COLORING AND BOUNDARY INVARIANTS FOR POLYOMINOES

Juri Kirillov
Clare Hall, Herschel Road, Cambridge, United Kingdom

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Abstract

Using Gaussian integers we construct a coloring for polyominoes and find necessary and sufficient conditions for the existence of a tiling of rectangles. First, we review a known coloring where the unit squares are colored. Then, we show that another type of coloring is possible – where the edges of unit squares are colored.

1. Introduction

One of the ideas to find a coloring for polyominoes is to use the bijection $g : \mathbb{Z}^2 \to \mathbb{Z}[i]$, given by $g(x, y) \to x + iy$ to obtain the coloring of type

$$\begin{aligned} f : \mathcal{P} \to \mathbb{Z}[i]/(v),\ v \in \mathbb{Z}[i], 1 \leqslant m < N(v), \\ f_m(x, y) \equiv (x + iy)^m \bmod (v), \\ \tau = \{z_1, \dots, z_s\}, \text{and} \\ f(\tau) = f(z_1) + \cdots + f(z_s), \end{aligned} \tag{1.1}$$

where $\mathcal{P}$ is the set of all polyominoes and $N(x+iy) = x^2+y^2$ is the norm of $x+iy$ [3]. A polyomino is defined as a connected set of unit squares of $\mathbb{Z}^2$. If $\tau = \{z_1, \dots, z_s\}$, then $f(\tau)$ is defined as the sum of its values on these unit squares. The colors in this coloring are residue classes modulo v. If the sum of colors $f(\tau)$ does not change under the usual transformations of τ (translation, rotation and reflection), then we get an invariant. In particular, if $v \in \mathbb{Z}$ and

$$s \equiv m \sum z_j \equiv \binom{m}{2} \sum z_j^2 \equiv \ldots \equiv \sum z_j^m \equiv 0 \pmod{v},$$

then f is an invariant (summation here is from 1 to s); see [3, Theorem 4.4].

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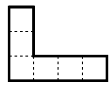
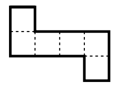
Let us consider one of the known results related to hexominoes [3, Theorem 5.5].

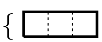
Theorem 1.1. *Let* $T = \{$  ,  $\}$. *A rectangle* $a \times b$ *can be tiled by the set* T *using translation, rotation and reflection if and only if the area of this rectangle is divisible by* 18 *and* $a, b \geqslant 3$.

Proof. Let us use coloring (1.1). As we can see, we have

$$f_1 = \sum z_j \equiv 0 \pmod 3$$

and

$$f_2 = \sum z_j^2 \equiv 0 \pmod 3$$

for each hexomino in T. The horizontal tromino $I_3 = \{$  $\}$ has

$$f_1 = \sum z_j = 0$$

and

$$f_2 = \sum z_j^2 = 2.$$

If a rectangle is tiled by T, then its area is divisible by 6, so we can tile it by I_3 using translation. Since $f_2(I_3) \not\equiv 0 \pmod 3$, it follows that the number of I_3 in this rectangle should be divisible by 3, so the area is divisible by 9. Since the area of a hexomino equals 6, it follows that the area of the rectangle is divisible by 18.

Sufficiency follows from the existence of tiling of rectangles 3×6, 4×9 and 5×18, and because all other rectangles $a \times b$ with $18 | ab$ and $a, b \geqslant 3$ can be tiled by these three rectangles. □

Let us consider another tile, the 15-omino τ in Fig. 1.1. It tiles the rectangle 30×30 [5].

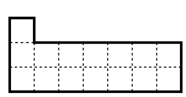


Figure 1.1: A 15-omino that tiles the rectangle 30×30.

Theorem 1.2. *If* τ *tiles a rectangle* $a \times b$, *then* $45 | ab$.

Proof. Let us use coloring (1.1). As we can see, we have

$$f_1 = \sum z_j = 3 + 9i \equiv 0 \pmod 3$$

and

$$f_2 = \sum z_j^2 = 54 + 12i \equiv 0 \pmod 3.$$

The horizontal tromino I_3 has

$$f_1 = \sum z_j = 0$$

and

$$f_2 = \sum z_j^2 = 2.$$

If a rectangle is tiled by τ, then its area is divisible by 3, so we can tile it by I_3 using translation. Since $f_2(I_3) \not\equiv 0 \pmod 3$, it follows that the number of I_3 in the rectangle should be divisible by 3 and hence the area is divisible by 9. □

2. Coloring on the Edges

We continue to develop the idea of using complex numbers to obtain a coloring. This time we assign colors to polyomino edges rather than to its unit squares. Let A be the infinite free group generated by elements a_i indexed by the points of lattice $\mathbb{Z}^2$,

$$A = \langle a_i \mid i \in \mathbb{Z}^2 \rangle.$$

Given a simply connected polyomino τ, we can assign to it the word

$$\partial\tau = a_1 a_2 \dots a_n \in A$$

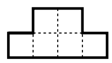
obtained by traversing the boundary of τ in the counterclockwise direction. An introduction to boundary words can be found in [10, Section 3]. The approach in [10] is a bit different – it uses the free group on two generators, while we define group A with an infinite number of generators. We use the same notation $\mathcal{P}$ for the set of all polyominoes. Depending on the context, assume that polyominoes to be of general type or simply-connected.

Let $\mathcal{T}$ be a set of polyominoes and suppose that there is a group G and a homomorphism $\varphi : A \to G$. We are interested in finding a map $f : \mathcal{P} \to G$ given as $f(\tau) = a_1 a_2 \dots a_n$ such that $f(\tau) = e$ (the identity element) for all $\tau \in \mathcal{T}$ and such that $f(\tau) = f(\tau')$, where τ' is τ after translation, rotation and reflection. Then f is a boundary invariant. If the group G is abelian, it is known that f is equivalent to a coloring (defined on the unit squares); see [1] or [10, Section 4]. Michael Reid gives many examples where the image of the homomorphism φ is a finite group [10]. An example of infinite group is given in [4].

Let us take $G = \mathbb{Z}[i]$ and consider the map $f : \mathcal{P} \to \mathbb{Z}[i]$ given as

$$f(\tau) = S_m := z_1^m - z_2^m + z_3^m - z_4^m + \ldots + (-1)^{n+1} z_n^m, \tag{2.1}$$

where m is a positive integer and $z_1, z_2, \ldots, z_n$ are Gaussian integers that represent the edges of τ, and which are taken along the boundary of τ in the counterclockwise direction. We assume that the absolute value of the difference between numbers that represent the adjacent edges equals 2. Notice that we use $\mathbb{Z}[i]$ as a ring, not only as a group.

Example 2.1. Let $\tau = \{$  $\}$. The boundary of τ can be given as $\{0, 2, 4, 6, 7+i, 6+2i, 5+3i, 4+4i, 2+4i, 1+3i, 2i, -1+i\}$. We can compute S_m for $m = 1, 2, 3, 4$ by formula (2.1): $S_1 = 0, S_2 = 0, S_3 = 72$ and $S_4 = 3i(1+i)^{11}(-7+2i)$.

One may note that S_1 is always zero for any polyomino. Therefore, we can compute S_m starting from $m = 2$. We see that $S_2 \equiv S_3 \equiv S_4 \equiv 0 \pmod 3$ and $S_2 \equiv S_3 \equiv 0 \pmod 9$. Similar to the invariant based on the map (1.1), we may suppose that if $S_2 = \ldots = S_m = 0$ for $m \geqslant 2$, then the map f in (2.1) is an invariant. Indeed, this is true, as shown by the next theorem.

Theorem 2.2. *Let τ be a polyomino with the boundary $\{z_1, z_2, \ldots, z_n\}$, m a positive integer, and $v \in \mathbb{Z}[i]$. The map $f : \mathcal{P} \to \mathbb{Z}[i]/(v)$ given as*

$$f(\tau) = S_m := z_1^m - z_2^m + z_3^m - \ldots + (-1)^{n+1} z_n^m \pmod v$$

is a boundary invariant for τ if the following conditions hold:

$$\binom{m}{2} S_2 \equiv \binom{m}{3} S_3 \equiv \ldots \equiv S_m \equiv 0 \pmod v. \tag{2.2}$$

Proof. We have to show that $f(\tau') \equiv f(\tau) \equiv 0 \pmod v$, where τ' is τ after translation and rotation. Note that if τ has the boundary $\{z_1, z_2, \ldots, z_n\}$, then τ' has the boundary $\{z_1\varepsilon + a, z_2\varepsilon + a, \ldots, z_n\varepsilon + a\}$ with $\varepsilon \in \{\pm 1, \pm i\}$ and $a \in \mathbb{Z}[i]/(v)$. We have

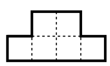
$$\begin{aligned} \sum_{j=1}^{s} (z_j\varepsilon + a)^m = \varepsilon^m \sum z_j^m + \binom{m}{m-1} \varepsilon^{m-1} a \sum z_j^{m-1} \\ + \ldots + \binom{m}{2} \varepsilon^2 a^{m-2} \sum z_j^2 + m\varepsilon a^{m-1} \sum z_j + s a^m, \end{aligned} \tag{2.3}$$

where summation is from $j = 1$ to s. If (2.2) holds, then

$$f(\tau') = (z_1\varepsilon + a)^m - (z_2\varepsilon + a)^m + \ldots + (-1)^{m+1}(z_n\varepsilon + a)^m$$
$$= \sum_{j=1,\ j \text{ is odd}}^{n-1} (z_j\varepsilon + a)^m - \sum_{j=2,\ j \text{ is even}}^{n} (z_j\varepsilon + a)^m$$
$$\overset{(2.3)}{=} \varepsilon^m S_m + \binom{m}{m-1}\varepsilon^{m-1} a S_{m-1} + \ldots + \binom{m}{2}\varepsilon^2 a^{m-2} S_2$$
$$+ m\varepsilon a^{m-1} S_1 \overset{(2.2)}{\equiv} 0 \pmod{v}.$$

If we choose a different starting point for the boundary of the polyomino τ, then $f(\tau)$ may become $-f(\tau)$ in general, but since $f(\tau) = 0$, it follows that $f(\tau)$ is independent of the starting point. □

Notice that for f to be an invariant that includes reflection, we need that $v \in \mathbb{Z}$. Since the group $G = \mathbb{Z}[i]/(v)$ is abelian, it follows that boundary invariant f in (2.1) can be given by a coloring. Let us take a look at one known result (from Example 5.10 in [8]).

Theorem 2.3. *Let* $\mathcal{T} = \{$  $\}$. *The hexomino* $\mathcal{T}$ *tiles a rectangle* $a \times b$ *if and only if two conditions hold:*
(1) *either* a *or* b *is a multiple of* 4*, and*
(2) *either* a *or* b *is a multiple of* 6 *except when* $(a, b) = (12, 1), (12, 2), (12, 3)$ *or* $(12, 7)$.

Proof. Let us prove item (2) assuming that there is a tiling of the rectangle $a \times b$ by the hexomino $\mathcal{T}$. In Example 2.1 we saw that $S_2 \equiv S_3 \equiv 0 \pmod{9}$, hence the map $f : \mathcal{P} \to \mathbb{Z}[i]/(9)$ given by

$$f(\tau) = S_3 = z_1^3 - z_2^3 + z_3^3 - \ldots + (-1)^{n+1} z_n^3 \pmod{9}$$

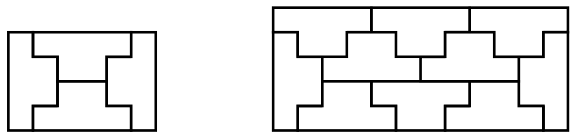
is an invariant for $\mathcal{T}$. We also have $S_2 \equiv S_3 \equiv 0 \pmod{9}$ for a 2×2 square and $S_2 = 0$ and $S_3 \equiv 6 \pmod{9}$ for horizontal domino I_2. Since the area of $\mathcal{T}$ is divisible by 6, the rectangle $a \times b$ is either $6k \times m$ or $2k \times 3m$ (with m odd). Let us consider the case $2k \times 3m$. Then we can tile the rectangle $a \times b$ by the horizontal domino I_2 using translation.

We have $f(2k \times b) = f(2k \times 1)$ because the rectangle 2×2 has the trivial boundary – it is equal to the identity element in the group $\mathbb{Z}[i]/(9)$, i.e., $f(2 \times 2) = 0 \pmod{9}$. If we form the rectangle $2k \times 1$ by adding together k copies of I_2, then we have $f(2k \times 1) \equiv 6k \pmod{9}$, so $3|k$.

Since the group $\mathbb{Z}[i]/(9)$ is abelian, the above result can be proved by a coloring. One such coloring is given in [8, Proposition 5.11].

Item (1) is proved in [8, Theorem 5.12] using a boundary invariant which maps generators of the group $\langle x, y \rangle$ into permutations of the symmetric group S_{32}.

From Figure 2.1 we see that there exist tilings of rectangles 4×6 and 5×12 by $\mathcal{T}$. As there is a tiling of rectangle 4×6, then $\mathcal{T}$ tiles any rectangle $4k \times 6\ell$. So we



(a) 4×6 (b) 5×12

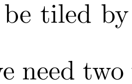
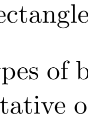
Figure 2.1: Tiling of rectangles 4×6 and 5×12.

need to check if there is a tiling of a rectangle $12k \times n$. If n is not equal to $1, 2, 3$ or 7, then this rectangle can be tiled by rectangles 4×6 and 5×12. □

In the above theorem we need two types of boundary invariants – a commutative invariant (a coloring) and a noncommutative one. The coloring can be found using Theorem 2.2. That is not a unique way to obtain a coloring. As mentioned in the proof, yet another coloring can be defined on the unit squares. This corresponds to the group $G = \mathbb{Z}[i]$ being abelian, and therefore any map defined on the edges and giving an invariant can also be given by a map defined on the unit squares. Our approach is more algorithmic – for any polyomino we can use Theorem 2.2 to compute the coloring. On the other hand, it is not obvious how to find the coloring given in [8, Proposition 5.11].

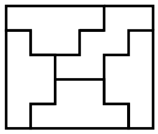
Also, notice that in the approach by Conway and Lagarias, where the boundary invariants are defined using a group on two generators, the image of the homomorphism φ can be abelian but the invariant itself is not equivalent to any coloring. It is possible if the map $f : \mathcal{P} \to G$ is not defined on the edges but only on polyominoes. An example of such an invariant is the *winding number* [1, 6].

The next result gives an example where the coloring is enough to completely solve the tiling problem. Together with Theorem 2.5 and 2.6, it improves the known results on tiling where only partial results are given – either an incomplete list of tilings or just necessary conditions without the sufficient ones.

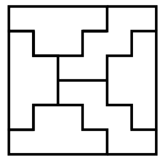
Theorem 2.4. *Let* $\mathcal{T} = \{$  ,  $\}$.

A rectangle $a \times b$ *can be tiled by the set* $\mathcal{T}$ *if and only if* a *or* b *is a multiple of* 6, *except when* $a \times b$ *is* $1 \times 6, 2 \times 6,$ *or* 3×6.

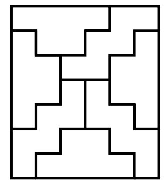
Proof. The necessity can be proved in the same way as Theorem 2.3 (b), observing that the second hexomino in the set $\mathcal{T}$ also has $S_2 \equiv S_3 \equiv 0 \pmod 9$. The sufficiency follows from the existence of tilings shown in Figure 2.1 and Figure 2.2. □



(a) 5×6



(b) 6×6



(c) 7×6

Figure 2.2: Tiling of rectangles by the set $\mathcal{T}$.

Our next example is for dekomino τ shown in Fig. 2.3. This dekomino has at least 15 prime rectangles. To remind the reader, the *prime rectangles* are those that can be tiled by the given polyomino P but cannot be composed from the smaller rectangles tileable by P [8].

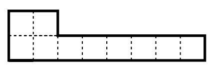


Figure 2.3: A dekomino that tiles rectangles 2×10, 11×120, 15×66 and others.

Theorem 2.5. *For dekomino τ to tile a rectangle, it is necessary that this rectangle is one of the following types:*

- $30t \times b$,
- $5k \times 6m$, *or*
- $10n \times 2q$.

If we also have one of the following:

- $b \geqslant 135$ *and* $t \geqslant 1$,
- $b \geqslant 11$ *and* $t \geqslant 4$,
- $b \geqslant 13$ *and* $t \geqslant 2$,
- $k \geqslant 27$ *and* $m \geqslant 2$,
- $2 \leqslant k \leqslant 26$, $m \geqslant 8$, $m \neq 9$,
- $k \geqslant 14$ *and* $m = 9$, *or*
- $q \geqslant 1$ *and* $n \geqslant 1$,

then the above conditions are sufficient.

Proof. We can check that $S_2 \equiv S_3 \equiv S_4 \equiv S_5 \equiv 0 \pmod{12}$ for τ. It means that the map $f : \mathcal{P} \to \mathbb{Z}[i]/(12)$ given by

$$f(\tau) = S_5 = z_1^5 - z_2^5 + z_3^5 - \ldots + (-1)^{n+1} z_n^5 \pmod{12}$$

is an invariant for τ. We also have $S_2 \equiv S_3 \equiv S_4 \equiv S_5 \equiv 0 \pmod{12}$ for the square 2×2, and $S_2 \equiv S_3 \equiv S_4 \equiv 0$, $S_5 \equiv 8 \pmod{12}$ for the rectangle 10×1, and also $S_2 \equiv S_3 \equiv S_4 \equiv 0$, $S_5 \equiv 8i \pmod{12}$ for the rectangle 5×2.

Since the area of a rectangle $a \times b$ tileable by τ is divisible by 10, we have two cases: $10k \times b$ and $5k \times 2m$.

In the first case, if b is odd, we get $f(10k \times b) = f(10k \times 1)$ because $f(10 \times 2) = f(2 \times 2) \equiv 0 \pmod{12}$. Since $f(10k \times 1) = kf(10 \times 1)$ and $f(10 \times 1) \equiv 8 \pmod{12}$, it follows that $3|k$.

In the second case, $f(5k \times 2m) = f(5 \times 2m)$ because $f(10 \times 2) \equiv 0 \pmod{12}$. Since $f(5 \times 2m) = mf(5 \times 2)$ and $f(5 \times 2) \equiv 8i \pmod{12}$, it follows that $3|m$.

For sufficiency, the reader can take a look at the web page [9] that gives a list of 15 prime rectangles for dekomino τ. It is unknown if this list is complete. That web page gives the actual tiling for the rectangle 15×66; for others only the dimensions are given. Most of the actual tilings can be found in [2]. Here is the list of prime rectangles from [9]:

- 2×10,
- $11 \times 120, 11 \times 150, 11 \times 180, 11 \times 210$,
- 12×135,
- $13 \times 60, 13 \times 90$,
- $15 \times 48, 15 \times 66, 15 \times 72, 15 \times 78, 15 \times 84, 15 \times 102$, and
- 18×75.

The rectangle $30k \times b$, $b \geqslant 135$, and the rectangle $5k \times 6m$, $k \geqslant 27$, $m \geqslant 2$, can be tiled by the rectangles 12×135, 18×75 and 2×10.

In the case of the rectangle $5k \times 6m$ with $2 \leqslant k \leqslant 26$, we can tile it when $m = 8t + d$ with $t \geqslant 1$ and $d = 0, 1, 2, \ldots, 7$ except $t = 1, d = 1$, because we have prime rectangles $15 \times 48, 15 \times 66$, $15 \times 72, 15 \times 78, 15 \times 84, 15 \times 102, 13 \times 60, 13 \times 90$ and 2×10 that give all possible residues modulo 8. We also have the rectangle 75×54, so we can tile the rectangle $5k \times 54$ for $k \geqslant 14$. □

As we can see, the coloring is enough to completely sort out all rectangles and to reduce the question to a finite number of cases. To find a complete solution, we only need to check if there is a tiling by dekomino τ for the following rectangles (it remains an open problem):

- $b \times 30$ for $b = 19, 21, \ldots, 133$, and
- $5k \times 54$ for $k = 5, 7, 9, 11, 13$.

This type of results for tiling problems is similar to the ones from [7], where finitely presented groups are used to obtain boundary invariants.

A proof for the necessary conditions for the 14-omino shown in Fig. 2.4 follows the same pattern. However, in this case we lack tilings to obtain the sufficient conditions. Only the tiling of rectangle 66×84 is known [9, 11].

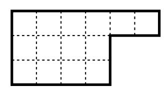


Figure 2.4: A 14-omino that tiles the rectangle 66×84.

Theorem 2.6. *If the 14-omino τ tiles a rectangle, then it must be one of the following:* $42k \times b$, $7k \times 6m$ *or* $14k \times 2m$.

Proof. We can check that $S_2 \equiv S_3 \equiv S_4 \equiv S_5 \equiv 0 \pmod{12}$ for τ. This means that the map $f : \mathcal{P} \to \mathbb{Z}[i]/(12)$ given by

$$f(\tau) = S_5 = z_1^5 - z_2^5 + z_3^5 - \ldots + (-1)^{n+1} z_n^5 \pmod{12}$$

is an invariant for τ. We also have $S_2 \equiv S_3 \equiv S_4 \equiv S_5 \equiv 0 \pmod{12}$ for the square 2×2, and $S_2 \equiv S_3 \equiv S_4 \equiv 0 \pmod{12}$, $S_5 \equiv 8 \pmod{12}$ for the rectangle 14×1. In a similar way, we have $S_2 \equiv S_3 \equiv S_4 \equiv 0 \pmod{12}$ and $S_5 \equiv 4i \pmod{12}$ for the rectangle 7×2.

As the area of a rectangle $a \times b$ tileable by τ is divisible by 14, we have two cases: $14k \times b$ and $7k \times 2m$. In the first case, if b is odd, we get $f(14k \times b) = f(14k \times 1)$ because $f(14 \times 2) = f(2 \times 2) \equiv 0 \pmod{12}$. Since $f(14k \times 1) = kf(14 \times 1)$ and $f(14 \times 1) \equiv 8 \pmod{12}$, it follows that $3|k$. In the second case, $f(7k \times 2m) = f(7 \times 2m)$ because $f(14 \times 2) \equiv 0 \pmod{12}$. Since $f(7 \times 2m) = mf(7 \times 2)$ and $f(7 \times 2) \equiv 4i \pmod{12}$, it follows that $3|m$. □

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