smallest possible size. On the other hand, we use an elementary argument to prove a non-trivial lower bound for $D_4(S)$, namely $|D_4(S)| \geq \frac{5}{4}n - 1$. For sufficiently large values of i, we are able to prove a non-trivial bound that grows with i using incidence geometry.

1. Introduction

Let $S \subseteq \mathbb{R}$ be a finite set and write $S = \{s_1, s_2, \ldots, s_n\}$ such that $s_i < s_{i+1}$ for all $i \in [n-1]$. S is said to be a *convex set* if $s_{i+1} - s_i < s_{i+2} - s_{i+1}$ for all $i \in [n-2]$.

In the spirit of the sum-product problem, Erdős raised the question of whether $A + A$ and $A - A$ are guaranteed to be large for any convex set. Progress has been made towards this question via a combination of methods in incidence geometry (see for instance [2]) and elementary methods (see [3], [8] and [4]), but the question of determining the best possible lower bounds remains open. The current state-of-the-art results state that, for some absolute constant $c > 0$, the bounds

$$|S - S| \gg \frac{n^{8/5}}{(\log n)^c}, \quad |S + S| \gg \frac{n^{30/19}}{(\log n)^c} \tag{1}$$

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hold for any convex set $A \subset \mathbb{R}$, see [10] and [7] respectively. The notation $\gg$ is used to absorb a multiplicative constant. That is $X \gg Y$ denotes that there exists an absolute constant $C > 0$ such that $X \geq CY$.

In this paper, we introduce a new *local* variant of this problem, where only differences between close elements of A are considered. We define

$$D_i(S) := \{s_x - s_y : 1 \leq x - y \leq i\}.$$

Moreover, for any set $\mathcal{I} \subset [n-1]$, define

$$D_{\mathcal{I}}(S) := \{s_x - s_y \mid x - y \in \mathcal{I}\}.$$

We expect that $D_i(S)$ is large (in terms of i) and that $D_{\mathcal{I}}(S)$ is large (in terms of $|\mathcal{I}|$) for any convex set $S \subset \mathbb{R}$. We begin with some trivial observations.

- For any convex set S, $|D_1(S)| = |S| - 1$. Indeed, it is an immediate consequence of the definition of a convex set that S has *distinct consecutive differences*. Additive properties of such sets were considered in [8] and [9].

- For any convex set S, we have $|D_2(S)| \geq |S|$. This can be seen by observing that $s_n - s_{n-2} \in D_{\{2\}}(S) \setminus D_{\{1\}}(S)$, and so

$$|D_2(S)| = |D_{\{1\}}(S)| + |D_{\{2\}}(S) \setminus D_{\{1\}}(S)| \geq n - 1 + 1 = n. \tag{2}$$

  On the other hand, if we consider the set

$$S = \{f_{i+3} : i \in [n]\}, \tag{3}$$

  where f_i is the i-th Fibonacci number, we see that $|D_2(S)| = n$, and so the lower bound (2) is optimal.

- Setting $i = n-1$, we see that the set $D_i(S)$ is equal to the set of positive elements of the difference set $S - S$. Therefore, the question of obtaining lower bounds for $D_{n-1}(S)$ is equivalent to that of lower bounding the size of $S - S$.

The main results of this paper concern the minimum possible sizes of $D_3(S)$ and $D_4(S)$ when S is convex. Let $\mathcal{S}(n)$ be the set of all convex sets of size n. Define

$$D_i(n) := \min\{|D_i(S)| : S \in \mathcal{S}(n)\}.$$

The discussion above shows that $D_1(n) = n - 1$ and $D_2(n) = n$. For $D_3(n)$ we will provide a constructive proof of the following result.

Theorem 1.1. *For any integer $n \geq 5$,*

$$D_3(n) = n + 2.$$

However, there is an interesting change of behaviour that occurs when i goes from 3 to 4, and $D_4(n)$ is not of the form $n+O(1)$. In this paper we will prove the following result.

Theorem 1.2. *For any $n \in \mathbb{N}$,*

$$D_4(n) \geq \frac{5}{4}n - 1.$$

We also consider a stronger notion of convexity, namely k-convexity, which was one of the main considerations of the paper [4].

Definition 1.3. A set S is *k-convex* if $D_1(S)$ is a $(k-1)$-convex set. A convex set is also called a 1-convex set.

Let $\mathcal{S}^2(n)$ be the set of all 2-convex sets of size n. Define

$$D_i^2(n) := \min\{|D_i(S)| : S \in \mathcal{S}^2(n)\}.$$

Note that $D_1^2(n) = n-1$. The set S defined in (3) is a 2-convex set with $|D_2(S)| = n$, and it therefore follows that $D_2^2(n) = n$.

Similarly to Theorem 1.1 we will prove the following result.

Theorem 1.4. *For any integer $n \geq 5$,*

$$D_3^2(n) = n + 2. \tag{4}$$

We give a slightly better lower bound for $D_4^2(n)$ than that for $D_4(n)$ given in Theorem 1.2.

Theorem 1.5. *For any $n \in \mathbb{N}$,*

$$D_4^2(n) \geq \frac{4}{3}n - \frac{4}{3}.$$

We expect that the values of $D_i(n)$ and $D_i^2(n)$ increase with i, and it may even be possible that a lower bound of the form $\Omega((ni)^{1-\epsilon})$ holds. However, we are not able to generalize the elementary techniques used to prove Theorems 1.2 and 1.4 to obtain better bounds by considering larger i.

On the other hand, for sufficiently large i, a non-trivial lower bound for $D_i(n)$ can be obtained by a trivial application of the lower bound for $S-S$ in (1). Indeed, let S be a convex set with cardinality n and let $S' \subset S$ denote the subset formed by taking the i smallest elements. Then

$$|D_i(S)| \geq |D_i(S')| \gg |S' - S'| \gg \frac{i^{8/5}}{(\log n)^c} \tag{5}$$

In particular, if $i \geq n^{5/8+\epsilon}$, then (5) gives a lower bound of the form

$$D_i(n) \gg n^{1+\epsilon'}. \tag{6}$$

It appears plausible that (6) holds under the weaker assumption that $i \geq n^{\epsilon}$.

In this paper, we use incidence geometric techniques to give a more general result, see the forthcoming Theorem 2.1. In particular, this gives us a good lower bound for the size of $D_{\mathcal{I}}(S)$ when $|\mathcal{I}|$ is large.

2. Proofs of the Main Results

Proof of Theorems 1.1 and 1.4. To prove that $D_3(n) \geq n+2$, we need to prove that

$$|D_3(S)| \geq n+2 \tag{7}$$

holds for an arbitrary convex set S with $|S| = n$. Observe that $|D_3(S)| \geq |D_2(S)|+1$, since $s_n - s_{n-3} \in D_3(S) \setminus D_2(S)$. Hence $|D_3(S)| \geq n+1$. Suppose for a contradiction that $|D_3(S)| = n+1$ for some $n \geq 5$. Observe that

$$s_2 - s_1 < s_3 - s_2 < s_3 - s_1 < s_4 - s_2 < s_4 - s_1 < s_5 - s_2 < \cdots < s_n - s_{n-3}$$

and

$$s_2 - s_1 < s_3 - s_2 < s_4 - s_3 < s_4 - s_2 < s_5 - s_3 < s_5 - s_2 < \cdots < s_n - s_{n-3}.$$

We have identified two increasing sequences of length $n+1$ in $D_3(S)$, and so the sequences must be identical. Comparing the third terms of the sequences, we get

$$s_4 - s_3 = s_3 - s_1 \tag{8}$$

and comparing the fifth terms of the sequences, we get

$$s_4 - s_1 = s_5 - s_3. \tag{9}$$

From Equations (8) and (9) we get

$$s_5 - s_4 = s_3 - s_1 = s_4 - s_3.$$

But as S is a convex set this is not possible. Hence our assumption that $|D_3(S)| = n+1$ was wrong, and so $|D_3(S)| \geq n+2$, proving (7).

Since every 2-convex set is also convex, it follows that $|D_3(S)| \geq n+2$ for all 2-convex sets S. Therefore, $D_3^2(n) \geq n+2$.

To prove that $D_3(n) \leq n+2$ and $D_3^2(n) \leq n+2$, we provide an example of a 2-convex (and thus also convex) set with $|D_3(S)| = n+2$. Consider the set $S = \{s_1, s_2, \ldots, s_n\}$ where

$$s_1 = 0,\ s_2 = 10,\ s_3 = 23,\ s_4 = 40,$$
$$s_i := s_{i-1} + s_{i-2} - s_{i-4} \text{ for } i = 5, \ldots, n.$$

Indeed, $|D_3(S)| = n+2$. Observe that

$$s_{i+5} - s_{i+4} = s_{i+2} - s_i \forall i \in [1, n-5]. \tag{10}$$

We will prove that S is 2-convex by induction on n. The base case, with $n = 5, 6, 7$, can be checked by a direct calculation.

Now let $n \geq 8$. To show S is 2-convex it is sufficient to show that

$$(s_{i+3} - s_{i+2}) - (s_{i+2} - s_{i+1}) > (s_{i+2} - s_{i+1}) - (s_{i+1} - s_i) \tag{11}$$

for all $i \in [1, n-3]$. We know from the induction hypothesis that (11) holds for $i = 1, \ldots, n-4$, and so we only need to check that it is valid for $i = n-3$.

From Equation (10) we know that

$$\begin{aligned}(s_n - s_{n-1}) - (s_{n-1} - s_{n-2}) &= (s_{n-3} - s_{n-5}) - (s_{n-4} - s_{n-6}) \\ &= (s_{n-3} - s_{n-4}) - (s_{n-5} - s_{n-6}).\end{aligned} \tag{12}$$

Analogously,

$$(s_{n-1} - s_{n-2}) - (s_{n-2} - s_{n-3}) = (s_{n-4} - s_{n-5}) - (s_{n-6} - s_{n-7}). \tag{13}$$

Applying (11) twice (with $i = n-6$ and $i = n-7$), it follows that

$$(s_{n-3} - s_{n-4}) - (s_{n-4} - s_{n-5}) > (s_{n-5} - s_{n-6}) - (s_{n-6} - s_{n-7}).$$

Rearranging gives

$$(s_{n-3} - s_{n-4}) - (s_{n-5} - s_{n-6}) > (s_{n-4} - s_{n-5}) - (s_{n-6} - s_{n-7}).$$

We then apply (12) and (13) to conclude that

$$(s_n - s_{n-1}) - (s_{n-1} - s_{n-2}) > (s_{n-1} - s_{n-2}) - (s_{n-2} - s_{n-3}).$$

This proves (11) for the case $i = n-3$, as required. □

Taking a closer look at the construction of the set S defined above, we see that there is additional structure, as the set $D_{\{5\}}(S)$ overlaps significantly with $D_3(S)$. To be precise, we have

$$|D_{\{1,2,3,5\}}(S)| = n+4.$$

Since the two largest elements of $D_{\{1,2,3,5\}}(S)$ are not in $D_3(S)$, it follows from Theorem 1.1 that $D_{\{1,2,3,5\}}(S) \geq n+4$ for any convex set S, and so this construction is optimal in this regard.

Proof of Theorem 1.2. To prove Theorem 1.2, we need to show that

$$|D_4(S)| \geq \frac{5}{4}n - 1 \tag{14}$$

holds for an arbitrary convex set S with size n. Indeed, let S be such a set. The proof of (14) is split into two cases.

`Case 1`. Suppose that $|D_{\{2\}}(S) \setminus D_{\{1\}}(S)| \geq \frac{n}{4} - 2$. It follows that

$$\begin{aligned}|D_4(S)| \geq |D_2(S)| + 2 &= |D_{\{1\}}(S) \cup [D_{\{2\}}(S) \setminus D_{\{1\}}(S)]| + 2 \\ &\geq (n-1) + (n/4 - 2) + 2 = \frac{5n}{4} - 1.\end{aligned}$$

`Case 2`. Suppose that $|D_{\{2\}}(S) \setminus D_{\{1\}}(S)| < n/4 - 2$. It follows that

$$s_{i+2} - s_i, s_{i+3} - s_{i+1} \in D_{\{1\}}(S)$$

holds for at least

$$(n-3) - 2(n/4 - 2) = \frac{n}{2} + 1$$

values of $i \in [n-3]$. Let I be the set of all such i. Then, for all $i \in I$,

$$s_{i+2} - s_i = s_{j+1} - s_j, \quad s_{i+3} - s_{i+1} = s_{j'+1} - s_j,$$

for some $j, j' \in [n-1]$.

`Case 2a`. Suppose that, for at least $\frac{n}{4} + 2$ of the elements $i \in I$, we have $j' = j+1$. Then $s_{j+2} - s_j$ is in $D_{\{2\}}(S)$, but it is also strictly between two consecutive elements of $D_{\{4\}}(S)$. Indeed

$$\begin{aligned}s_{i+3} - s_{i-1} &= (s_{i+3} - s_{i+1}) + (s_{i+1} - s_{i-1}) \\ &< (s_{i+3} - s_{i+1}) + (s_{i+2} - s_i) \\ &= (s_{j+2} - s_{j+1}) + (s_{j+1} - s_j) \\ &= s_{j+2} - s_j \\ &< (s_{i+4} - s_{i+2}) + (s_{i+2} - s_i) \\ &= s_{i+4} - s_i.\end{aligned}$$

It follows that $|D_{\{2\}}(S) \setminus D_{\{4\}}(S)| \geq \frac{n}{4} + 2$. Therefore,

$$\begin{aligned}|D_4(S)| \geq 1 + |D_{\{2,4\}}(S)| &= 1 + |D_{\{4\}}(S) \cup [D_{\{2\}}(S) \setminus D_{\{4\}}(S)]| \\ &\geq 1 + n - 4 + \frac{n}{4} + 2 = \frac{5n}{4} - 1.\end{aligned}$$

`Case 2b`. Suppose that we are not in case 2a, and hence, for at least $\frac{n}{4} - 1$ of the elements of I, we have $j' \geq j + 2$. Then $s_{j+2} - s_{j+1}$ lies strictly between two consecutive elements of $D_{\{2\}}$, namely $s_{i+2} - s_i$ and $s_{i+3} - s_{i+1}$. Therefore, $|D_{\{1\}}(S) \setminus D_{\{2\}}(S)| \geq \frac{n}{4} - 1$, and thus

$$\begin{aligned}|D_4(S)| \geq |D_2(S)| + 2 &= |D_{\{2\}}(S) \cup [D_{\{1\}}(S) \setminus D_{\{2\}}(S)]| + 2\\ &\geq (n-2) + (n/4 - 1) + 2 = \frac{5n}{4} - 1.\end{aligned}$$

□

A closer look at the proof of Theorem 1.2 reveals that we have barely used the elements of $D_{\{3\}}(S)$ anywhere in the proof of (14). By modifying the proof slightly, we obtain the bound

$$|D_{\{1,2,4\}}(S)| \geq \frac{5}{4}n - 2$$

for any convex set S with cardinality n.

The construction of the set S from Theorem 1.1 yields the bound

$$|D_4(S)| \leq |D_3(S)| + (n-4) = 2n - 2.$$

Combining this observation with the result of Theorem 1.2, we see that

$$\frac{5}{4}n - 1 \leq D_4(n) \leq 2n - 2.$$

We now proceed to the proof of Theorem 1.5. The proof is largely the same as that of Theorem 1.2. The main difference is that we can use the 2-convex condition to show that Case 2b does not occur.

Proof of Theorem 1.5. To prove Theorem 1.5, we need to show that

$$|D_4(S)| \geq \frac{4}{3}n - \frac{4}{3} \tag{15}$$

holds for an arbitrary 2-convex set S with size n. Indeed, let S be such a set. The proof is split into two cases.

`Case 1`. Suppose that $|D_{\{2\}}(S) \setminus D_{\{1\}}(S)| \geq \frac{n}{3} - \frac{7}{3}$. It follows that

$$\begin{aligned}|D_4(S)| \geq |D_2(S)| + 2 &= |D_{\{1\}}(S) \cup [D_{\{2\}}(S) \setminus D_{\{1\}}(S)]| + 2\\ &\geq (n-1) + \left(\frac{n}{3} - \frac{7}{3}\right) + 2 = \frac{4}{3}n - \frac{4}{3}.\end{aligned}$$

`Case 2`. Suppose that $|D_{\{2\}}(S) \setminus D_{\{1\}}(S)| < \frac{n}{3} - \frac{7}{3}$. It follows that

$$s_{i+2} - s_i, s_{i+3} - s_{i+1} \in D_{\{1\}}(S)$$

holds for at least

$$(n-3) - 2\left(\frac{n}{3} - \frac{7}{3}\right) = \frac{n}{3} + \frac{5}{3}$$

values of $i \in [n-3]$. Let I be the set of all such i. Then, for all $i \in I$,

$$s_{i+2} - s_i = s_{j+1} - s_j, \quad s_{i+3} - s_{i+1} = s_{j'+1} - s_j,$$

for some $j, j' \in [n-1]$ satisfying $j' > j > i$. We claim now that it must be the case that $j' = j+1$. Indeed, suppose for a contradiction that $j' \geq j+2$. It then follows that

$$\begin{aligned}(s_{j+3} - s_{j+2}) - (s_{j+1} - s_j) &\leq (s_{j'+1} - s_{j'}) - (s_{j+1} - s_j) \\ &= (s_{i+3} - s_{i+1}) - (s_{i+2} - s_i) \\ &= (s_{i+3} - s_{i+2}) - (s_{i+1} - s_i).\end{aligned}$$

However, since $j > i$, this contradicts the assumption that S is 2-convex. Indeed, write the convex set $D_1(S) = \{d_1 < d_2 < \cdots < d_{n-1}\}$, and so $d_i = s_{i+1} - s_i$. Then the previous inequality can be written as

$$d_{j+2} - d_j \leq d_{i+2} - d_i.$$

But this inequality cannot hold if $D_1(S)$ is convex, which proves the claim.

As was the case in the proof of Theorem 1.2, $s_{j+2} - s_j$ is in $D_{\{2\}}(S)$, but it is also strictly in-between two consecutive elements of $D_{\{4\}}(S)$. Indeed

$$\begin{aligned}s_{i+3} - s_{i-1} &= (s_{i+3} - s_{i+1}) + (s_{i+1} - s_{i-1}) \\ &< (s_{i+3} - s_{i+1}) + (s_{i+2} - s_i) \\ &= (s_{j+2} - s_{j+1}) + (s_{j+1} - s_j) \\ &= s_{j+2} - s_j \\ &< (s_{i+4} - s_{i+2}) + (s_{i+2} - s_i) \\ &= s_{i+4} - s_i.\end{aligned}$$

It follows that $|D_{\{2\}}(S) \setminus D_{\{4\}}(S)| \geq \frac{n}{3} + \frac{5}{3}$. Therefore,

$$\begin{aligned}|D_4(S)| \geq 1 + |D_{\{2,4\}}(S)| &= 1 + |D_{\{4\}}(S) \cup [D_{\{2\}}(S) \setminus D_{\{4\}}(S)]| \\ &\geq 1 + n - 4 + \frac{n}{3} + \frac{5}{3} = \frac{4}{3}n - \frac{4}{3}.\end{aligned}$$

□

We now turn to the case of bounding $D_{\mathcal{I}}(S)$ for large $\mathcal{I}$, giving a modification of the proof of the main result in [2].

Theorem 2.1. *Let $S = \{s_1 < s_2 < \cdots < s_n\}$ be convex and let $G \subset [n] \times [n]$. Then*

$$|\{s_x - s_y : (x,y) \in G\}| \gg \left(\frac{|G|}{n}\right)^{3/2}. \tag{16}$$

Similar sum-product type results for restricted pairs have been considered in, for instance, [1] and [5]. Note that the bound in (16) becomes meaningful when $|G|$ is significantly larger than n. A construction of a convex set with a rich difference in [6] shows that the set $\{s_x - s_y : (x,y) \in G\}$ can have cardinality as small as one when G has cardinality as large as $n/2$.

Proof of Theorem 2.1. We may assume that $|G| \geq 2n$, as otherwise the bound (16) becomes trivial since the right hand side of the inequality is constant.

Denote

$$S -_G S := \{s_x - s_y : (x,y) \in G\}.$$

We will prove that

$$|S -_G S| \gg \left(\frac{|G|}{n}\right)^{3/2}. \tag{17}$$

Since S is convex, it follows that there exists a strictly convex function $f : \mathbb{R} \to \mathbb{R}$ such that $f(i) = s_i$.

Define P to be the point set

$$P = \{-n, -n+1, \ldots, n-1, n\} \times (S -_G S).$$

Let $\ell_{a,b}$ denote the curve with equation $y = f(x+a) - b$. Define L to be the set of curves

$$L = \{\ell_{j,s_h} : 1 \leq j, h \leq n\}.$$

The set L consists of translates of the same convex curve, and it therefore follows that we have the Szemerédi-Trotter type bound

$$I(P,L) \leq 4|P|^{2/3}|L|^{2/3} + 4|P| + |L|, \tag{18}$$

where

$$I(P,L) := \{(p,\ell) \in P \times L : p \in \ell\}.$$

For each $\ell_{j,s_h} \in L$, and for any k such that $(k,h) \in G$, observe that

$$(k-j, s_k - s_h) \in P \cap \ell_{j,s_h}.$$

This implies that

$$I(P,L) = \sum_{\ell \in L} |\ell \cap P| \geq \sum_{1 \leq j \leq n} \sum_{1 \leq h \leq n} |\{k : (k,h) \in G\}| = n|G|.$$

Comparing this bound with (18) yields

$$n|G| \leq 4n^2 |S -_G S|^{2/3} + 4n|S -_G S| + n^2.$$

The assumption that $|G| \geq 2n$ can then be used to deduce that

$$n|G| \ll n^2|S -_G S|^{2/3} + n|S -_G S| \ll n^2 |S -_G S|^{2/3},$$

where the latter inequality makes use of the (very) trivial bound $|S -_G S| \leq n^3$. A rearrangement gives the claimed bound (17). $\square$

3. Concluding Remarks

3.1. Sums Instead of Differences

One may also consider a version of this question with sums. Let $E_i(S)$ denote the set

$$E_i(S) = \{s_x + s_y : 1 \le x - y \le i\}.$$

However, it turns out that this modification to the question makes it rather straightforward to prove a non-trivial bound that grows with i. If we split S into disjoint consecutive blocks $S_1 \cup S_2 \cup \cdots \cup S_t$, with each block having i elements (we possibly discard some elements to ensure that all blocks have exactly the same size, and so $t = \lfloor n/i \rfloor$), then the sum sets $S_j + S_j$ are pairwise disjoint. It then follows from the lower bound for $S_j + S_j$ given in (1) that

$$|S + S| \ge \sum_{j=1}^{t} |S_j + S_j| \gg \frac{n}{i} \cdot \frac{i^{30/19}}{(\log n)^c} = \frac{ni^{11/19}}{(\log n)^c}.$$

3.2. Sets $\mathcal{I}$ such that $D_{\mathcal{I}} = n + O(1)$

In Theorem 1.1, we have seen that $D_{\{1,2,3\}}(n) = n + 2$. There are other examples of sets $\mathcal{I} \subset [n-1]$ with cardinality 3 such that $D_{\mathcal{I}}(n) = n + O(1)$. For example, we can define S using the recurrence relation

$$s_n = s_{n-2} + s_{n-3} - s_{n-6}.$$

By choosing the initial elements of S suitably, the set S is convex. However, this recurrence relations gives rise to the system of equations

$$s_j - s_{j-2} = s_{j-3} - s_{j-6} = s_{j-5} - s_{j-9}, \quad \forall\ 10 \le j \le n.$$

This implies that the elements of $D_{\{2\}}(S), D_{\{3\}}(S)$ and $D_{\{4\}}(S)$ are largely the same, with just a few exceptions occurring at the extremes of the three sets. It follows that $D_{\{2,3,4\}}(S) = n + C$, for some absolute constant C. It appears likely that the same argument can be used to show that $D_{\{k,k+d,k+2d\}}(n) = n + O_{k,d}(1)$.

On the other hand, we have seen in this paper that $D_{\{1,2,4\}}(n) = \frac{5n}{4} - O(1)$. This raises the following question: can we classify the sets $\mathcal{I}$ with cardinality 3 with the property that $D_{\mathcal{I}}(n) = n + C$, where C is some constant (which may depend on the elements of $\mathcal{I}$)?

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