SOME CONGRUENCES FOR REGULAR PARTITIONS WITH DISTINCT ODD PARTS

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Abstract

Let $\mathrm{pod}_\ell(n)$ denote the number of ℓ-regular partitions of a positive integer n into distinct odd parts. In this article, we find some congruences for $\mathrm{pod}_5(n)$, $\mathrm{pod}_7(n)$, $\mathrm{pod}_9(n)$ and $\mathrm{pod}_{25}(n)$ by using the congruence properties of $t_4(n)$, $t_6(n)$, $t_8(n)$ and $t_{24}(n)$, respectively, where $t_k(n)$ is the number of representations of n as a sum of k triangular numbers.

1. Introduction

Throughout this article, we let $q = e^{2\pi i z}$ with Im $z > 0$. We also define the *q-shifted factorials* as

$$(a;q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n).$$

A special case of $(a;q)_\infty$ is given by

$$f_k = (q^k;q^k)_\infty = \prod_{n=1}^{\infty} (1 - q^{nk}), \quad \text{for any } k \geq 1.$$

Furthermore, let $t_k(n)$ denote the *number of representations* as the sum of k triangular numbers of a positive integer n. The corresponding generating function of $t_k(n)$ is given by

$$\psi^k(q) = \sum_{n=0}^{\infty} t_k(n) q^n.$$

A *partition* of a positive integer n is any nonincreasing sequence of positive integers whose sum is n. We say that a partition of n is *ℓ-regular* if none of its parts

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are divisible by ℓ. Moreover, let $\mathrm{pod}_\ell(n)$ denote the number of *ℓ-regular partitions with distinct odd parts*. The generating function of $\mathrm{pod}_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} \mathrm{pod}_\ell(n) q^n = \frac{\psi(-q^\ell)}{\psi(-q)}. \tag{1}$$

Recently, the arithmetic properties of $\mathrm{pod}_\ell(n)$ have been widely studied. See, for example, [1]-[5]. In this article, by using the congruence properties of $t_4(n)$, $t_6(n)$, $t_8(n)$ and $t_{24}(n)$, we prove some congruence relations for $\mathrm{pod}_5(n)$, $\mathrm{pod}_7(n)$, $\mathrm{pod}_9(n)$ and $\mathrm{pod}_{25}(n)$, respectively.

2. Main Results

In order to obtain the main results, we first prove the following lemmas.

Lemma 1. *For any $k \geq 0$, let p be an odd prime. Then we have the following:*

(i) *if $p \equiv 1 \pmod 3$, then*

$$t_4\left(\frac{p^k-1}{2}\right) \equiv t_8(p^k) \equiv k+1 \pmod 3; \tag{2}$$

(ii) *if $p \equiv 2 \pmod 3$, then*

$$t_4\left(\frac{p^{2k}-1}{2}\right) \equiv t_8(p^{2k}) \equiv 1 \pmod 3 \tag{3}$$

and

$$t_4\left(\frac{p^{2k+1}-1}{2}\right) \equiv t_8(p^{2k+1}) \equiv 0 \pmod 3; \tag{4}$$

(iii) *if $p \equiv 0 \pmod 3$, then*

$$t_4\left(\frac{p^k-1}{2}\right) \equiv t_8(p^k) \equiv 0 \pmod 3. \tag{5}$$

Proof. We write

$$\sigma_t(p^k) = \sum_{i=0}^{k} \left(p^t\right)^i = \frac{\left(p^t\right)^{k+1}-1}{p^t-1}.$$

If t is odd, then for $p \equiv l \pmod 3$ with $l = 0$, 1, 2, we have

$$p^t \equiv l \pmod 3.$$

Consequently, we can deduce that

$$\sigma_t(p^k) \equiv k+1 \pmod 3, \quad p \equiv 1 \pmod 3; \tag{6}$$

$$\sigma_t(p^{2k}) \equiv 0 \pmod 3, \quad \sigma_t(p^{2k+1}) \equiv 1 \pmod 3, \quad p \equiv 2 \pmod 3; \tag{7}$$

$$\sigma_t(p^k) \equiv 1 \pmod 3, \quad p \equiv 0 \pmod 3. \tag{8}$$

We make use of the following identity from [6]:

$$t_4(n) = \sigma_1(2n+1), \tag{9}$$

where $\sigma_k(n) = \sum_{d|n} d^k$ is the standard divisor function. Replacing n by $(p^k-1)/2$ in (9), we obtain

$$t_4\left(\frac{p^k-1}{2}\right) = \sigma_1(p^k). \tag{10}$$

We also use the following identity from [6]:

$$t_8(n) = \sigma_3(n) - \sigma_3\left(\frac{n}{2}\right). \tag{11}$$

Similarly, replacing n by p^k in (11), we obtain

$$t_8(p^k) = \sigma_3(p^k). \tag{12}$$

Combining (6), (7), (8), (10) and (12), we deduce (2)-(5). □

The proof of the following lemma is similar to the the proof of Lemma 1, and we omit the proof.

Lemma 2. *For any $k \geq 0$, let p be an odd prime. Then we have the following:*

(i) *if $p \equiv 1 \pmod 5$, then*

$$t_4\left(\frac{p^k-1}{2}\right) \equiv k+1 \pmod 5;$$

(ii) *if $p \equiv 2 \pmod 5$, then*

$$t_4\left(\frac{p^{4k}-1}{2}\right) \equiv 1 \pmod 5;$$

$$t_4\left(\frac{p^{4k+1}-1}{2}\right) \equiv 3 \pmod 5;$$

$$t_4\left(\frac{p^{4k+2}-1}{2}\right) \equiv 2 \pmod 5$$

and

$$t_4\left(\frac{p^{4k+3}-1}{2}\right) \equiv 0 \pmod 5.$$

(iii) *if* $p \equiv 3 \pmod 5$, *then*

$$t_4\left(\frac{p^{4k}-1}{2}\right) \equiv 1 \pmod 5;$$

$$t_4\left(\frac{p^{4k+1}-1}{2}\right) \equiv 4 \pmod 5;$$

$$t_4\left(\frac{p^{4k+2}-1}{2}\right) \equiv 3 \pmod 5$$

and

$$t_4\left(\frac{p^{4k+3}-1}{2}\right) \equiv 0 \pmod 5;$$

(iv) *if* $p \equiv 4 \pmod 5$, *then*

$$t_4\left(\frac{p^{2k}-1}{2}\right) \equiv 1 \pmod 5$$

and

$$t_4\left(\frac{p^{2k+1}-1}{2}\right) \equiv 0 \pmod 5;$$

(v) *if* $p \equiv 0 \pmod 5$, *then*

$$t_4\left(\frac{p^{k}-1}{2}\right) \equiv 1 \pmod 5.$$

Lemma 3. *For any* $k \geq 0$, *let* p *be an odd prime. The following congruences hold:*

(i) *if* $p \equiv 1 \pmod 7$, *then*

$$t_6\left(\frac{3(p^{2k}-1)}{4}\right) \equiv 2k+1 \pmod 7; \tag{13}$$

(ii) *if* $p \equiv 2,\ 5 \pmod 7$, *then*

$$t_6\left(\frac{3(p^{6k}-1)}{4}\right) \equiv 1 \pmod 7; \tag{14}$$

$$t_6\left(\frac{3(p^{6k+4}-1)}{4}\right) \equiv 5 \pmod 7 \tag{15}$$

and

$$t_6\left(\frac{3(p^{6k+2}-1)}{4}\right) \equiv 0 \pmod 7; \tag{16}$$

(iii) *if* $p \equiv 3,\ 4,\ 6 \pmod 7$*, then*

$$t_6\left(\frac{3(p^{6k}-1)}{4}\right) \equiv 1 \pmod 7; \tag{17}$$

$$t_6\left(\frac{3(p^{6k+4}-1)}{4}\right) \equiv 3 \pmod 7 \tag{18}$$

and

$$t_6\left(\frac{3(p^{6k+2}-1)}{4}\right) \equiv 0 \pmod 7; \tag{19}$$

(iv) *if* $p \equiv 0 \pmod 7$*, then*

$$t_6\left(\frac{3(p^{2k}-1)}{4}\right) \equiv 1 \pmod 7. \tag{20}$$

Proof. We use the following identity for $t_6(n)$, which was proved in [6] using the theory of elliptic functions:

$$t_6(n) = \frac{1}{8}\left(\sum_{d|4n+3,\, d\equiv 3 \pmod 4} d^2 - \sum_{d|4n+3,\, d\equiv 1 \pmod 4} d^2\right). \tag{21}$$

If p is an odd prime, then for $k \geq 0$, we have

$$p^{2k} \equiv 1 \pmod 4.$$

Consequently, replacing n by $3(p^{2k}-1)/4$ in (21), we can conclude that

$$t_6\left(\frac{3(p^{2k}-1)}{4}\right) = \sum_{i=1}^{2k} p^{2i} = \frac{p^{2(2k+1)}-1}{p^2-1}. \tag{22}$$

Hence equation (22) gives the congruences (13)-(20). □

Lemma 4. *For any* $k \geq 0$*, let* p *be an odd prime. Then we have the following:*

(i) *if* $p \equiv 1 \pmod 5$*, then*

$$t_{24}(5p^k - 3) \equiv k+1 \pmod 5; \tag{23}$$

(ii) *if* $p \equiv 2 \pmod 5$*, then*

$$t_{24}(5p^{4k} - 3) \equiv 1 \pmod 5; \tag{24}$$

$$t_{24}(5p^{4k+1} - 3) \equiv 4 \pmod 5; \tag{25}$$

$$t_{24}(5p^{4k+2} - 3) \equiv 3 \pmod 5 \tag{26}$$

and

$$t_{24}(5p^{4k+3} - 3) \equiv 0 \pmod 5; \tag{27}$$

(iii) *if* $p \equiv 3 \pmod 5$*, then*

$$t_{24}(5p^{4k}-3) \equiv 1 \pmod 5; \tag{28}$$

$$t_{24}(5p^{4k+1}-3) \equiv 3 \pmod 5; \tag{29}$$

$$t_{24}(5p^{4k+2}-3) \equiv 2 \pmod 5 \tag{30}$$

and

$$t_{24}(5p^{4k+3}-3) \equiv 0 \pmod 5; \tag{31}$$

(iv) *if* $p \equiv 4 \pmod 5$*, then*

$$t_{24}(5p^{2k}-3) \equiv 1 \pmod 5 \tag{32}$$

and

$$t_{24}(5p^{2k+1}-3) \equiv 0 \pmod 5; \tag{33}$$

(v) *if* $p \equiv 0 \pmod 5$*, then*

$$t_{24}(5p^{k}-3) \equiv 1 \pmod 5. \tag{34}$$

Proof. We also use the following identity for $t_{24}(n)$, which was proved in [6] using the theory of elliptic functions:

$$176896t_{24}(n-3) = \sigma_{11}(n) - \sigma_{11}\left(\frac{n}{2}\right) - \tau(n) - 2072\tau\left(\frac{n}{2}\right), \tag{35}$$

where $\tau(n)$ is the Ramanujan tau function. Let p be an odd prime and $k \geq 0$. Replacing n by $5p^k$ in (35), we obtain

$$t_{24}(5p^k-3) \equiv \sigma_{11}(5p^k) - \tau(5p^k) \pmod 5. \tag{36}$$

We note that

$$\sigma_{11}(5p^k) = \sum_{i=0}^{k} p^{11i} + \sum_{j=0}^{k} 5p^{11j} \equiv \sum_{i=0}^{k} p^{11i} = \frac{p^{11(k+1)}-1}{p^{11}-1} \pmod 5. \tag{37}$$

Using the fact that $\tau(5p^k) \equiv 0 \pmod 5$, we have

$$t_{24}(5p^k-3) \equiv \sigma_{11}(5p^k) \equiv \sum_{i=0}^{k} p^{11i} = \frac{p^{11(k+1)}-1}{p^{11}-1} \pmod 5. \tag{38}$$

Similarly as in the preceding discussion, Equation (38) gives the congruences (23)-(34). □

We now state and prove our main results of this article.

Theorem 1. *For any $k \geq 0$, let p be an odd prime. Then we have the following:*

(i) *if* $p \equiv 1 \pmod 3$, *then*

$$\text{pod}_9(p^k - 1) \equiv \text{pod}_9(2p^k + 1) \equiv k + 1 \pmod 3; \tag{39}$$

(ii) *if* $p \equiv 2 \pmod 3$, *then*

$$\text{pod}_9(p^{2k} - 1) \equiv \text{pod}_9(2p^{2k} + 1) \equiv 1 \pmod 3 \tag{40}$$

and

$$\text{pod}_9(p^{2k+1} - 1) \equiv \text{pod}_9(2p^{2k+1} + 1) \equiv 0 \pmod 3; \tag{41}$$

(iii) *if* $p \equiv 0 \pmod 3$, *then*

$$\text{pod}_9(p^k - 1) \equiv \text{pod}_9(2p^k + 1) \equiv 0 \pmod 3. \tag{42}$$

Proof. Setting $\ell = 9$ in (1), we obtain

$$\sum_{n=0}^{\infty} \text{pod}_9(n) q^n = \frac{\psi(-q^9)}{\psi(-q)}. \tag{43}$$

Replacing q by $-q$ in (43), we have

$$\sum_{n=0}^{\infty} \text{pod}_9(n)(-1)^n q^n = \frac{\psi(q^9)}{\psi(q)} = \frac{f_{18}^2}{f_9} \cdot \frac{f_1}{f_2^2} \equiv f_2^{16} \cdot \frac{f_1}{f_9} \pmod 3. \tag{44}$$

We make use of the following identity from [7]:

$$\frac{f_1}{f_9} = \frac{f_2 f_{12}^3}{f_4 f_6 f_{18}^2} - q \frac{f_4 f_6 f_{36}^2}{f_{12} f_{18}^3}. \tag{45}$$

By substituting (45) in (44), we can rewrite (44) as

$$\sum_{n=0}^{\infty} \text{pod}_9(n)(-1)^n q^n \equiv f_2^{16} \left(\frac{f_2 f_{12}^3}{f_4 f_6 f_{18}^2} - q \frac{f_4 f_6 f_{36}^2}{f_{12} f_{18}^3} \right) \pmod 3. \tag{46}$$

Extracting the terms involving q^{2n} in (46) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \text{pod}_9(2n) q^n \equiv \frac{f_2^{16} f_2 f_{12}^3}{f_4 f_6 f_{18}^2} \equiv \frac{f_2^8}{f_1^4} \pmod 3.$$

Equating the coefficients of q^n, we find that

$$\text{pod}_9(2n) \equiv t_4(n) \pmod 3. \tag{47}$$

Similarly, extracting the terms involving q^{2n+1} in (46), we obtain

$$-\sum_{n=0}^{\infty} \mathrm{pod}_9(2n+1)q^{2n+1} \equiv -q\frac{f_2^{16}f_4f_6f_{36}^2}{f_{12}f_{18}^3} \equiv -q\frac{f_4^{16}}{f_2^8} \pmod{3}. \tag{48}$$

Dividing (48) by $-q$ and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \mathrm{pod}_9(2n+1)q^n \equiv \frac{f_2^{16}}{f_1^8} \pmod{3}. \tag{49}$$

Equating the coefficients of q^n in (49), we find that

$$\mathrm{pod}_9(2n+1) \equiv t_8(n) \pmod{3}. \tag{50}$$

Congruences (39)-(42) follows from (47), (50) and Lemma 1. □

To prove the remaining theorems, we need to state the following congruence. If p is prime, then for $k \geq 0$, we have

$$\sum_{n=0}^{\infty} \mathrm{pod}_{p^k}(n)(-1)^n q^n = \frac{\psi(q^{p^k})}{\psi(q)} \equiv \psi^{p^k-1}(q) = \sum_{n=0}^{\infty} t_{p^k-1}(n)q^n \pmod{p},$$

which implies

$$\mathrm{pod}_{p^k}(n)(-1)^n \equiv t_{p^k-1}(n) \pmod{p}. \tag{51}$$

Combining (51) with Lemma 2, Lemma 3 and Lemma 4, we can derive the following theorems.

Theorem 2. *For any $k \geq 0$, let p be an odd prime. Then we have the following:*

(i) *if $p \equiv 1 \pmod{5}$, then*

$$\mathrm{pod}_5\left(\frac{p^k-1}{2}\right) \equiv k+1 \pmod{5};$$

(ii) *if $p \equiv 2 \pmod{5}$, then*

$$\mathrm{pod}_5\left(\frac{p^{4k}-1}{2}\right) \equiv 1 \pmod{5};$$

$$\mathrm{pod}_5\left(\frac{p^{4k+1}-1}{2}\right) \equiv 3 \pmod{5};$$

$$\mathrm{pod}_5\left(\frac{p^{4k+2}-1}{2}\right) \equiv 2 \pmod{5}$$

and

$$\mathrm{pod}_5\left(\frac{p^{4k+3}-1}{2}\right) \equiv 0 \pmod{5};$$

(iii) *if* $p \equiv 3 \pmod 5$, *then*

$$\mathrm{pod}_5\left(\frac{p^{4k}-1}{2}\right) \equiv 1 \pmod 5;$$

$$\mathrm{pod}_5\left(\frac{p^{4k+1}-1}{2}\right) \equiv 4 \pmod 5;$$

$$\mathrm{pod}_5\left(\frac{p^{4k+2}-1}{2}\right) \equiv 3 \pmod 5$$

and

$$\mathrm{pod}_5\left(\frac{p^{4k+3}-1}{2}\right) \equiv 0 \pmod 5;$$

if $p \equiv 4 \pmod 5$, *then*

$$\mathrm{pod}_5\left(\frac{p^{2k}-1}{2}\right) \equiv 1 \pmod 5$$

and

$$\mathrm{pod}_5\left(\frac{p^{2k+1}-1}{2}\right) \equiv 0 \pmod 5;$$

if $p \equiv 0 \pmod 5$, *then*

$$\mathrm{pod}_5\left(\frac{p^{k}-1}{2}\right) \equiv 1 \pmod 5.$$

Theorem 3. *For any* $k \geq 0$, *let* p *be an odd prime. Then we have the following:*

(i) *if* $p \equiv 1 \pmod 7$, *then*

$$\mathrm{pod}_7\left(\frac{3(p^{2k}-1)}{4}\right) \equiv 2k+1 \pmod 7;$$

(ii) *if* $p \equiv 2,\ 5 \pmod 7$, *then*

$$\mathrm{pod}_7\left(\frac{3(p^{6k}-1)}{4}\right) \equiv 1 \pmod 7;$$

$$\mathrm{pod}_7\left(\frac{3(p^{6k+4}-1)}{4}\right) \equiv 5 \pmod 7$$

and

$$\mathrm{pod}_7\left(\frac{3(p^{6k+2}-1)}{4}\right) \equiv 0 \pmod 7;$$

(iii) *if* $p \equiv 3,\ 4,\ 6 \pmod 7$*, then*

$$\mathrm{pod}_7\left(\frac{3(p^{6k}-1)}{4}\right) \equiv 1 \pmod 7;$$

$$\mathrm{pod}_7\left(\frac{3(p^{6k+4}-1)}{4}\right) \equiv 3 \pmod 7$$

and

$$\mathrm{pod}_7\left(\frac{3(p^{6k+2}-1)}{4}\right) \equiv 0 \pmod 7;$$

(iv) *if* $p \equiv 0 \pmod 7$*, then*

$$\mathrm{pod}_7\left(\frac{3(p^{2k}-1)}{4}\right) \equiv 1 \pmod 7.$$

Theorem 4. *For any* $k \geq 0$*, let* p *be an odd prime. Then we have the following:*

(i) *if* $p \equiv 1 \pmod 5$*, then*

$$\mathrm{pod}_{25}(5p^k - 3) \equiv k + 1 \pmod 5;$$

(ii) *if* $p \equiv 2 \pmod 5$*, then*

$$\mathrm{pod}_{25}(5p^{4k} - 3) \equiv 1 \pmod 5;$$

$$\mathrm{pod}_{25}(5p^{4k+1} - 3) \equiv 4 \pmod 5;$$

$$\mathrm{pod}_{25}(5p^{4k+2} - 3) \equiv 3 \pmod 5$$

and

$$\mathrm{pod}_{25}(5p^{4k+3} - 3) \equiv 0 \pmod 5;$$

(iii) *if* $p \equiv 3 \pmod 5$*, then*

$$\mathrm{pod}_{25}(5p^{4k} - 3) \equiv 1 \pmod 5;$$

$$\mathrm{pod}_{25}(5p^{4k+1} - 3) \equiv 3 \pmod 5;$$

$$\mathrm{pod}_{25}(5p^{4k+2} - 3) \equiv 2 \pmod 5$$

and

$$\mathrm{pod}_{25}(5p^{4k+3} - 3) \equiv 0 \pmod 5;$$

(iv) *if* $p \equiv 4 \pmod 5$*, then*

$$\mathrm{pod}_{25}(5p^{2k} - 3) \equiv 1 \pmod 5$$

and

$$\mathrm{pod}_{25}(5p^{2k+1} - 3) \equiv 0 \pmod 5;$$

(v) *if* $p \equiv 0 \pmod 5$, *then*

$$\mathrm{pod}_{25}(5p^k - 3) \equiv 1 \pmod 5.$$

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