ON THE RIEMANN HYPOTHESIS AND THE DEDEKIND PSI FUNCTION

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Abstract

Let $N_n = 2 \cdot 3 \cdots p_n$ be the primorial of order n and Ψ the Dedekind Psi function. Solé and Planat (2011) proved that the Riemann Hypothesis is true if and only if $\Psi(N_n)/(N_n \log\log N_n) > e^\gamma/\zeta(2)$ for all $n \geq 3$. We investigate the possibility of a reformulation of this criterion, where the term $\log N_n$ is replaced by the nth prime p_n. Actually, we prove that if $\Psi(N_n)/(N_n \log p_n) > e^\gamma/\zeta(2)$ for all $n \geq 3$, then the Riemann Hypothesis is true. Let φ denote the Euler totient function. As a consequence of the previous result, we obtain that if $N_n/\varphi(N_n) > e^\gamma \log p_n$ for all $n \geq 3$, then the Riemann Hypothesis is true.

1. Introduction

The Riemann Hypothesis, stated in 1859 [10], concerns the complex zeros of the Riemann zeta function. This function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges for $\Re(s) > 1$ and has an analytic continuation to the complex plane with one singularity, a simple pole with residue 1 at $s = 1$. The Riemann Hypothesis, RH for short, states that all the nonreal zeros of the Riemann zeta function $\zeta(s)$ lie on the line $\Re(s) = 1/2$. This problem is of great interest due to its connection with the distribution of prime numbers.

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The connection of the Riemann Hypothesis with prime numbers was considered by Gauss. Let $\pi(x)$ count the number of primes p with $1 < p \le x$. It is known that the Riemann Hypothesis is equivalent to the assertion that

$$|\pi(x) - Li(x)| \le C_\varepsilon x^{\frac{1}{2}+\varepsilon}$$

for all $x \ge 2$ and for all $\varepsilon > 0$, where

$$Li(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$$

is a logarithmic integral and C_ε is a positive constant (see for instance [5]). There are many other equivalent elementary formulations.

G. Robin, in [11], proved the following criterion connected with the sum of divisors function $\sigma(n) = \sum_{d|n} d$.

Theorem 1 (Robin Criterion)**.** *The Riemann Hypothesis is true if and only if*

$$\sigma(n) < e^\gamma n \log\log n,$$

for all $n \ge 5041$, *where* $\gamma = \lim_{n\to\infty}(\sum_{k=1}^n 1/k - \log n) \approx 0.57721\ldots$ *is the Euler-Mascheroni constant.*

The above inequality was checked for many infinite families of integers (see for instance [1, 2, 8, 14]). Let N_n be the *primorial number* of index n, i.e., the product of the first n primes

$$N_n = \prod_{k=1}^n p_k.$$

Thus $N_1 = 2, N_2 = 6$, and so on. Consider the ratio

$$S(n) = \frac{n}{\varphi(n)\log\log n},$$

where $\varphi(n)$ is the Euler totient function. In [9], J.L. Nicolas proved the following criterion.

Theorem 2 (Nicolas Criterion)**.** *The Riemann Hypothesis is true if and only if*

$$S(N_n) > e^\gamma$$

for all $n \ge 3$.

In [13], P. Solé and M. Planat provide a new formulation where the Euler function φ is replaced by the Dedekind Psi function Ψ. Let

$$R(n) = \frac{\Psi(n)}{n\log\log n}.$$

The following theorem holds.

Theorem 3 (Solé and Planat Criterion)**.** *The Riemann Hypothesis is true if and only if*

$$R(N_n) > \frac{e^{\gamma}}{\zeta(2)}$$

for all $n \geq 3$.

Notice that one has

$$S(N_n) = \frac{N_n}{\varphi(N_n) \log \theta(p_n)} \quad \text{and} \quad R(N_n) = \frac{\Psi(N_n)}{N_n \log \theta(p_n)},$$

where $\theta(x) = \sum_{p \leq x} \log p$ is Chebyshev's first summatory function. The aim of this paper is to study the functions obtained replacing $\theta(p_n)$ by p_n in the above expressions. In particular, we investigate whether properties similar to those stated in Theorems 2 and 3 still hold with these modified expressions.

More precisely, let

$$D(n) = \frac{\Psi(N_n)}{N_n \log p_n} = \frac{\prod_{i=1}^{n} \left(1 + \frac{1}{p_i}\right)}{\log p_n}$$

and

$$E(n) = \frac{N_n}{\varphi(N_n) \log p_n} = \frac{1}{\log p_n \prod_{i=1}^{n} \left(1 - \frac{1}{p_i}\right)}.$$

We prove the following result.

Theorem 4. *If*

$$D(n) > \frac{e^{\gamma}}{\zeta(2)} \tag{1}$$

for all $n \geq 3$, *then the Riemann Hypothesis is true.*

Moreover, as a consequence of the previous theorem, we obtain the following theorem.

Theorem 5. *If*

$$E(n) > e^{\gamma} \tag{2}$$

for all $n \geq 3$, *then the Riemann Hypothesis is true.*

Thus, each of the two Inequalities (1) and (2) is a sufficient condition for the truth of the Riemann Hypothesis. One may ask whether, in the case that the Riemann Hypothesis is true, Inequalities (1) and (2) are verified. On this question, we give partial answers. Indeed, we prove the following result.

Proposition 1. *If the Riemann Hypothesis is true and $\theta(p_n) \geq p_n$, then*

$$D(n) > \frac{e^\gamma}{\zeta(2)} \quad \textit{and} \quad E(n) > e^\gamma.$$

The main tool for establishing these results is a study of the monotonicity of the sequences $R(N_n)$ and $S(N_n)$. Indeed, we show that when $p_n < \theta(p_{n-1})$ one has $R(N_n) > R(N_{n-1})$ and $S(N_n) > S(N_{n-1})$, while when $p_n > \theta(p_n)$ one has $R(N_n) < R(N_{n-1})$ and $S(N_n) < S(N_{n-1})$. The paper is organized as follows. Section 2 contains the notions necessary for our work and some preliminary lemmata. In Section 3 we study the behavior of the functions R and S. In Section 4 we prove the main results of the paper. Some further results are presented is Section 5. In particular, we prove that Inequalities (1) and (2) considered in Theorems 4 and 5 are satisfied for infinitely many integers n.

2. Notation and Preliminary Results

In this section we recall some definitions and results useful for our work. Moreover, we establish a result (see Lemma 2) related to the Solé and Planat Criterion, concerning the case that the Riemann Hypothesis is not true.

The Dedekind function Ψ is an arithmetic multiplicative function defined for every integer $n > 0$ by

$$\Psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

The Euler totient function φ is an arithmetic multiplicative function defined for every integer $n > 0$ by

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

In these equations, as usual, the notation $\prod_{p|n}$ means that the product is extended to all prime divisors p of n.

We recall that $\theta(x) = \sum_{p \leq x} \log p$ is Chebyshev's first summatory function [6]. In particular, for all integer n, $\theta(p_n) = \sum_{k=1}^{n} \log p_k = \log N_n$.

Consider the functions R and S introduced in the previous section. One easily verifies that

$$R(N_n) = \frac{\prod_{k=1}^{n} \left(1 + \frac{1}{p_k}\right)}{\log \theta(p_n)} \quad \textit{and} \quad S(N_n) = \frac{1}{\log \theta(p_n) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right)}. \tag{3}$$

The following technical lemmata will be useful to prove our main results.

Lemma 1. *For all $x \geq 3$, one has*

$$\sum_{p>x} \log\left(1 - \frac{1}{p^2}\right) \geq -\frac{2}{x}.$$

Proof. See [2, Lemma 6.4], with $t = 2$. □

Consider the function

$$f(x) = e^\gamma \log\theta(x) \prod_{p\leq x}\left(1 - \frac{1}{p}\right). \tag{4}$$

The following useful result was established in the proof of [9, Proposition 3].

Lemma 2. *Suppose that RH is false and denote by Θ the upper bound of the real parts of the zeros of the Riemann zeta function. For all $b \in [1-\Theta, 1/2]$ there exist positive constants C, D_+ and D_- such that*

1. *there are arbitrarily large x such that $\log f(x) > C/x^b$ and $\theta(x) - x < -D_- x^{(\Theta+1-b)/2}$,*
2. *there are arbitrarily large x such that $\log f(x) > C/x^b$, and $\theta(x) - x > D_+ x^{(\Theta+1-b)/2}$.*

Now, we study the behavior of $R(N_n)$ in the case that the Riemann Hypothesis is false.

Lemma 3. *If RH is false, there are infinitely many n such that*

$$R(N_{n-1}) < \frac{e^\gamma}{\zeta(2)} \quad \text{and} \quad \theta(p_n) < p_n, \tag{5}$$

and infinitely many n such that

$$R(N_n) < \frac{e^\gamma}{\zeta(2)} \quad \text{and} \quad \theta(p_n) > p_n. \tag{6}$$

Proof. Let f be the function defined by (4) and set

$$g(x) = \frac{e^\gamma}{\zeta(2)} \log\theta(x) \prod_{p\leq x}\left(1 + \frac{1}{p}\right)^{-1}. \tag{7}$$

Taking into account that $\zeta(2) = \prod_p 1/(1-p^{-2})$, one easily finds that

$$\frac{g(x)}{f(x)} = \frac{1}{\zeta(2)\prod_{p\leq x}\left(1 - \dfrac{1}{p^2}\right)} = \prod_{p>x}\left(1 - \frac{1}{p^2}\right).$$

Thus, in view of Lemma 1,

$$\log g(x) - \log f(x) = \sum_{p>x} \log\left(1 - \frac{1}{p^2}\right) \geq -\frac{2}{x}. \tag{8}$$

Now we suppose that RH is false and show that there are infinitely many n satisfying (5).

From Lemma 2, one easily derives that there are arbitrarily large x such that

$$\log f(x) > 2/x \quad \text{and} \quad \theta(x) - x < -\log(2x). \tag{9}$$

In view of (8), for such x's one has $\log g(x) > 0$, that is, $g(x) > 1$. Denote by p_n the least prime larger than x. Then, the right-hand side of (7) is equal to $e^\gamma/(\zeta(2)R(N_{n-1}))$, so that $e^\gamma/(\zeta(2)R(N_{n-1})) > 1$, that is,

$$R(N_{n-1}) < e^\gamma/\zeta(2).$$

Moreover, in view of the so-called Bertrand's postulate (see, e.g., [6]), one has $\lfloor x \rfloor < p_n < 2\lfloor x \rfloor$ and, therefore, $p_{n-1} \leq x < p_n < 2x$. Consequently, in view of (9),

$$\theta(p_n) = \theta(p_{n-1}) + \log(p_n) < \theta(x) + \log(2x) < x.$$

This proves that there are infinitely many n satisfying (5).

Now we show that, assuming RH false, there are infinitely many n satisfying (6). From Lemma 2, one easily derives that there are arbitrarily large x such that

$$\log f(x) > 2/x \quad \text{and} \quad \theta(x) - x > 0.$$

Denote by p_n the least prime not larger than x. By an argument similar to that used for the condition (5), one obtains that $R(N_n) < e^\gamma/\zeta(2)$. Moreover, $\theta(p_n) = \theta(x) > x \geq p_n$. This completes the proof. □

3. Monotonic Properties of the Functions *R* and *S*

In [13] and [9], respectively, it is proved that

$$\lim_{n\to+\infty} R(N_n) = \frac{e^\gamma}{\zeta(2)}.$$

and

$$\lim_{n\to+\infty} S(N_n) = e^\gamma.$$

Unfortunately, the functions R and S are not ultimately decreasing. Actually, also the sequences $R(N_n)$ and $S(N_n)$ are not ultimately monotonic [3].

Below we prove some conditions ensuring that the sequences $R(N_n)$ and $S(N_n)$ increase or decrease at some point.

Proposition 2. *For all $n \geq 2$ such that $\theta(p_n) < p_n$, one has $R(N_n) < R(N_{n-1})$.*

Proof. By (3), one has

$$\frac{R(N_n)}{R(N_{n-1})} = \left(1 + \frac{1}{p_n}\right) \frac{\log \theta(p_{n-1})}{\log \theta(p_n)}.$$

Thus, we are reduced to proving that the right-hand side of the equation above is smaller than 1, or, equivalently,

$$\frac{\log \theta(p_{n-1})}{p_n} < \log \theta(p_n) - \log \theta(p_{n-1}). \tag{10}$$

Recalling that for $b > a > 0$, one has $\log b - \log a > (b-a)/b$, one obtains

$$\log \theta(p_n) - \log \theta(p_{n-1}) > \frac{\theta(p_n) - \theta(p_{n-1})}{\theta(p_n)} = \frac{\log p_n}{\theta(p_n)}.$$

Moreover, since in our hypotheses $p_n > \theta(p_n) > \theta(p_{n-1})$, one has

$$\frac{\log p_n}{\theta(p_n)} > \frac{\log \theta(p_{n-1})}{p_n}.$$

Equation (10) is a straightforward consequence of the last two inequalities. This concludes the proof. □

Proposition 3. *For all $n \geq 2$ such that $\theta(p_n) \geq p_n + \log p_n$, one has $R(N_n) > R(N_{n-1})$.*

Proof. Proceeding similarly to the proof of the previous proposition one is reduced to verifying that

$$\frac{\log \theta(p_{n-1})}{p_n} > \log \theta(p_n) - \log \theta(p_{n-1}).$$

Recalling that for $b > a > 0$, one has $\log b - \log a < (b-a)/a$ and that in our hypotheses $p_n \leq \theta(p_n) - \log p_n = \theta(p_{n-1})$, one obtains

$$\log \theta(p_n) - \log \theta(p_{n-1}) < \frac{\theta(p_n) - \theta(p_{n-1})}{\theta(p_{n-1})} = \frac{\log p_n}{\theta(p_{n-1})} \leq \frac{\log \theta(p_{n-1})}{p_n}.$$

This concludes the proof. □

Proposition 4. *For all $n \geq 2$ such that $\theta(p_n) < p_n$, one has $S(N_n) < S(N_{n-1})$.*

Proof. By (3), one has

$$\frac{S(N_{n-1})}{S(N_n)} = \left(1 - \frac{1}{p_n}\right) \frac{\log \theta(p_n)}{\log \theta(p_{n-1})}.$$

Thus, we are reduced to proving that the right-hand side of the equation above is larger than 1, or, equivalently,

$$\frac{\log \theta(p_n)}{p_n} < \log \theta(p_n) - \log \theta(p_{n-1}).$$

This result can be easily achieved proceeding as in the proof of Proposition 2. □

Proposition 5. *For all $n \geq 2$ such that $\theta(p_n) \geq p_n + \log p_n$, one has $S(N_n) > S(N_{n-1})$.*

Proof. Proceeding similarly to the proof of the previous proposition one is reduced to verifying that

$$\frac{\log \theta(p_n)}{p_n} > \log \theta(p_n) - \log \theta(p_{n-1})$$

and, again, this result can be achieved proceeding as in the proof of Proposition 3. □

We note that these bounds are significant in view of the following theorem (see [7, Theorem 34]). We recall, preliminarily, that for two real-valued functions f and g of a real variable x, with $g(x) > 0$, we write $f(x) = \Omega_\pm(g(x))$, if $\limsup_{x\to\infty} f(x)/g(x) > 0$ and $\liminf_{x\to\infty} f(x)/g(x) < 0$.

Theorem 6 (Littlewood Oscillation Theorem)**.** *One has*

$$\theta(p_n) - p_n = \Omega_\pm(\sqrt{p_n} \log\log\log p_n).$$

Thus, there are infinitely many integers n such that $\theta(p_n) < p_n$ and infinitely many integers n such that $\theta(p_n) \geq p_n + \log p_n$.

4. Proofs of the Main Results

In this section we prove our main results.

Proof of Theorem 4. By contradiction, suppose that $D(n) > e^\gamma/\zeta(2)$ for all $n \geq 3$ and that the Riemann Hypothesis is false. Let $n \geq 3$ be an integer such that $\theta(p_n) < p_n$ and $R(N_{n-1}) < e^\gamma/\zeta(2)$. Such an integer exists by Lemma 3. From the former inequality and the definitions of the functions R and D, one obtains that $R(N_n) > D(n)$. Moreover, in view of Proposition 2, $R(N_{n-1}) > R(N_n)$. One derives $R(N_{n-1}) > D(n) > e^\gamma/\zeta(2)$, contradicting our assumption. □

Proof of Theorem 5. As one easily checks,

$$\frac{D(n)}{E(n)} = \Psi(N_n)\varphi(N_n) = \prod_{i=1}^{n} \left(1 - \frac{1}{p_i^2}\right).$$

Taking into account that $1/\zeta(2) = \prod_{i=1}^{\infty}\left(1 - 1/p_i^2\right) < \prod_{i=1}^{n}\left(1 - 1/p_i^2\right)$, one derives

$$D(n) > \frac{E(n)}{\zeta(2)}.$$

Thus, if $E(n) > e^{\gamma}$ for all $n \geq 3$, one has also $D(n) > e^{\gamma}/\zeta(2)$ and the Riemann Hypothesis holds true by Theorem 4. □

Proof of Proposition 1. In the hypothesis that $\theta(p_n) \geq p_n$ one has $D(n) \geq R(N_n)$ and $E(n) \geq S(N_n)$. Thus, the statement is a straightforward consequence of Theorems 3 and 2. □

5. Further Remarks

In this section, we will prove that Inequalities (1) and (2) considered in Theorems 4 and 5 are satisfied for infinitely many integers n.

Passing to logarithms on both sides, Inequality (1) becomes

$$\sum_{p \leq p_n} \log\left(1 + \frac{1}{p}\right) > \gamma - \log(\zeta(2)) + \log\log p_n.$$

In [4], the following theorem has been established.

Theorem 7. *For every positive number K, there are arbitrarily large x such that*

$$-\sum_{p \leq x} \log\left(1 - \frac{1}{p}\right) - \log\log x - \gamma > K/(\sqrt{x}\log x) \tag{11}$$

and arbitrarily large x such that

$$-\sum_{p \leq x} \log\left(1 - \frac{1}{p}\right) - \log\log x - \gamma < -K/(\sqrt{x}\log x).$$

When x satisfies (11), exponentiating both sides, one obtains

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right) e^{\gamma}\log x < e^{-K/(\sqrt{x}\log x)} < 1.$$

One easily derives that if p_m is the largest prime smaller than or equal to x, then

$$\prod_{i=1}^{m}\left(1 - \frac{1}{p_i}\right) < \frac{1}{e^{\gamma}\log p_m}.$$

By Theorem 7, the equation above is satisfied by arbitrarily large m. Taking into account that $1/\zeta(2) < \prod_{p \le x}(1 - 1/p^2)$, for all such m's, one has

$$\prod_{i=1}^{m}\left(1+\frac{1}{p_i}\right) = \frac{\prod_{i=1}^{m}\left(1-\frac{1}{p_i^2}\right)}{\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right)} > \frac{e^{\gamma}\log p_m}{\zeta(2)}.$$

One immediately derives the following corollary.

Corollary 1. *There are infinitely many integers n such that*

$$D(n) > \frac{e^{\gamma}}{\zeta(2)}$$

Finally, it is easy to verify that a similar result also holds for Inequality (2). Thus, the following statement also holds true.

Corollary 2. *There are infinitely many integers n such that*

$$E(n) > e^{\gamma}.$$

In conclusion, the results obtained so far and, in particular, the monotonic properties of the functions $R(N_n)$ and $S(N_n)$ seen in Section 3, lead the authors to hope that some refinement of the techniques introduced in this paper may lead to proving the equivalence of the Riemann Hypothesis with Inequalities (1) and (2).

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