STIRLING NUMBERS AND THE PARTITION METHOD FOR A POWER SERIES EXPANSION

Victor Kowalenko
School of Mathematics and Statistics, The University of Melbourne, Victoria, Australia

Received: 6/15/22, Revised: 3/29/23, Accepted: 10/5/23, Published: 11/6/23

Abstract

The partition method for a power series expansion is a method that utilizes standard integer partitions to evaluate the coefficients in power series expansions and generating functions. Here it is shown how an existing code based on the method can be adapted to deal with partitions homologous to standard integer partitions. Consequently, with the aid of further processing in Mathematica this work presents polynomial expressions for both kinds of Stirling numbers in two cases where: (1) the secondary variable is fixed and (2) it becomes a variable. Interestingly, the second case requires the results produced in the first case for the Stirling numbers of the first kind. In the second case the highest power in the primary variable is found to be dependent upon the secondary variable and the coefficients become polynomials in terms of this variable, whereas in the first case the coefficients are rational. The results represent a major advance on already published results of both kinds of the Stirling numbers due to the introduction of partitions into the analysis. Finally, new results for the related Worpitzky numbers and Stirling polynomials are also presented.

1. Introduction

Introduced in the 18th century, the Stirling numbers arise in an extensive number of analytic and combinatorial problems [1, 22, 30]. There are basically two sets of these numbers, known as the Stirling numbers of the first and second kinds, and they yield completely different values from each other. The Stirling numbers of the first kind are often denoted as $s(n,k)$ or $s_n^{(k)}$, while their unsigned values or magnitudes represent the number of permutations of n elements that contain exactly k cycles.

DOI: 10.5281/zenodo.10075878

The Stirling numbers of the second kind, which are often denoted by $S(n,k)$, yield the number of ways to partition n elements into k non-empty subsets or groups. Yet, despite the fact that they appear in a vast number of combinatorial problems, there are very few expressions or general formulas that yield their values. Consequently, to this day, there is very little known about the structure of both sets of numbers.

A step in this regard was accomplished by Comtet in [5, page 217], who developed polynomials where the coefficients are given by the Stirling numbers of first kind. These polynomials, which are known as Pochhammer polynomials, are discussed in Chapters 24 and 18 of [1] and [30], respectively. They are defined as

$$(y)_n = \frac{\Gamma(y+n)}{\Gamma(y)} \doteqdot (-1)^n \sum_{k=0}^{n} (-1)^k s(n,k)\, y^k. \tag{1}$$

Where required, we shall refer to n as the primary variable and k as the secondary variable. From this definition we see that $s(0,0)=1$, $s(n,0)=0$ for $n \geq 1$, and $s(n,1) = (-1)^n (n-1)!$, but beyond these results, it becomes a more formidable exercise to determine expressions or formulas for them. However, by applying Bell polynomials of the second kind [31] to the above result, Comtet derived the $k=2$ to 4 results in terms of the harmonic numbers, $H_n \doteqdot \sum_{j=1}^{n} 1/j$, which were given in their unsigned form as

$$|s(n+1,2)| = n!(1+1/2+\cdots+1/n) = \Gamma(n+1)H_n, \tag{2}$$

$$|s(n+1,3)| = n!\Big(H_n^2 - \big(1+1/2^2+\cdots+1/n^2\big)\Big)/2, \tag{3}$$

and

$$\begin{aligned} |s(n+1,4)| =& n!\Big(H_n^3 - 3H_n\left(1+1/2^2+\cdots+1/n^2\right)\Big) \\ & + 2\left(1+1/2^3+\cdots+1/n^3\right)\Big)/6. \end{aligned} \tag{4}$$

It should also be noted that the harmonic numbers are related to the digamma function $\psi(n+1)$ since $H_n = \psi(n+1)+\gamma$, where γ is the Euler-Mascheroni constant. Moreover, they have been generalized as described in [29]. The generalized harmonic numbers are defined as $H_n^{(r)} \doteqdot \sum_{j=1}^{n} 1/j^r$ and are themselves a subject of mathematical research, e.g., [4] and [8]. Consequently, the sum over inverse squares in the second result represents $H_n^{(2)}$, which are known as Wolstenholme numbers [32], while the last sum in the third result represents $H_n^{(3)}$. These results in addition to $s(n+1,5)$ and a very general formula were later obtained by applying the partition method for a power series expansion to Equation (1) as described in the appendix of [12] and also in [13]. These results will be extended here.

More recently, in Chapter 14 of [25], Quaintance and Gould consider the sum of the products of integers of the first n integers taken j at a time without repetitions or

$$S_1(n,k) \doteqdot \sum_{1\le j_1<j_2<\cdots<j_n\le n} \prod_{i=1}^{k} j_i,$$

Next they cast the above result in a symbolic computational program to obtain explicit formulas for the first seven values of k, which are listed in the chapter as polynomials of degree $2k$ with rational coefficients. Then they show that the $S_1(n,k)$ are related to the Stirling numbers of the first kind by

$$S_1(n,k) = (-1)^k s(n+1, n+1-k). \tag{5}$$

Surprisingly, they do not list the ensuing formulas for the Stirling numbers of the first kind. Unbeknownst to them, the appendix in [12] presents the following results

$$s(n,n) = 1, \quad s(n,n-1) = -\binom{n}{2}, \quad s(n,n-2) = \Big(\frac{3n-1}{4}\Big)\binom{n}{3}$$
$$s(n,n-3) = -\Big(\frac{n(n-1)}{2}\Big)\binom{n}{4},$$

and

$$s(n,n-4) = \Big(\frac{15n^3-30n^2+5n+2}{48}\Big)\binom{n}{5}. \tag{6}$$

The last of these results has a typographical error in [12], where 336 in the denominator has now been replaced by 48. These results have been obtained by expanding the left-hand side (lhs) of Equation (1) in decreasing powers of y, which yields

$$s(n,n-k) = (-1)^k \sum_{i_j=j}^{k-1} i_j \sum_{i_{j-1}=j-1}^{i_j-1} i_{j-1} \sum_{i_{j-2}=j-2}^{i_{j-1}-1} i_{j-2} \cdots \sum_{i_1=1}^{i_2-1} i_1. \tag{7}$$

Then Equation (7) is introduced into Mathematica for specific values of k. The above result is not only cumbersome for large values of k, but also does not provide information about general behaviour of k with n. Surprisingly, we shall observe that in order to determine the general k-behaviour for both kinds of Stirling numbers, we shall require the results for specific values of k.

The situation regarding Stirling numbers of the second kind is slightly better than their first kind counterparts since general forms of $S(n,k)$ for fixed (low integer) values of k can be determined by using Euler's formula as it has been referred to in [9] and [25]. This result also appears as Theorem 8.4 in [3] and is given by

$$S(k,j) = \frac{1}{j!}\sum_{i=1}^{j}(-1)^{j-i}\binom{j}{i} i^k. \tag{8}$$

In a similar manner to the Stirling numbers of the first kind, k and j will be referred to as the primary and secondary variables, respectively. In addition, in [14] a Kronecker delta was introduced into the result to ensure $S(0,0) = 1$, which is not really necessary if 0^0 is taken to be equal to unity. Hence one obtains

$$S(k,1) = 1, \quad S(k,2) = \frac{1}{2}\left(2^k - 1\right), \quad S(k,3) = \frac{1}{2}\Big(3^{k-1} - 2^k + 1\Big),$$
$$\text{and} \quad S(k,4) = \frac{1}{6}\left(4^{k-1} - 3^k + 3 \times 2^{k-1} - 1\right).$$

In the case of the Stirling numbers of the second kind, Quaintance and Gould introduce the sum, $S_2(n,k)$, also in Chapter 14 of [25], where it is defined as the sum of the products of the first n integers taken k at a time with repetitions or

$$S_2(n,k) \doteqdot \sum_{1 \le j_1 \le j_2 \le \cdots \le j_n \le n} \prod_{i=1}^{k} j_i.$$

Next they give the forms of $S_2(n,k)$ for k ranging from 1 to 6. Although they relate the Stirling numbers of the second kind to $S_2(n,k)$ by noting that $S(n,k) = S_2(k, n-k)$, they do not give any of the resulting forms for $S(n,k)$.

With the exception of Euler's formula for the Stirling numbers of the second kind, all the preceding results can be developed further by using the partition method for a power series expansion, which is described in detail in [14] and [15]. In fact, the results for both kinds of Stirling numbers have been tabulated up to $k = 10$ in the first of these references. The problem with these results is that the secondary variable has had to be fixed. Therefore, it has not been possible to obtain the general formulas for the coefficients of the polynomials for $s(n, n-k)$ and $S(n, n-k)$ as functions of both n and k. Consequently, one cannot relate the polynomials for different values of n with each other. That is, they just appear as distinct polynomials with numerical values for their coefficients without any understanding of the dependence on k. However, this situation has changed with the advent of [14]. There it has been sketched out how the first few leading order terms of $S(n, n-k)$ and $s(n, n-k)$ can be obtained in Chapters 2 and 6, respectively, via the partition method for a power series expansion. In this paper we shall extend the partition method for a power series expansion to far more orders of the above results or polynomials by adapting and developing the method further. Surprisingly, in order to obtain the general coefficients, we shall still require the fixed k values of $s(n, n-k)$ for both kinds of Stirling numbers. Furthermore, computer codes discussed in [14] will need to be modified to determine contributions due to other classes of partitions. These new codes appear in their entirety in the appendix of this paper. Moreover, as a result of the analysis on the refined rencontres numbers presented in [6], which are, in turn, related to the signless Stirling numbers of the first kind, the limited results for $s(n,k)$ appearing in Equation (2) will be developed further, thereby resulting in a comprehensive account of the structure and properties of both kinds of Stirling numbers.

2. The Partition Method for a Power Series Expansion

In order to understand the following material, it will be necessary to summarise the relevant points of the partition method for a power series expansion, which is described in great detail in Chapters 4 and 6 of [14]. Basically, we begin with two functions, $f(z)$ and $g(z)$, each of which can be expressed in terms of a power series. The power series for $f(z)$, referred to as the inner power series, is given by $\sum_{k=0}^{\infty} p_k\, y^k$, where $y = z^{\mu}$, while the outer power series for $g(z)$ is given by $h(z) \sum_{k=0}^{\infty} q_k\, z^k$, where $h(z)$ can be an arbitrary number or even a function. From the general theorem, viz. Theorem 4.1 in [14], we have

$$\frac{g\left(af(z)\right)}{h\left(af(z)\right)} \equiv \sum_{k=0}^{\infty} D_k(a)\, y^k, \tag{9}$$

where for $p_0 = 0$, the coefficients in Result (9) are given by

$$D_k(a) = L_{P,k}\left[q_{l(\boldsymbol{\lambda}_k)} a^{l(\boldsymbol{\lambda}_k)} l(\boldsymbol{\lambda}_k)! \prod_{i=1}^{k} \frac{p_i^{\lambda_i}}{\lambda_i!} \right]. \tag{10}$$

Representing the sum over all partitions summing to k, the partition operator $L_{P,k}\Big[\cdot\Big]$ in Equation (10) is defined as

$$L_{P,k}\Big[\cdot\Big] \doteqdot \sum_{\boldsymbol{\lambda}_k} = \sum_{\substack{i_1, i_2, i_3, \dots, i_k = 0 \\ \sum_{i=1}^{k} i\lambda_i = k}}^{k, \lfloor k/2 \rfloor, \lfloor k/3 \rfloor, \dots, 1} \Big(\cdot\Big), \tag{11}$$

where $\lfloor\ \rfloor$ represents the floor function, $\boldsymbol{\lambda}_k$ represents a partition summing to k and λ_i, the multiplicity or number of occurrences of the part i in a partition. The total number of parts or length of each partition, $l(\boldsymbol{\lambda}_k)$ or N_k in [14], is equal to the sum of the multiplicities, i.e. $l(\boldsymbol{\lambda}_k) = \sum_{i=1}^{k} \lambda_i$. The length ranges from unity corresponding to the one-part partition, $\{k\}$, to k, corresponding to the partition with k ones, which is denoted here by $\{1_k\}$. That is, it should be stated that in this work partitions will be expressed in terms of the shorthand notation of $\{1_{\lambda_1}, 2_{\lambda_2}, 3_{\lambda_3}, \cdots, k_{\lambda_k}\}$ for each $\lambda_k > 1$. For example, in accordance with this notation, the partition $\{1,1,1,1,2,2,3,6,6\}$ is represented as $\{1_4, 2_2, 3, 6_2\}$.

As in the case of Taylor series expansions, the power series in Result (9) need not be convergent in specific sectors of the complex plane. Nor do both the inner and outer power series need to be absolutely convergent. However, to compensate for these cases, an equivalence symbol appears in the result rather than an equals sign. From here on, Result (9) will be referred to as an equivalence statement, not an equation. When it is established that a result is convergent over a sector of the complex plane, the equivalence symbol can be replaced by an equals sign. Then we have an equation for that sector. This will become clearer later because in order to derive new results from Equivalence (9), we will require equals signs or equations in regions of validity.

According to [5,page 133] and [26], the Bell polynomials of the first kind are defined by the generating function of the exponential, $\exp\left(x(e^t-1)\right)$. That is,

$$\sum_{k=0} \mathcal{B}_k(x)\,\frac{t^k}{k!} = \exp\Big(x\left(e^t-1\right)\Big), \tag{12}$$

where the notation $\mathcal{B}_k(x)$ has been used instead of $B_k(x)$ to avoid confusion with the more famous Bernoulli polynomials. Expanding the exponential on the right-hand side (rhs) of Equation (12) yields

$$\sum_{k=0} \mathcal{B}_k(x)\,\frac{t^k}{k!} = \sum_{k=0}^{\infty}\big(t+t^2/2!+t^3/3!+t^4/4!+\cdots\big)^k\,\frac{x^k}{k!}. \tag{13}$$

In this situation the powers of t represent the the inner series with $p_k = 1/k!$, while the powers of x represent the outer series with $q_k = 1/k!$. Except for $p_0 = 0$, this is an unusual example as the coefficients of both the inner and outer power series are identical. If we introduce the forms for p_k and q_k into Equation (10) together with $a = x$, then by equating the $D_k(a)$ with the Bell polynomials on the lhs of Equation (13), we arrive at

$$\mathcal{B}_k(x) = k!\,L_{P,k}\left[x^{l(\boldsymbol{\lambda}_k)}\prod_{i=1}^{k}\frac{1}{i!^{\lambda_i}\lambda_i!}\right]. \tag{14}$$

Thus, we observe that since $l(\boldsymbol{\lambda}_k)$ ranges from unity to k, the $D_k(x)$ represent polynomials in x of degree k. Moreover, by setting $x = 1$, we obtain the Bell numbers, which can be expressed in terms of the Stirling numbers of the second kind as

$$\mathcal{B}_k = \sum_{j=1}^{k} S(k,j).$$

From these results one can see that the Stirling numbers of the second kind $S(k,j)$ represent the coefficient of the j-th power in Equation (14). In other words, $S(k,j)$ represents the value obtained by summing all the quantities in Equation (14) when the length or total number of parts, i.e., $l(\boldsymbol{\lambda}_k)$, is only equal to j. This means that the partition operator must be modified so that it evaluates the contributions from the partitions summing to k with only j parts. To accomplish this, we introduce an extra constraint or restriction, $\sum_{i=1}^{k} \lambda_i = j$, into the partition operator. Hence we have the following operator:

$$L_{P,k}^{(j)}\Big[\cdot\Big] \doteqdot \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k=0 \\ \sum_{i=1}^{k} i\lambda_i=k,\ \sum_{i=1}^{k}\lambda_i=j}}^{k,\lfloor k/2\rfloor,\lfloor k/3\rfloor,\ldots,1} \Big(\cdot\Big), \tag{15}$$

Note that summing this operator from $j=1$ to k produces the partition operator. In the appendix we present the program **numparts** which is virtually a C-coded version of the above operator in that it only considers partitions with j parts. Thus, j must be specified by the user as input in order to run the code. As a result of Definition (15), the Stirling numbers of the second kind can be expressed as

$$S(k,j) = k!\, L_{P,k}^{(j)}\left[\prod_{i=1}^{k} \frac{1}{i!^{\lambda_i}\lambda_i!}\right]. \tag{16}$$

It should be mentioned here that the Worpitzky numbers [25, 35] are also related to the Stirling numbers of the second kind. These numbers, which are denoted by $W_{j,m}^{(n)}$, are defined as

$$W_{j,m}^{(n)} \doteqdot \sum_{k=0}^{j} (-1)^k \binom{m}{k} (j-k)^n.$$

In addition, they satisfy the following recurrence relations:

$$W_{j,m+1}^{(n)} - (-1)^{m+n} W_{m-j+1,m+1}^{(n)} = \begin{cases} 0, & m \geq n \geq 1, \\ (-1)^j \binom{m+1}{j}, & m \geq n,\ n=0, \end{cases}$$

$$W_{j,m+1}^{(n+1)} = (m-j+1) W_{j-1,m}^{(n)} + j W_{j,m}^{(n)}, \quad m \geq n+1 \geq 1,$$

and

$$W_{j,m+1}^{(n)} = W_{j,m}^{(n)} - W_{j-1,m}^{(n)}.$$

The first two relations are derived in Chapter 11 of [25], while the last one is obtained by introducing the following identity into Equation (2):

$$\binom{m}{k} = \binom{m+1}{k} - \binom{m}{k-1}.$$

From Equation (8), we find that

$$W_{j,j}^{(n)} = j! S(n,j). \tag{17}$$

Therefore, if expressions for the Stirling numbers of the second kind can be determined, then it follows that they will yield expressions for the special case of

Worpitzky numbers, where j and m are equal to each other. In addition, by using Equation (16), we arrive at

$$W_{j,j}^{(k)} = j!k!\, L_{P,k}^{(j)} \left[\prod_{i=1}^{k} \frac{1}{i!^{\lambda_i} \lambda_i!} \cdot \right]. \tag{18}$$

The strength of the partition method for a power series expansion is its ability to yield power series expansions for intractable problems where standard methods such as Taylor series break down. Even when the power series is known, the method can offer a different perspective and produce new results. A typical example is the binomial theorem, which can be expressed as

$$\sum_{k=0}^{\infty} \frac{\Gamma(k-a)}{\Gamma(-a)\, k!} (-z)^k \equiv (z+1)^a. \tag{19}$$

In Appendix A of [14] it is shown that the binomial series is absolutely convergent for $|z| < 1$, and conditionally convergent for $\Re z > -1$ and $|z| > 1$. For these values of z, the equivalence symbol in Equivalence (19) can be replaced by an equals sign. That is, we have an equation. For the remaining values of z, the equivalence statement is divergent with the rhs representing the regularized value of the series. On the other hand, the rhs of Equivalence (19) can be written as

$$\exp\bigl(a \ln(1+z)\bigr) = \sum_{k=0}^{\infty} \Bigl(z - z^2/2 + z^3/3 + \cdots\Bigr)^k \frac{a^k}{k!}. \tag{20}$$

In obtaining this result, $\ln(1+z)$ has been replaced by its Taylor series expansion, which is again absolutely convergent for $|z| < 1$, conditionally convergent for $\Re z > -1$ and $|z| > 1$, and divergent elsewhere. This is discussed at length in [16]. Therefore, the rhs of Equation (20) can be expressed in the form of Equivalence (9), where according to Equation (10), the coefficients $D_k(a)$ are given by

$$D_k(a) = (-1)^k\, L_{P,k} \left[(-a)^{l(\boldsymbol{\lambda}_k)} \prod_{i=1}^{k} \frac{1}{i^{\lambda_i} \lambda_i!} \right].$$

This is a similar situation to the Stirling numbers of the second kind except the coefficients of the inner series are now given by $p_k = (-1)^{k+1}/k$ as opposed to $1/k!$. For $\Re z > -1$, we can replace the equivalence symbol by an equals sign in Equivalence (19) and then equate Equivalence (19) with the resulting form of Equivalence (9) with $y = z$. Hence we arrive at

$$\sum_{k=0}^{\infty} \frac{\Gamma(k-a)}{\Gamma(-a)\, k!} (-z)^k = \sum_{k=0}^{\infty} (-1)^k\, L_{P,k} \left[(-a)^{l(\boldsymbol{\lambda})_k} \prod_{i=1}^{k} \frac{1}{i^{\lambda_i} \lambda_i!} \right] z^k.$$

Since z is still fairly arbitrary or admits an infinite number of solutions to the above equation, we can equate like powers on both sides of Equivalence (19), which, in turn, yields

$$L_{P,k}\left[(-a)^{l(\boldsymbol{\lambda}_k)}\prod_{i=1}^{k}\frac{1}{i^{\lambda_i}\,\lambda_i!}\right]=\frac{\Gamma(k-a)}{\Gamma(-a)\,k!}. \tag{21}$$

If we introduce Equation (1) into the rhs of Equation (21) and fix the partition length to j, then we can equate like powers of a, thereby obtaining

$$L^{(j)}_{P,k}\left[\prod_{i=1}^{k}\frac{1}{i^{\lambda_i}\,\lambda_i!}\right]=\frac{(-1)^{k+j}}{k!}\,s(k,j). \tag{22}$$

By comparing the above result with the corresponding result for the Stirling numbers of the second kind, viz. Equation (16), we see that aside from the phase factor of $(-1)^{k+j}$, the main difference between them is that there is no factorial on the parts inside the product for the Stirling numbers of the first kind. This means that the analysis for one kind of the Stirling numbers will be similar to that for the other kind, even though the final values will be vastly different from each other. Another interesting property of both results is that j must be less than or equal to k or both kinds of numbers do not exist.

To conclude this section, we now discuss the Stirling polynomials [7, 27], which are related to both kinds of Stirling numbers. The Stirling polynomials $S_n(x)$ are defined in terms of the following generating function:

$$\sum_{n=0}^{\infty} S_n(x)\,\frac{t^n}{n!}\equiv\Bigl(\frac{t}{1-e^{-t}}\Bigr)^{x+1}. \tag{23}$$

Note the appearance of the equivalence symbol because as we shall see, the lhs can become divergent, while the rhs is always convergent. We now apply Theorem 4.1 from [14] by expressing the rhs as

$$\Bigl(\frac{t}{1-e^{-t}}\Bigr)^{x+1}=\Bigl(\sum_{n=1}^{\infty}\frac{(-t)^{n-1}}{n!}\Bigr)^{-x-1}. \tag{24}$$

In this instance the coefficients of the inner series or the bracketed series in Equation (24) are given by $p_k=(-1)^k/(k+1)!$. Furthermore, the series is convergent for all values of t. However, the outer series becomes the binomial series, which as stated earlier can become divergent. Hence the equivalence symbol appears in Equivalence (23). Thus, the coefficients of the outer series are given by $q_k=(-1)^k(x+1)_k/k!$, where $(x)_k$ represents the Pochhammer notation for $\Gamma(k+x)/\Gamma(x)$. By applying Theorem 4.1 of [14] with $a=1$ to the above, we arrive at

$$\sum_{k=0}^{\infty} D_k\,t^k\equiv\Bigl(\frac{t}{1-e^{-t}}\Bigr)^{x+1}. \tag{25}$$

where

$$D_k = (-1)^k L_{P,k} \left[(-1)^{l(\boldsymbol{\lambda}_k)} (x+1)_{l(\boldsymbol{\lambda}_k)} \prod_{i=1}^{k} \frac{1}{(i+1)!^{\lambda_i} \lambda_i!} \right].$$

Since we know that there is a region in the complex plane or an infinite number of values of t, where the equivalence symbols in Equivalences (23) and (25) can be replaced by an equals sign, we can set the lhs's of both statements equal to one another for these values of t. Then we find that

$$\sum_{k=0}^{\infty} S_k(x) \frac{t^k}{k!} = \sum_{k=0}^{\infty} D_k t^k. \tag{26}$$

Moreover, since t is fairly arbitrary, we can equate like powers on both sides of Equation (26), thereby yielding

$$S_k(x) = (-1)^k k!\, L_{P,k} \left[(-1)^{l(\boldsymbol{\lambda}_k)} (x+1)_{l(\boldsymbol{\lambda}_k)} \prod_{i=1}^{k} \frac{1}{(i+1)!^{\lambda_i} \lambda_i!} \right]. \tag{27}$$

This result yields the values of the Stirling polynomials listed in [33].

According to [27], the Stirling polynomials are related to the Stirling numbers of the first kind by

$$S_k(m) = (-1)^k \binom{m}{k}^{-1} s(m+1, m-k+1), \tag{28}$$

for $m \geq k$. Thus, with the aid of Equation (27) we arrive at

$$s(m-k+1, m+1) = \frac{1}{(m-k)!} L_{P,k} \left[(-1)^{l(\boldsymbol{\lambda}_k)} \big(m + l(\boldsymbol{\lambda}_k)\big)! \prod_{i=1}^{k} \frac{1}{(i+1)!^{\lambda_i} \lambda_i!} \right].$$

On the other hand, in [20] the Stirling polynomials are found to be related to the Stirling numbers of the second kind by

$$S(k+n, n) = (-1)^k \binom{k+n}{n} S_k(-n-1). \tag{29}$$

Then from Equation (26), we find that

$$S(k+n, n) = (n+1)_k L_{P,k} \left[(-1)^{l(\boldsymbol{\lambda}_k)} (-n)_{l(\boldsymbol{\lambda}_k)} \prod_{i=1}^{k} \frac{1}{(i+1)!^{\lambda_i} \lambda_i!} \right].$$

3. Stirling Numbers of the First Kind

In the previous section we presented the two main results for both kinds of the Stirling numbers via the application of the partition method for a power series expansion. In the next two sections we shall reveal the large amount of information that these results possess. Before doing so, however, we shall indicate how the partition method for a power series expansion can be applied to the Pochhammer polynomials [30], thereby yielding the results below Equation (1). As mentioned previously, these results have been obtained by Comtet by using Bell polynomials of the second kind. By applying the partition method for a power series expansion, however, we shall be able to tabulate far more results than those presented in the previous section.

It should also be pointed out that while Comtet's approach and the partition method for a power series both utilize integer partitions, they are, in fact, quite different. In particular, Chapter 4 of [14] describes how the partition method for a power series is actually the more general approach since the expressions that Comtet uses, viz., the Bell polynomials of the second kind or partial Bell polynomials as he refers to them, can be derived as a special case of Theorem 4.1.

To obtain the forms for the Stirling numbers of the first kind in terms of the generalized harmonic numbers, we begin by exponentiating the logarithm of the lhs of Equation (1), which gives

$$\begin{aligned}(y)_n = y\,\exp\Big(&\ln(1+y) + \ln 2 + \ln(1+y/2) + \ln 3 + \ln(1+y/3)\cdots + \ln(n-1)\\ &+\ \ln\big(1+y/(n-1)\big)\Big).\end{aligned} \tag{30}$$

Next we expand each logarithm as a Taylor series. From [16], the Taylor series expansion for $\ln(1+z)$ is absolutely convergent for $|z|<1$ and conditionally convergent for $|z|>1$ and $\Re z > -1$. For the other values of z, it is divergent. Because the first logarithm in the above result possesses the most restrictive domain of convergence, it determines the convergence conditions for the exponential when all the logarithms are expanded into their Taylor series. Thus, we find that

$$(y)_n \equiv y\,\Gamma(n)\exp\Big(H_{n-1}\,y - H_{n-1}^{(2)}\,\frac{y^2}{2} + H_{n-1}^{(3)}\,\frac{y^3}{3} - \cdots\Big). \tag{31}$$

where the generalized harmonic numbers $H_{n-1}^{(r)}$ are defined below (4) in the introduction. Note that the introduction of the Taylor series expansion for $\ln(1+z)$ has resulted in Equation (30) becoming an equivalence statement.

The rhs of Equivalence (31) is now in a form resembling Equivalence (9) except that the coefficients of y^k are now $D_{k-1}(a)$ because of the external factor of y. Nevertheless, the coefficients of the inner series, given earlier by p_k, are equal to $(-1)^{k+1}H_{n-1}^{(k)}/k$ and those for the outer series are again given by $q_k = 1/k!$. Since

$a = 1$, the coefficients of y^k are given by

$$D_{k-1} = (-1)^{k-1} L_{P,k-1} \left[(-1)^{l(\boldsymbol{\lambda}_{k-1})} \prod_{i=1}^{k-1} \frac{1}{\lambda_i!} \Big(\frac{H_{n-1}^{(i)}}{i} \Big)^{\lambda_i} \right],$$

while Equivalence (31) reduces to

$$(y)_n \equiv \sum_{k=1}^{n} \Gamma(n)\, D_{k-1}\, y^k. \tag{32}$$

Ultimately, the aim is to equate Equivalence (32) with Equation (1) in order to derive a formula for the Stirling numbers of the first kind, but this is not possible since Equivalence (32) is not an equation That is, Equivalence (32) can become divergent. Hence we need to establish the region or values of y where Equivalence (32) is convergent. Then both statements can be equated to one another in that region. By introducing Equation (32) into Equivalence (32), one obtains a series with the following general form:

$$\mathcal{S}_{k-1}(y) \equiv \sum_{k=1}^{\infty} (-y)^k \prod_{i=1}^{k-1} \frac{1}{\lambda_i! i^{\lambda_i}} \left(H_{n-1}^{(i)} \right)^{\lambda_i}.$$

The series can be bounded by noting that

$$\mathcal{S}_{k-1}(y) \leq \sum_{k=1}^{\infty} |y|^k \prod_{i=1}^{k-1} \frac{1}{\lambda_i! i^{\lambda_i}} \left(H_{n-1}^{(i)} \right)^{\lambda_i}.$$

Moreover, $H_{n-1}^{(i)} \leq H_{n-1}$ for all i. Hence the above inequality can be expressed more simply as

$$\mathcal{S}_{k-1}(y) \leq \sum_{k=1}^{\infty} |y|^k \prod_{i=1}^{k-1} \frac{1}{\lambda_i! i^{\lambda_i}} \left(H_{n-1} \right)^{\lambda_i} \leq \sum_{k=1}^{\infty} |y|^k H_{n-1}^{k-1}.$$

The last series in the above result is effectively the geometric series with the variable equal to $|y|H_{n-1}$, which according to [14, 16, 17, 18], is absolutely convergent for $|yH_{n-1}| < 1$ and conditionally convergent for $|yH_{n-1}| > 1$ and $|\Re\, yH_{n-1}| < 1$. Hence $\mathcal{S}_{k-1}(y)$ is absolutely convergent for $|y| < 1/H_{n-1}$. This means that we can replace the equivalence symbol in Equivalence (32) by an equals sign at least for these values of y. Then the resulting equation can be equated to Equation (1). Since y is still fairly arbitrary, like powers of y can be equated with each other on both sides of the resulting equation, thereby yielding

$$s(n,k) = (-1)^{n+1} \frac{\Gamma(n)}{\Gamma(k)} L_{P,k-1} \left[(-1)^{l(\boldsymbol{\lambda}_{k-1})} \prod_{i=1}^{k-1} \frac{\Gamma(k)}{\lambda_i!} \Big(\frac{H_{n-1}^{(i)}}{i} \Big)^{\lambda_i} \right]. \tag{33}$$

$\boldsymbol{\lambda}_6$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	$l(\boldsymbol{\lambda}_6)$	$T(\boldsymbol{\lambda}_6)$
$\{1_6\}$	6						6	1
$\{1_4, 2\}$	4	1					5	15
$\{1_3, 3\}$	3		1				4	40
$\{1_2, 2_2\}$	2	2					4	45
$\{1_2, 4\}$	2			1			3	90
$\{1, 2, 3\}$	1	1	1				3	120
$\{2_3\}$		3					3	15
$\{1, 5\}$	1				1		2	144
$\{2, 4\}$		1		1			2	90
$\{3_2\}$			2				2	40
$\{6\}$						1	1	120

Table 1: Partitions summing to 6 with their multiplicities, lengths and refined rencontres numbers.

Thus, we see that $s(n, k)$ is determined by the contributions from the partitions summing to the secondary variable, or more specifically, $k - 1$.

To develop an understanding of Equation (33), let us consider the evaluation of $s(n, 7)$, which requires calculating the contributions made by each partition summing to 6. Table 1 presents the 11 partitions summing to 6 together with their multiplicities and lengths. The final column gives the values of $T(\boldsymbol{\lambda}_k) = \prod_{i=1}^{k} k!/(i^{\lambda_i} \lambda_i!)$. When one types these numbers for $k = 1$ to 4 into the online encyclopedia of integer sequences, one is referred to the sequence A181897, which are known as the triangle of rencontres numbers [23]. Interestingly, these numbers appear in connection with the higher degree symmetric polynomials derived by summing over the entire sequence of quadratic or square powers of integers [6]. Furthermore, from Corollary 12.1 in [3], the number of permutations of k elements that can be decomposed into j cycles yields the signless Stirling numbers of the first kind or $|s(k, j)|$. Thus, if there is only one partition with j parts, then $s(n, k) = T(\boldsymbol{\lambda}_k)$. Unfortunately, there are only a few instances where there is a single partition for a fixed number of parts. These are: (1) the partitions composed of only ones as displayed in the second row of Table 1, (2) the partitions with a single part as at the bottom of the table, and (3) the partition with one two and the remaining parts equal to unity, which is the only partition with five parts as displayed in the third row of the table.

Alternatively, if we sum the values of $T(\boldsymbol{\lambda}_k)$, where the lengths of the partitions are set to a fixed value j, then we find that $|s(k, j)| = \sum_{l(\boldsymbol{\lambda}_k)=j} T(\boldsymbol{\lambda}_k)$. From Table 1 we observe that there are two partitions with four parts and three partitions with three parts. Combining the values of $T(\boldsymbol{\lambda}_6)$ for $l(\boldsymbol{\lambda}_6) = 4$, i.e., $\boldsymbol{\lambda}_6$ is set equal to the partitions $\{1_3,3\}$ and $\{1_2,2_2\}$ yields a value of 85 (40+45), which is indeed equal to $|s(6,4)|$, while doing the same whenever $l(\boldsymbol{\lambda}_6)=3$ yields 225 (90+120+15) or $|s(6,3)|$.

Returning to Equation (33), we note that to obtain $s(n,k)$, we require the partitions summing to $k-1$. That is, the value of n in the refined rencontres numbers is equal to $k-1$, while n only appears in the generalized harmonic numbers. As a result, the refined rencontres numbers become the coefficients of $\prod_{i=1}^{k-1}\big(H_{n-1}^{(i)}\big)^{\lambda_i}$ for each partition. Hence Equation (33) reduces to

$$s(n,k)=(-1)^{n+1}\frac{\Gamma(n)}{\Gamma(k)}\sum_{\boldsymbol{\lambda}_{k-1}}(-1)^{l(\boldsymbol{\lambda}_{k-1})}T(\boldsymbol{\lambda}_{k-1})\prod_{i=1}^{k-1}\big(H_{n-1}^{(i)}\big)^{\lambda_i}. \tag{34}$$

For $k=7$ and using the results in Table 1, we find that Equation (34) yields

$$\begin{aligned}s(n,7)=&(-1)^{n+1}\frac{\Gamma(n)}{\Gamma(7)}\Big(H_{n-1}^6-15H_{n-1}^4H_{n-1}^{(2)}+40H_{n-1}^3H_{n-1}^{(3)}+45H_{n-1}^2(H_{n-1}^{(2)})^2\\&-\;90H_{n-1}^2H_{n-1}^{(4)}-120H_{n-1}H_{n-1}^{(2)}H_{n-1}^{(3)}-15(H_{n-1}^{(2)})^3+144H_{n-1}H_{n-1}^{(5)}\\&+\;90H_{n-1}^{(2)}H_{n-1}^{(4)}+40(H_{n-1}^{(3)})^2-120H_{n-1}^{(6)}\Big).\end{aligned}\tag{35}$$

The above result appears as the $k=7$ result in Table 2, which displays the first ten formulas of $s(n,k)$ from Equation (33). Note that the number of terms in the polynomials is equal to $p(k)$, where $p(k)$ represents the number of partitions summing to k. They can also be obtained via the PartitionsP[k] routine in Mathematica. Moreover, summing the powers and orders of the harmonic numbers yields $k-1$, which can also serve a check on the validity of the results in Table 2.

The results in Table 2 can be introduced into Mathematica to yield the values of $s(n,k)$ for $k\le 10$. All that is required is to replace $H_{n-1}^{(k)}$ in the tabulated results by the routine, HarmonicNumber[n-1,k], in the software package. Then the major problem becomes determining the refined rencontres numbers, especially for large values of k where it is no longer feasible to evaluate them following the approach of Table 1. Although not necessary, it is better to list these numbers in the same order as they appear in the table because it would not be possible to identify them as the refined rencontres numbers. Another advantage of this ordering is that the partitions with the same length appear in groups or clusters and are, thus, more easily combined to yield the signless Stirling numbers of the first kind.

The results for specific values of n and k in $s(n,k)$ can be also obtained by deriving the exponential complete Bell polynomials, Y_n, from the exponential, $\exp\big(\sum_{k=1}^{\infty}x_k t_k/k!\big)$. These polynomials are discussed in Chapter 3.3 of [5] and Chapter 2.8 of [26] where they are derived in terms of the partition operator. In actual fact, the complete Bell polynomials represent a special case of the partition method for a power series expansion as discussed in Chapter 4 of [14]. Therefore, we

k	$s(n,k)$
1	$(-1)^{n+1}(n-1)!$
2	$(-1)^{n}(n-1)!H_{n-1}$
3	$(-1)^{n+1}(n-1)!\left(H_{n-1}^{2}-H_{n-1}^{(2)}\right)/2$
4	$(-1)^{n}(n-1)!\left(H_{n-1}^{3}-3H_{n-1}H_{n-1}^{(2)}+2H_{n-1}^{(3)}\right)/6$
5	$(-1)^{n+1}(n-1)!\left(H_{n-1}^{4}-6H_{n-1}^{2}H_{n-1}^{(2)}+8H_{n-1}H_{n-1}^{(3)}+3(H_{n-1}^{(2)})^{2}-6H_{n-1}^{(4)}\right)/24$
6	$(-1)^{n}(n-1)!\Big(H_{n-1}^{5}-10H_{n-1}^{3}H_{n-1}^{(2)}+20H_{n-1}^{2}H_{n-1}^{(3)}+15H_{n-1}(H_{n-1}^{(2)})^{2}$ $-30H_{n-1}H_{n-1}^{(4)}-20H_{n-1}^{(2)}H_{n-1}^{(3)}+24H_{n-1}^{(5)}\Big)/5!$
7	$(-1)^{n+1}(n-1)!\Big(H_{n-1}^{6}-15H_{n-1}^{4}H_{n-1}^{(2)}+40H_{n-1}^{3}H_{n-1}^{(3)}+45H_{n-1}^{2}(H_{n-1}^{(2)})^{2}$ $-90H_{n-1}^{2}H_{n-1}^{(4)}-120H_{n-1}H_{n-1}^{(2)}H_{n-1}^{(3)}-15(H_{n-1}^{(2)})^{3}+144H_{n-1}H_{n-1}^{(5)}$ $+90H_{n-1}^{(2)}H_{n-1}^{(4)}+40(H_{n-1}^{(3)})^{2}-120H_{n-1}^{(6)}\Big)/6!$
8	$(-1)^{n}(n-1)!\Big(H_{n-1}^{7}-21H_{n-1}^{5}H_{n-1}^{(2)}+70H_{n-1}^{4}H_{n-1}^{(3)}+105H_{n-1}^{(3)}(H_{n-1}^{(2)})^{2}$ $-210H_{n-1}^{3}H_{n-1}^{(4)}-420H_{n-1}^{2}H_{n-1}^{(2)}H_{n-1}^{(3)}-105H_{n-1}(H_{n-1}^{(2)})^{3}+504H_{n-1}^{2}H_{n-1}^{(5)}$ $+630H_{n-1}H_{n-1}^{(2)}H_{n-1}^{(4)}+280H_{n-1}(H_{n-1}^{(3)})^{2}+210(H_{n-1}^{(2)})^{2}H_{n-1}^{(3)}-840H_{n-1}H_{n-1}^{(6)}$ $-504H_{n-1}^{(2)}H_{n-1}^{(5)}-420H_{n-1}^{(3)}H_{n-1}^{(4)}+720H_{n-1}^{(7)}\Big)/7!$
9	$(-1)^{n+1}(n-1)!\Big(H_{n-1}^{8}-28H_{n-1}^{6}H_{n-1}^{(2)}+112H_{n-1}^{5}H_{n-1}^{(3)}+210H_{n-1}^{4}(H_{n-1}^{(2)})^{2}$ $-420H_{n-1}^{4}H_{n-1}^{(4)}-1120H_{n-1}^{3}H_{n-1}^{(2)}H_{n-1}^{(3)}-420H_{n-1}^{2}(H_{n-1}^{(2)})^{3}+1344H_{n-1}^{3}H_{n-1}^{(5)}$ $+2520H_{n-1}^{2}H_{n-1}^{(2)}H_{n-1}^{(4)}+1120H_{n-1}^{2}(H_{n-1}^{(3)})^{2}+1680H_{n-1}(H_{n-1}^{(2)})^{2}H_{n-1}^{(3)}$ $+105(H_{n-1}^{(2)})^{4}-3360H_{n-1}^{2}H_{n-1}^{(6)}-4032H_{n-1}H_{n-1}^{(2)}H_{n-1}^{(5)}-3360H_{n-1}H_{n-1}^{(3)}H_{n-1}^{(4)}$ $-1260(H_{n-1}^{(2)})^{2}H_{n-1}^{(4)}-1120H_{n-1}^{(2)}(H_{n-1}^{(3)})^{2}+5760H_{n-1}H_{n-1}^{(7)}+3360H_{n-1}^{(2)}H_{n-1}^{(6)}$ $+2688H_{n-1}^{(3)}H_{n-1}^{(5)}+1260(H_{n-1}^{(4)})^{2}-5040H_{n-1}^{(8)}\Big)/8!$
10	$(-1)^{n}(n-1)!\Big(H_{n-1}^{9}-36H_{n-1}^{7}H_{n-1}^{(2)}+168H_{n-1}^{6}H_{n-1}^{(3)}+378H_{n-1}^{5}(H_{n-1}^{(2)})^{2}$ $-756H_{n-1}^{5}H_{n-1}^{(4)}-2520H_{n-1}^{4}H_{n-1}^{(2)}H_{n-1}^{(3)}-1260H_{n-1}^{3}(H_{n-1}^{(2)})^{3}+3024H_{n-1}^{4}H_{n-1}^{(5)}$ $+7560H_{n-1}^{3}H_{n-1}^{(2)}H_{n-1}^{(4)}+3360H_{n-1}^{3}(H_{n-1}^{(3)})^{2}+7560H_{n-1}^{2}(H_{n-1}^{(2)})^{2}H_{n-1}^{(3)}$ $+945H_{n-1}(H_{n-1}^{(2)})^{4}-10080H_{n-1}^{3}H_{n-1}^{(6)}-18144H_{n-1}^{2}H_{n-1}^{(2)}H_{n-1}^{(5)}$ $-15120H_{n-1}^{2}H_{n-1}^{(3)}H_{n-1}^{(4)}-11340H_{n-1}(H_{n-1}^{(2)})^{2}H_{n-1}^{(4)}-10080H_{n-1}H_{n-1}^{(2)}(H_{n-1}^{(3)})^{2}$ $-2520(H_{n-1}^{(2)})^{3}H_{n-1}^{(3)}+25920H_{n-1}^{2}H_{n-1}^{(7)}+30420H_{n-1}H_{n-1}^{(2)}H_{n-1}^{(6)}$ $+24192H_{n-1}H_{n-1}^{(3)}H_{n-1}^{(5)}+11340H_{n-1}(H_{n-1}^{(4)})^{2}+9072(H_{n-1}^{(2)})^{2}H_{n-1}^{(5)}$ $+15120H_{n-1}^{(2)}H_{n-1}^{(3)}H_{n-1}^{(4)}+2240(H_{n-1}^{(3)})^{3}-45360H_{n-1}H_{n-1}^{(8)}$ $-25920H_{n-1}^{(2)}H_{n-1}^{(7)}-20160H_{n-1}^{(3)}H_{n-1}^{(6)}-18144H_{n-1}^{(4)}H_{n-1}^{(5)}+40320H_{n-1}^{(9)}\Big)/9!$

Table 2: The first 10 expressions for the Stirling numbers of the first kind in terms of the generalized harmonic numbers.

can apply Equation (10) to the exponential to determine the exponential complete Bell polynomials, which are given by

$$\exp\Bigl(\sum_{k=1}^{\infty} x_k\,\frac{t^k}{k!}\Bigr) = \sum_{k=0}^{\infty} Y_k(x_1, x_2, \ldots, x_k)\,\frac{t^k}{k!},$$

with $Y_0 = 1$. In order to apply Equation (10) to Equation (3), we set $p_k = x_k/k!$, $q_k = 1/k!$, and $a = 1$. Then we find that

$$Y_k(x_1, x_2, \ldots, x_k) = k!L_{P,k}\left[\prod_{i=1}^{k} \frac{x_i^{\lambda_i}}{i!^{\lambda_i}\lambda_i!}\right].$$

The first eleven polynomials are listed in Table 3. On the other hand, if one differentiates Equation (3), then one obtains

$$\sum_{k=1}^{\infty} Y_k(x_1, x_2, \ldots, x_k)\frac{t^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} Y_k(x_1, x_2, \ldots, x_k)\frac{t^k}{k!}\sum_{k=1}^{\infty} x_k\,\frac{t^{k-1}}{(k-1)!}.$$

Multiplying both series on the rhs of the above result and equating like powers of t, one finally arrives at

$$Y_k(x_1, x_2, \ldots, x_k) = \sum_{j=0}^{k-1} \binom{k-1}{j} x_{k-j}\, Y(x_1, x_2, \ldots, x_j). \tag{36}$$

If we replace n by $n+1$ in Equation (31) and introduce Equation (1), then the following result is obtained

$$\sum_{k=0}^{n+1} (-1)^{n+1-k} s(n+1, k)\, y^{k-1} = n!\exp\Bigl(H_n\, y - H_n^{(2)}\,\frac{y^2}{2} + H_n^{(3)}\,\frac{y^3}{3} - \cdots\Bigr). \tag{37}$$

On the lhs of the above result we replace k by k+1, while we note that the exponential on the rhs can be expressed as

$$\exp\Bigl(-0!H_n\,\frac{(-y)}{1!} - 1!H_n^{(2)}\,\frac{(-y)^2}{2!} - 2!H_n^{(3)}\,\frac{(-y)^3}{3!} - \cdots\Bigr).$$

In other words, the above result has the same exponential form as Equation (3) with $x_k = -(k-1)!H_n^{(k)}$ and $t = -y$. Therefore, we can replace the exponential on the rhs of Equation (37) by the rhs of Equation (3). Equating like powers of t yields

$$(-1)^n s(n+1, k+1) = \frac{n!}{k!} Y\bigl(-H_n, -H_n^{(1)}, -2!H_n^{(3)}, \ldots, -(k-1)!H_n^{(k)}\bigr). \tag{38}$$

The quantities on the lhs of Equation (38) represent the unsigned Stirling numbers. Moreover, by introducing the above result into Equation (36), we eventually arrive at

$$s(n+1,k+1) = -\frac{1}{k}\sum_{j=0}^{k-1} H_n^{(k-j)} s(n+1,j+1), \quad \text{for } n,k \geq 1. \tag{39}$$

Though elegant, the above recurrence relation is of limited use for the purposes of this work. First, when using the rhs to evaluate the lhs, one requires all the previous values of Stirling numbers of the first kind from $j = 0$ to $k-1$ for the same arbitrary value of the primary variable n as on the lhs of the equation. That is, if we want to keep n as a variable, one must have expressions in terms of n for $s(n+1,j+1)$ for each value of j between 0 and $k-1$. Software packages such as Mathematica possess routines for determining the Stirling numbers provided only numerical values for both n and k are given. Likewise a similar situation applies to the harmonic numbers, where again the input variables need to be specified. One could, perhaps, argue that one can replace the HarmonicNumber routine in Mathematica by one's own form, say H[n,k-j], but then Mathematica would not be able to provide values for $s(n+1,j+1)$, when n is a variable. Finally, from a purist's point of view, Equation (34), which is valid for any value of n and k, does not require previous values of the Stirling numbers in the evaluation of the Stirling numbers of the first kind as in Equation (39). In fact, a simple calculator is all that is needed to produce the results in Table 2.

If one looks closely at the order of the partitions in Table 1, then one can see that the partitions have been listed with decreasing lengths $l(\boldsymbol{\lambda}_k)$ or by decrementing the total number of parts. Ordering partitions by their lengths is regarded as non-standard since most approaches for generating partitions are based on a lexicographic order [14]. However, the partitions and their corresponding refined rencontres numbers can be generated simply by implementing partition trees as described in Chapters 3 and 6 of [14] and also here in the appendix. As can be seen from the appendix, this means modifying the program **numparts**, which is discussed in [14, page 180], but only appears here in its entirety in the appendix. To run this program, the user must specify both the sum of the parts of a partition and the number of parts. Basically, the code represents the computer implementation of the operator in Equation (15). When the argument is set equal to unity in the operator, it yields the number of partitions with a set number of parts or length. For example, to run the code, one simply types a line such as ./numparts 9 3. This results in the generation of those partitions summing to 9 with only three parts, namely $\{1_2,7\}$, $\{1,2,6\}$, $\{1,3,5\}$, $\{1,4_2\}$, $\{2_2,5\}$, $\{2,3,4\}$ and $\{3_3\}$. Hence, we observe that $L_{P,9}^{(3)}[1] = 7$.

As explained in the appendix, there are two necessary modifications before **numparts** can print out the refined rencontres numbers or $T(\boldsymbol{\lambda}_k)$ for each partition. First, instead of the user specifying the number of parts, the main function

k	$Y_k(x_1, x_2, \ldots, x_k)$
0	1
1	x_1
2	$x_1^2 + x_2$
3	$x_1^3 + 3x_1x_2 + x_3$
4	$x_1^4 + 6x_1^2x_2 + 3x_2^2 + 4x_1x_3 + x_4$
5	$x_1^5 + 10x_1^3x_2 + 15x_1x_2^2 + 10x_1^2x_3 + 10x_2x_3 + 5x_1x_4 + x_5$
6	$x_1^6 + 15x_1^4x_2 + 15x_2^3 + 20x_1^3x_3 + 10x_3^2 + 15x_2x_4 + 45x_1^2x_2^2 + 15x_1^2x_4$ $+60x_1x_2x_3 + 6x_1x_5 + x_6$
7	$x_1^7 + 21x_1^5x_2 + 35x_1^4x_3 + 105x_2^2x_3 + 35x_3x_4 + 35x_1^3 + 105x_1^3x_2^2 + 35x_1^3x_4$ $+21x_2x_5 + 210x_1^2x_2x_3 + 21x_1^2x_5 + 105x_1x_2^3 + 70x_1x_3^2 + 105x_1x_2x_4$ $+105x_1x_2x_4 + 7x_1x_6 + x_7$
8	$x_1^8 + 28x_1^6x_2 + 105x_2^4 + 56x_1^5x_3 + 210x_2^2x_4 + 35x_4^2 + 210x_1^4x_2^2 + 70x_1^4x_4$ $+56x_3x_5 + 560x_1^3x_2x_3 + 56x_1^3x_5 + 280x_2x_3^2 + 28x_2x_6 + 420x_1^2x_2^3$ $+280x_1^2x_3^2 + 420x_1^2x_2x_4 + 28x_1^2x_6 + 840x_1x_2^2x_3 + 280x_1x_3x_4$ $+168x_1x_2x_5 + 8x_1x_7 + x_8$
9	$x_1^9 + 36x_1^7x_2 + 84x_1^6x_3 + 1260x_2^3x_3 + 280x_3^3 + 378x_1^5x_2^2 + 126x_1^5x_4$ $+378x_2^2x_5 + 126x_4x_5 + 1260x_1^4x_2x_3 + 126x_1^4x_5 + 84x_3x_6 + 1260x_1^3x_2^3$ $+840x_1^3x_3^2 + 1260x_1^3x_2x_4 + 84x_1^3x_6 + 1260x_2x_3x_4 + 36x_2x_7$ $+3780x_1^2x_2^2x_3 + 1260x_1^2x_3x_4 + 756x_1^2x_2x_5 + 36x_1^2x_7 + 945x_1x_2^4$ $+1890x_1x_2^2x_4 + 315x_1x_4^2 + 504x_1x_3x_5 + 2520x_1x_2x_3^2 + 252x_1x_2x_6$ $+9x_1x_8 + x_9$
10	$x_1^{10} + 45x_1^8x_2 + 945x_2^5 + 120x_1^7x_3 + 3150x_2^3x_4 + 2100x_3^2x_4 + 630x_1^6x_2^2$ $+210x_1^6x_4 + 126x_5^2 + 2520x_1^5x_2x_3 + 252x_1^5x_5 + 210x_4x_6 + 6300x_2^2x_3^2$ $+630x_2^2x_6 + 3150x_1^4x_2^3 + 2100x_1^4x_3^2 + 3150x_1^4x_2x_4 + 210x_1^4x_6 + 120x_3x_7$ $+12600x_1^3x_2^2x_3 + 4200x_1^3x_3x_4 + 2520x_1^3x_2x_5 + 120x_1^3x_7 + 1575x_2x_4^2$ $+2520x_2x_3x_5 + 45x_2x_8 + 4725x_1^2x_2^4 + 9450x_1^2x_2^2x_4 + 1575x_1^2x_4^2$ $+2520x_1^2x_3x_5 + 12600x_1^2x_2x_3^2 + 1260x_1^2x_2x_6 + 45x_1^2x_8 + 12600x_1x_2^3x_3$ $+2800x_1x_3^3 + 3780x_1x_2^2x_5 + 1260x_1x_4x_5 + 840x_1x_3x_6 + 12600x_1x_2x_3x_4$ $+360x_1x_2x_7 + 10x_1x_9 + x_{10}$

Table 3: The exponential complete Bell polynomials, $Y(x_1, x_2, \ldots, x_k)$, up to $k = 10$.

program called **main** now includes another for loop where the variable *numparts* is decremented from *tot* parts down to unity. This enables all the refined rencontres numbers of the partitions to be printed out. The second modification is the inclusion of another for loop in the function subprogram called **termgen**, which prints out the refined rencontres number in symbolic form with each partition. The program called **refined_rencontres** appears as the second code in the appendix. As an example, when the code reaches the fourteenth partition summing to 9 or encounters the partition $\{1_2,2,5\}$, it prints out

14: 2(1) 1(2) 1(5), and the refined rencontres number is: 9!/((2! 1^(2))(1! 2^(1))(1! 5^(1))).

Then the symbolic forms for each refined rencontres number can be imported into Mathematica to yield the actual integer value. When this is done with the above output, a value of 18144 is printed out in Mathematica.

In the introduction it was stated that the general forms of $s(n, n-k)$ for $k = 1$ to $k = 4$ in Equation (6) were evaluated via Equation (7), which, in turn, resulted from expanding the lhs of Equation (1) in powers of y. Now we utilize Equation (22) to develop more results in addition to investigating whether we can develop an understanding of the coefficients of the resulting polynomials. If one puts $j = k$ in Equation (22), then one obtains $L_{P,k}^{(k)}\left[\prod_{i=1}^{k} 1/\lambda_i! i^{\lambda_i}\right] = 1/k! = s(k,k)/k!$ or $s(k,k) = 1$. This is obvious because there is only one partition with k parts or $\{1_k\}$. Similarly, there is only one partition with $j = k-1$ parts, namely, $\{1_{k-2},2\}$. Then one finds that $L_{P,k}^{(k-1)}\left[\prod_{i=1}^{k} 1/\lambda_i! i^{\lambda_i}\right] = 1/(2(k-2)!) = -s(k,k-1)/k!$, which leads to $s(k,k-1) = -\binom{k}{2}$. For $j = k$–2 parts, there are two partitions, $\{1_{k-3},3\}$ and $\{1_{k-4},2_2\}$, whereas for $j = k-3$ parts, three partitions exist: $\{1_{k-4},4\}$, $\{1_{k-5},2,3\}$ and $\{1_{k-6},2_3\}$. Thus, Equation (22) yields

$$L_{P,k}^{(k-2)}\left[\prod_{i=1}^{k} \frac{1}{i^{\lambda_i}\lambda_i!}\right] = \frac{1}{3(k-3)!} + \frac{1}{2^2 \times 2!(k-4)!} = \frac{s(k,k-2)}{k!}, \tag{40}$$

and

$$L_{P,k}^{(k-3)}\left[\prod_{i=1}^{k} \frac{1}{i^{\lambda_i}\lambda_i!}\right] = \frac{1}{4(k-4)!} + \frac{1}{2 \times 3(k-5)!} + \frac{1}{2^3 \times 3!(k-6)!}$$
$$= -\frac{s(k,k-3)}{k!}. \tag{41}$$

Further simplification of the lhs's of both results produces the third and fourth results displayed in Equation (6).

From these results we observe that determining $s(k, k-j)$ requires $p(j)$ partitions, where $p(j)$ again represents the number of partitions summing to j. Since $p(4) = 5$, we expect five distinct contributions to $s(k, k-4)$. Specifically, these are the partitions, $\{1_{k-5},5\}$, $\{1_{k-6},2,4\}$, $\{1_{k-6},3_2\}$, $\{1_{k-7},2_2,3\}$, and $\{1_{k-8},2_4\}$. Then Equation (22) gives

$$L_{P,k}^{(k-4)}\left[\prod_{i=1}^{k}\frac{1}{i^{\lambda_i}\lambda_i!}\right]=\frac{1}{(k-5)!}\Big(\frac{1}{5}+(k-5)\left(\frac{1}{8}+\frac{1}{18}\right)+\frac{1}{24}(k-5)(k-6)+\frac{1}{384}(k-5)(k-6)(k-7)\Big)=\frac{s(k,k-4)}{k!}. \tag{42}$$

At this stage we need to modify the definition of the partition operator in order to allow for the fact that we may wish to exclude certain parts from being included in the analysis. For example, the coefficients of the product terms in the bracketed term of Equation (42), viz., those terms of the form of $\prod_{i=1}^{j}(k-4-i)$ for j ranging from 1 to 4, involve partitions that exclude unity since they have been removed by the factor of $1/(k-5)!$. Hence the partition operator must exclude unity when evaluating the coefficients. More generally, we modify the operator in Definition (11) so that it now excludes the part l in partitions, which is achieved by ensuring that i_l is always equal to zero. In other words, the index i_l is effectively excluded in the sum over all partitions summing to k with j parts. Therefore, the new operator is defined as

$$L_{P,k/\{l\}}^{(j)}\Big[\cdot\Big]\doteqdot\sum_{\substack{i_1,i_2,i_3,\ldots,i_l=0,\ldots,i_k=0\\ \sum_{i=1}^{k}i\lambda_i=k,\ \sum_{i=1}^{k}\lambda_i=j}}^{k,\lfloor k/2\rfloor,\lfloor k/3\rfloor,\ldots,0,\ldots,1}\Big(\cdot\Big). \tag{43}$$

For example, to exclude unity from appearing in the partitions, the above result becomes $L_{P,k/\{1\}}^{(j)}[\cdot]$. Moreover, the definition can be used to restrict multiple parts simply by introducing them in the subscripted curly brackets and setting their respective indices in the summation to zero. Therefore, by employing this definition, one can express Equation (42) as

$$s(k,k-4)=\binom{k}{5}\sum_{j=1}^{4}L_{P,(j+4)/\{1\}}^{(j)}\left[\prod_{i=2}^{j+4}\frac{5!}{i^{\lambda_i}\lambda_i!}\right]\prod_{i=1}^{j-1}(k-i-4). \tag{44}$$

In Equation (44) the final product yields unity when $j=1$. In addition, Equation (42) can be simplified by using the Simplify routine in Mathematica, which yields

$$s(k,k-4)=\frac{1}{48}\binom{k}{5}\big(15k^3-30k^2+5k+2\big).$$

This agrees with the final result in (6).

We can generalize Equation (44) further by replacing 4 with l. Then we arrive at

$$s(k,k-l)=\binom{k}{l+1}\sum_{j=1}^{l}L_{P,(j+l)/\{1\}}^{(j)}\left[\prod_{i=2}^{j+l}\frac{(l+1)!}{i^{\lambda_i}\lambda_i!}\right]\prod_{i=1}^{j-1}(k-i-l). \tag{45}$$

For $l = 5$, Equation (45) gives

$$\begin{aligned} s(k,k-5) = &\binom{k}{6} 6! \sum_{j=1}^{5} L^{(j)}_{P,(j+5)/\{1\}} \left[\prod_{i=2}^{j+5} \frac{1}{i^{\lambda_i}\lambda_i!}\right] \prod_{i=1}^{j-1} (k-i-5) = \binom{k}{6} 6! \left[\frac{1}{6}\right.\\ &+ \Big(\frac{1}{2}\frac{1}{5} + \frac{1}{3}\frac{1}{4}\Big)(k-6) + \Big(\frac{1}{2^2\cdot 2!}\frac{1}{4} + \frac{1}{2}\frac{1}{3^2\cdot 2!}\Big)(k-6)(k-7)\\ &+ \frac{1}{2^3\cdot 3!}\frac{1}{3}(k-6)(k-7)(k-8) + \frac{1}{2^5\cdot 5!}\\ &\left.\times (k-6)(k-7)(k-8)(k-9)\right]. \end{aligned} \tag{46}$$

Via the Simplify routine in Mathematica, one finds that the above result reduces to

$$\begin{aligned} s(k,k-5) &= -\frac{1}{16}\binom{k}{6} k(k-1)\big(3k^2-7k-2\big)\\ &= \frac{1}{16}\binom{k}{6}(-3k^4+10k^3-5k^2-2k). \end{aligned} \tag{47}$$

From the above results, the Stirling numbers of the first kind can be expressed as

$$s(k,k-l) = (-1)^l \binom{k}{l+1} r_l(k), \tag{48}$$

where $r_l(k)$ are polynomials of degree $l-1$ in k. The latter polynomials are displayed in Table 6.1 of [14]. For odd values of l, the polynomials begin at first order, while for even values, they possess a constant. As an aside, it should be mentioned here that there are transcription errors in some of the coefficients of the two lowest order terms in the $l = 8$ result, while the $l = 9$ result is missing a minus sign and a factor of $(k-1)$. Therefore, the correct forms should be

$$\begin{aligned} s(k,k-8) = \frac{1}{3840}\binom{k}{9}\Big(&135k^7 - 1260k^6 + 3150k^5 - 840k^4 - 2345k^3\\ &- 540k^2 + 404k + 144\Big), \end{aligned} \tag{49}$$

and

$$\begin{aligned} s(k,k-9) = &-\frac{1}{768}\binom{k}{10} k(k-1)\Big(15k^6 - 165k^5 + 465k^4 + 17k^3 - 648k^2 - 548k\\ &- 144\Big), \end{aligned}$$

or

$$\begin{aligned} s(k,k-9) = \frac{1}{768}\binom{k}{10}\Big(&-15k^8 + 180k^7 - 630k^6 + 448k^5 + 665k^4 - 100k^3\\ &- 404k^2 - 144k\Big). \end{aligned} \tag{50}$$

Table 3 displays the $s(k,k-l)$ for fixed values of l up to 10 after they have been processed with the aid of the Expand, FullSimplify and FunctionExpand routines in Mathematica. They yield identical values for $s(k,k-l)$ to the results in Table 6.1 of [14]. The coefficients, $L^{(j)}_{P,(j+5)/\{1\}}$ in Equation (46), were obtained by running the program **numparts** in the appendix and discarding all partitions containing unity. From these results we observe that $s(k,k-l)$ are polynomials in k of degree $2l$. The surprising property of these results is that even though the coefficients are improper fractions, they always yield an integer for $s(k,k-l)$ for any value of k.

When the $r_l(k)$ polynomials are multiplied by k and are expressed in terms of a lowest common denominator with the resulting integer coefficients arranged from the lowest order k terms including the constant or k^0 term up to the highest order term, one obtains a sequence of integers. For example, in the case of $s(k,k-5)$ in Equation (47), this sequence would be [0,0,2,-5,10,-3], while for $s(k,k-8)$ in Equation (49), it is [0,144,404,-540,-2345,-840,3150,-1260,135]. If all these sequences are combined successively beginning with the $l=1$ polynomial, the infinite sequence becomes integer sequence A100655 in [28], where it is stated that the sequence represents the coefficients of the Nörlund polynomials, $B_l^{(z)}$, when they too share a common denominator. The Nörlund polynomials [21] are programmed in Mathematica with the built-in instruction NorlundB[n,x]. By applying the Together instruction, one can express all the coefficients of these polynomials as integers divided by their lowest common denominator. For example, if we type the following command into a notebook:

Together[NorlundB[7,k]],

then Mathematica prints out

$$\frac{(16k^2+42k^3-7k^4-105k^5+63k^6-9k^7)}{1152}.$$

On the other hand, if we apply the Factor instruction to the $l=7$ result in Table 3, then we find that Mathematica prints out:

$$-\frac{1}{5806080}(-7+k)(-6+k)(-5+k)(-4+k)(-3+k)(-2+k)(-1+k)^2k^2$$
$$(16+59k+51k^2-54k^3+9k^4).$$

In order to obtain $r_7(k)$ from this result, we need to remove $\binom{k}{8}$. In other words, one must multiply the above result by 8! and retain the quartic multiplied by $k(k-1)$.

l	$s(k,k-l)$
0	1
1	$-\frac{1}{2}\left(k^2-k\right)$
2	$\frac{1}{8}\left(k^4-\frac{10}{3}k^3+3k^2-\frac{2}{3}k\right)$
3	$-\frac{1}{48}\left(k^6-7k^5+17k^4-17k^3+6k^2\right)$
4	$\frac{1}{384}\left(k^8-12k^7+\frac{166}{3}k^6-\frac{616}{5}k^5+\frac{403}{3}k^4-60k^3+\frac{4}{3}k^2+\frac{16}{5}k\right)$
5	$-\frac{1}{3840}\left(k^{10}-\frac{55}{3}k^9+\frac{410}{3}k^8-\frac{1598}{3}k^7+\frac{3467}{3}k^6-\frac{4055}{3}k^5+\frac{2120}{3}k^4\right.$ $\left.-\frac{52}{3}k^3-80k^2\right)$
6	$\frac{1}{46080}\left(k^{12}-26k^{11}+285k^{10}-\frac{15422}{9}k^9+6143k^8-\frac{279142}{21}k^7\right.$ $\left.+16487k^6-\frac{28886}{3}k^5+156k^4+\frac{16232}{9}k^3-32k^2-\frac{1280}{7}k\right)$
7	$-\frac{1}{645120}\left(k^{14}-35k^{13}+\frac{1589}{3}k^{12}-\frac{40859}{9}k^{11}+\frac{217973}{9}k^{10}-\frac{249007}{3}k^9\right.$ $+\frac{542959}{3}k^8-\frac{707651}{3}k^7+149422k^6+\frac{5516}{9}k^5-\frac{374024}{9}k^4+288k^3$ $\left.+8960k^2\right)$
8	$\frac{1}{10321920}\left(k^{16}-\frac{136}{3}k^{15}+\frac{2716}{3}k^{14}-\frac{94304}{9}k^{13}+\frac{2098754}{27}k^{12}\right.$ $-385424k^{11}+\frac{173988644}{135}k^{10}-2845216k^9+\frac{34810009}{9}k^8-\frac{23420824}{9}k^7$ $-\frac{13192648}{135}k^6+\frac{3008192}{3}k^5+\frac{531632}{27}k^4-\frac{1089664}{3}k^3+\frac{56576}{15}k^2$ $\left.+43008k\right)$
9	$-\frac{1}{185794560}\left(k^{18}-57k^{17}+1452k^{16}-\frac{327148}{15}k^{15}+\frac{643538}{3}k^{14}\right.$ $-\frac{4348450}{3}k^{13}+\frac{102846644}{15}k^{12}-22634716k^{11}+50692273k^{10}$ $-\frac{214300123}{3}k^9+\frac{753452672}{15}k^8+\frac{63997064}{15}k^7-\frac{76757776}{3}k^6-1705872k^5$ $\left.+\frac{69859968}{5}k^4+\frac{407808}{5}k^3-3483648k^2\right)$
10	$\frac{1}{3715891200}\left(k^{20}-70k^{19}+2215k^{18}-\frac{125666}{3}k^{17}+527218k^{16}\right.$ $-\frac{41864540}{9}k^{15}+\frac{88466794}{3}k^{14}-\frac{1216903676}{9}k^{13}+443832229k^{12}$ $-\frac{99888047818}{99}k^{11}+\frac{4398344897}{3}k^{10}-\frac{9570348094}{9}k^9-\frac{469781368}{3}k^8$ $+\frac{6221692976}{9}k^7+95289008k^6-\frac{4782161632}{9}k^5-\frac{57906176}{3}k^4$ $\left.+222504448k^3-1413120k^2-\frac{309657600}{11}k\right)$

Table 4: Stirling numbers of the first kind, $s(k,k-l)$, for fixed values of l.

Then one arrives at

$$\begin{aligned} r_7(k) &= \frac{8!}{5806080} k(k-1)(16+59k+51k^2-54k^3+9k^4) \\ &= \frac{1}{144}(-16k-42k^2+7k^3+105k^4-63k^5+9k^6). \end{aligned}$$

Multiplying $r_7(k)$ by $k/8$ yields NorlundB[7,k] as given above with the same commom denominator of 1152. Hence we observe that

$$r_l(k) = \left(\frac{l+1}{k\,k!}\right) B_l^{(k)}. \tag{51}$$

Carlitz [2] defines the Nörlund polynomials by their generating function, which is

$$\sum_{n=0}^{\infty} B_n^{(s)} \frac{x^n}{n!} \doteqdot \left(\frac{x}{e^x-1}\right)^s. \tag{52}$$

For $s=1$, these polynomials reduce to the Bernoulli polynomials, whereas for other integer values, i.e., $s=n$, he expresses them in terms of the Stirling numbers of the first kind as

$$s(n-1,k) = (-1)^k \binom{n-1}{k} B_k^{(n)}. \tag{53}$$

We can determine another result by combining Equation (48) with Equation (51). Thus, it is found that

$$s(k,k-l) = \binom{k-1}{l} B_l^{(k)}. \tag{54}$$

To express the above result in a similar form to Carlitz's result, we replace k and l, respectively, by n and $n-k$ in Equation (54). This yields

$$s(n,k) = \binom{n-1}{n-k} B_{n-k}^{(n)}. \tag{55}$$

Let us verify the two different forms for the Stirling numbers of the first kind by putting $n=68$ and $k=13$. If we consider the Carlitz result first, then according to Mathematica, $s(67,13)$ is equal to

386337636331425756359010590303024925586624321085088329810852198453
1301219926445228629688320.

The value of the rhs or $(-1)^{(13)}\binom{67}{13}$NorlundB$[13,68]$ is given as

137204513009413360700275658925 8832.

This, unfortunately, is nowhere near the value obtained for the lhs of Equation (53). On the other hand, for the lhs of Equation (55), viz. $s(68,13)$, Mathematica prints out

-275870554911213017584867300082071301557129334314061522600050457372803918
468354582107956183040.

The rhs of Equation (55) becomes $\binom{67}{55}$NorlundB$[55, 68]$, whose value is

-275870554911213017584867300082071301557129334314061522600050457372803918
468354582107956183040.

Thus, we find that the Carlitz result is erroneous, while Equation (55), which has been determined by a combination of the partition method for a power series expansion and the sequence A100655 in [28], is correct. It should be mentioned that Equation (55) has been tested for numerous values of n and k. On each occasion both sides agreed with each other.

The $r_l(k)$ polynomials can also be obtained by applying the partition method for a power series expansion directly to Equation (52). To accomplish this, we need to identify the inner and outer series on the rhs of Equation (52) by expressing it as

$$\begin{aligned}\Big(\frac{x}{e^x-1}\Big)^z &= \Big(\frac{1}{1+x/2!+x^2/3!+\cdots}\Big)^z \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k+z)}{k!\Gamma(z)} \Big(\frac{x}{2!}+\frac{x^2}{3!}+\cdots\Big)^k. \end{aligned} \tag{56}$$

Note that when $|x^2/2! + x^3/3! + \cdots| > 1$, the equals sign in the final member of Equation (56) needs to be replaced by the equivalence symbol. Thus, we observe that the coefficients of the inner series are given by $p_k = 1/(k+1)!$, while the outer series has $q_k = (1)^k \Gamma(k+z)/k!\Gamma(z)$. Introducing these results into (10) with $a = 1$ yields

$$D_k = \frac{1}{k!}\, B_k^{(z)} = L_{P,k}\left[(-1)^{l(\boldsymbol{\lambda}_k)} \frac{\Gamma(l(\boldsymbol{\lambda}_k)+z)}{\Gamma(z)} \prod_{i=1}^{k} \frac{1}{\lambda_i!(i+1)!^{\lambda_i}}\right]. \tag{57}$$

As an example, if one inserts the multiplicities and lengths in Table 1 into Equation (57), then one obtains

$$\begin{aligned}B_6^{(z)} =& 6!\left[\frac{\Gamma(z+6)}{\Gamma(z)6!(2!)^6} - \frac{\Gamma(z+5)}{\Gamma(z)4!(2!)^4 1!3!} + \frac{\Gamma(z+4)}{\Gamma(z)3!(2!)^3 1!4!} + \frac{\Gamma(z+4)}{\Gamma(z)2!(2!)^2 2!(3!)^2}\right. \\ & - \frac{\Gamma(z+3)}{\Gamma(z)2!(2!)^2 1!5!} - \frac{\Gamma(z+3)}{\Gamma(z)1!2!1!3!1!4!} - \frac{\Gamma(z+3)}{\Gamma(z)3!(3!)^3} + \frac{\Gamma(z+2)}{\Gamma(z)1!2!1!6!} \\ & \left. + \frac{\Gamma(z+2)}{\Gamma(z)1!3!1!5!} + \frac{\Gamma(z+2)}{\Gamma(z)2!(4!)^2} - \frac{\Gamma(z+1)}{\Gamma(z)1!7!}\right].\end{aligned}$$

Furthermore, if one sets the rhs to Dk[6,1] in Mathematica and types in

Expand[FullSimplify[6! Dk[6,1]]],

then the following line is generated

$$-\frac{z}{252}-\frac{z^2}{96}+\frac{13z^3}{576}+\frac{5z^4}{64}-\frac{5z^5}{64}+\frac{z^6}{64}.$$

On the other hand, if the following line is typed into a notebook

Together[FullSimplify[6! Dk[6, 1] - NorlundB[6, z]]],

then the output is simply zero, which verifies that Equation (57) does yield the Nörlund polynomials.

It should also be noted that if one replaces k and l in Table 3 by $m+1$ and k respectively, then using Equation (28) one can determine the Stirling polynomials, $S_k(m)$. If the results in Table 3 are denoted as skminuslk[k_,l_] in Mathematica, where l is set to each value of l in the table, then typing

SP[m_,n_]:=((-1)^n (m-n)! n!/m!) skminuslk[m+1,n]

yields the Stirling polynomials. For example, typing

Expand[FullSimplify[SP[m, 10]]]

generates the following output for $S_{10}(m)$:

$$\frac{5}{66}+\frac{691m}{3168}-\frac{5m^2}{128}-\frac{1421m^3}{2304}-\frac{863m^4}{1536}+\frac{115m^5}{9216}+\frac{623m^6}{3072}+\frac{77m^7}{1536}\\-\frac{5m^8}{256}-\frac{5m^9}{1024}+\frac{m^{10}}{1024}.$$

Another interesting property is that one can develop general formulas for the highest and lowest order coefficients of the product terms in Equation (45). For $j=1$, $L^{(j)}_{P,(j+l)/\{1\}}\left[\prod_{i=2}^{j+l}\frac{(l+1)!}{i^{\lambda_i}\lambda_i!}\right]$ reduces to

$$L^{(1)}_{P,(l+1)/\{1\}}\left[\prod_{i=2}^{l+1}\frac{(l+1)!}{i^{\lambda_i}\lambda_i!}\right]=l!, \tag{58}$$

while for $j=2$, one finds that

$$L^{(2)}_{P,(l+1)/\{1\}}\left[\prod_{i=2}^{l+1}\frac{(l+1)!}{i^{\lambda_i}\lambda_i!}\right]=\frac{H_{l-1}-1}{l+1}-\frac{(1-(-1)^l)}{(l+1)^2}. \tag{59}$$

The final term in Equation (59) has been introduced to compensate for the fact that when $l+1$ is even, the multiplicity of the part in the central partition of $\{((l+1)/2)_2\}$ equals 2.

For $j=3$, the situation becomes more formidable since all the 3-part partitions summing to $l+3$ need to be considered. This is accomplished by fixing the first part in a partition and counting all the contributions from the 2-part partitions summing to a value of $l+3$ minus the value of the first part. For example, if the first part is equal to two, then one sums the contributions from the 2-part contributions summing to $l+1$. This is virtually a repeat of the process used to obtain Equation (59) except that one needs to account for the fact that the second part not only equals the first part at the bottom limit of the sum, but that it also equals the third part at the upper limit of the sum. For these partitions the multiplicity is equal to 2. So one compensates by subtracting half the values at the limits of the summation. After the first part has been set equal to two and the contributions from the 2-part partitions have been evaluated, one sets the first part equal to 3 and determines the contributions from the 2-part partitions summing to l with the parts greater than 2. Again, the second part will equal the first part initially, while it will equal the third part at the upper limit. Thus, half these contributions need to be removed. Then we set the first part equal to 4 and determine all the 2-part partitions summing to $l-1$, where the parts are greater than 3. The process continues until the first part is equal to $\lfloor l/3 \rfloor + 1$. This, however, is not the end of the matter. When $l+3$ is a multiple of 3, one of the partitions will be $\{(l/3+1)_3\}$, which has a multiplicity equal to 3. It turns out that this contribution must be inserted into the result for $L^{(3)}_{P,(l+3)/\{1\}}\left[\prod_{i=2}^{l+1} 1/i^{\lambda_i}\lambda_i!\right]$. Finally, one arrives at

$$
\begin{aligned}
L^{(3)}_{P,(l+3)/\{1\}}\left[\prod_{i=2}^{l+1}\frac{1}{i^{\lambda_i}\lambda_i!}\right] &= \sum_{i=2}^{\lfloor l/3\rfloor+1}\sum_{k=i}^{\lfloor (l+3-i)/2\rfloor}\frac{1}{ik(l+3-i-k)} \\
&\quad - \sum_{i=2}^{\lfloor l/3\rfloor+1}\frac{1}{2i^2(l+3-2i)} - \sum_{i=2}^{\lfloor l/3\rfloor+1}\frac{(1+(-1)^{l+3-i})}{i(l+3-i)^2} \\
&\quad - \left(\frac{1+2(-1)^l\cos(\pi l/3)}{3\cdot 3!(\lfloor \lfloor l/3\rfloor+1)^3}\right). \qquad (60)
\end{aligned}
$$

The second term on the rhs of Equation (60) compensates for the double counting in the first sum when $k=i$, while the third term, compensates for the double counting when $k=\lfloor (l+3-i)/2\rfloor$ and is an integer. The last term in Equation (60) accounts for the case when there are 3 parts of equal magnitude. The factor in the numerator yields a value of 3 only when $l \mod 3 \equiv 0$ or l is divisible by 3. It arises from putting $l=3$ in the following identity:

$$
\sum_{j=1}^{l} e^{2\pi ijk/l} = \begin{cases} l\,, & k\equiv 0 \pmod{l}\,, \\ 0\,, & k\,,\ \text{otherwise.} \end{cases} \qquad (61)
$$

For $l=14$ and $l=15$, Equation (60) yields values of 2913569/25225200 and 12867983/110073600 respectively in Mathematica, which agree with the results obtained by: (1) running Program **numparts** with input values of 17 and 18 and the number of parts set equal to 3 and (2) applying the lhs of Equation (60) to only the 3-part partitions that exclude unity. For example, the $l = 14$ case yields

$$
\begin{aligned}
L^{(3)}_{P,(17)/\{1\}}\left[\prod_{i=2}^{13}\frac{1}{i^{\lambda_i}\lambda_i!}\right] &= \frac{1}{2^2\cdot 2!\cdot 13}+\frac{1}{2\cdot 3\cdot 12}+\frac{1}{2\cdot 4\cdot 11}+\frac{1}{2\cdot 5\cdot 10}+\frac{1}{2\cdot 6\cdot 9}\\
&+\frac{1}{2\cdot 7\cdot 8}+\frac{1}{3^2\cdot 2!\cdot 11}+\frac{1}{3\cdot 4\cdot 10}+\frac{1}{3\cdot 5\cdot 9}+\frac{1}{3\cdot 6\cdot 8}+\frac{1}{3\cdot 7^2\cdot 2!}\\
&+\frac{1}{4^2\cdot 2!\cdot 9}+\frac{1}{4\cdot 5\cdot 8}+\frac{1}{4\cdot 6\cdot 7}+\frac{1}{5^2\cdot 2!\cdot 7}+\frac{1}{5\cdot 6^2\cdot 2!}\\
&=\frac{2913569}{25225200}.
\end{aligned}
$$

In principle, one can continue with this approach to evaluate the other coefficients of $\prod_{i=1}^{j-1}(k-i-l)$ in Equation (45). However, even for 4-part partitions it becomes cumbersome having to adjust the varying multiplicities whenever two, three or four parts are equal to one another. There is also the case when a partition is composed of two pairs of parts. As a consequence, there are far more sums to compensate for on the rhs than in Equation (60). On top of this, the equivalent of the first sum on the rhs of Equation (60) becomes a three-dimensional sum, which contributes greatly to the complexity of the problem. Nevertheless, after a substantial amount of algebra, for the 4-part partitions one eventually obtains

$$
\begin{aligned}
&L^{(4)}_{P,(l+4)/\{1\}}\left[\prod_{i=2}^{l-2}\frac{1}{i^{\lambda_i}\lambda_i!}\right] = \sum_{i_1=2}^{\lfloor (l+4)/4\rfloor}\sum_{i_2=i_1+1}^{\lfloor (l+4-i_1)/3\rfloor}\sum_{i_3=i_2}^{\lfloor (l+4-i_1-i_2)/2\rfloor}\frac{1}{i_1 i_2 i_3(l+4-i_1-i_2-i_3)}\\
&-\frac{1}{2}\sum_{i_1=2}^{\lfloor (l+4)/4\rfloor}\sum_{i_2=i_1+1}^{\lfloor (l+4-i_1)/3\rfloor}\frac{1}{i_1 i_2^2(l+4-i_1-2i_2)}-\sum_{i_1=2}^{\lfloor (l+4)/4\rfloor}\sum_{i_2=i_1+1}^{\lfloor (l+4-i_1)/3\rfloor}\frac{(1+(-1)^{l+4-i_1-i_2})}{i_1 i_2(l+4-i_1-i_2)^2}\\
&+\frac{1}{18}\sum_{i_1=2}^{\lfloor (l+4)/4\rfloor}\frac{(1+2(-1)^{l+4-i_1}\cos(\pi(l+4-i_1)/3))}{i_1((l+4-i_1)/3)^3}-\frac{1}{2}\sum_{i_1=2}^{\lfloor (l+4)/4\rfloor}\frac{(1+(-1)^{l+4-2i_1})}{i_1^2(l+4-2i_1)^2}\\
&+\frac{1}{2}\sum_{i_1=2}^{\lfloor (l+4)/4\rfloor}\sum_{i_2=i_1+1}^{\lfloor (l+4-2i_1)/2\rfloor}\frac{1}{i_1^2 i_2(l+4-2i_1-i_2)}+\frac{1}{6}\sum_{i_1=2}^{\lfloor (l+4)/4\rfloor}\frac{1}{i_1^3(l+4-3i_1)}\\
&-\left(\frac{(1+(-1)^l+2\cos(\pi(l+4)/2))}{4\times 4!((l+4)/4)^4}\right). \qquad (62)
\end{aligned}
$$

This result has been tested for various values of l in the same manner as Equation (60). For example, when $l = 12$, the last term on the rhs, which arises from the $l = 4$ case of Equation (61), contributes since there is now a partition with all parts equal to one another, viz. $\{4_4\}$. Then we find that $L^{(4)}_{P,16/\{1\}}[\prod_{i=2}^{12}1/i^{\lambda_i}\lambda_i!] = 1235677/29030400$.

So far, it has been observed that in order to derive general expressions for the Stirling numbers of the first kind, j in Equation (22) has had to be fixed. This is despite the fact that the calculations become more formidable for $j > 3$. However, if we wish to derive general expressions for the Stirling numbers in two variables, then we are going to have to consider $j = k - l$ or $s(k, k - l)$ again, but this time around we cannot fix l to specific values as in Table 3 since our aim is to determine the coefficients of the powers of k in terms of l. Surprisingly, to accomplish this, one requires the results in Table 3.

If we put $j = k - l$ in Equation (22), then we obtain

$$L_{P,k}^{(k-l)} \left[\prod_{i=1}^{k} \frac{1}{i^{\lambda_i} \lambda_i!} \right] = \frac{(-1)^l}{k!} \, s(k, k-l). \tag{63}$$

We have seen that the highest order term in the polynomials of Table 3, viz. $2l$, emanates from the partitions with $k-l$ parts. Moreover, this power occurs with the least number of ones in these partitions. In other words, this is the partition with the most number of twos or $\{1_{k-2l}, 2_l\}$. The contribution of this partition to the lhs of Equation (22) is simply $1/2^l l!(k-2l)!$. In order to obtain the contribution to $s(k, k-l)$, we need to multiply the above result by $(-1)^l k!$. Then we find that

$$\mathcal{C}_0 = \frac{(-1)^l}{2^l l!} \frac{\Gamma(k+1)}{\Gamma(k-2l+1)} = \frac{(-1)^l}{2^l l!} \, k(k-1)\cdots(k-2l+1). \tag{64}$$

Alternatively, we can use Equation (1) to express Equation (64) as

$$\mathcal{C}_0 = \frac{(-1)^l}{2^l l!} \frac{\Gamma(2l-k)}{\Gamma(-k)} = \frac{(-1)^l}{2^l l!} \sum_{j=0}^{2l} s(2l, j) k^j. \tag{65}$$

Both of the above results indicate that the contribution of $\{1_{k-2l}, 2_l\}$ to $s(k, k-l)$ is a polynomial in k of degree $2l$. Consequently, we shall denote the coefficients of k^i by $\mathcal{C}_{0,i}$. Thus, the coefficient $\mathcal{C}_{0,2l}$ is equal $(-1)^l/2^l \cdot l!$ since $s(2l, 2l) = 1$ according to Equation (6) or Table 3.

From Equation (65) we see that there will always be a contribution from the partition $\{1_{k-2l}, 2_l\}$ when we wish to evaluate the decreasing orders of k in $s(k, k-l)$. In addition, because k^{2l} is the highest power, $s(k, k-l)$ will be a polynomial in k of degree $2l$. Therefore, we can write as $s(k, k-l) = \sum_{j=1}^{2l} s_{2l,j}(l) k^{2l-j}$. Hence we observe that $s_{2l,0}(l) = (-1)^l/2^l l!$.

To determine the coefficients, $s_{2l,j}(l)$, for the other powers of k, we require the contributions from other partitions. For example, to calculate $s_{2l,2l-1}$, we need to evaluate the contribution from the next partition with the least number of ones, but still with $k - l$ parts, which is $\{1_{k-2l+1}, 2_{l-2}, 3\}$. The contribution from this partition is $1/(3 \cdot 2^{l-2}(l-2)!(k-2l+1)!$. When multiplied by $(-1)^k k!$, we obtain

$$\mathcal{C}_1 = \frac{(-1)^{k-1}}{3 \cdot 2^{l-2}(l-2)!} \, (-k)_{2l-1}.$$

Introducing Equation (1) into the above result gives

$$\mathcal{C}_1 = \frac{(-1)^l}{3 \cdot 2^{l-2}(l-2)!} \sum_{j=0}^{2l-1} s(2l-1, j)k^j. \tag{66}$$

As expected, the partition with the second most number of ones yields a polynomial in powers of k as we found previously with $\{k-2l, 2_l\}$, but in this case the degree of the polynomial is $2l-1$. Adopting a similar approach to Equation (65), we shall denote the coefficients of k^i in Equation (66) by $\mathcal{C}_{1,i}$. Thus, the coefficient of k^{2l-1} in $s(k, k-l)$ becomes by the sum of two quantities, the first from the coefficient of k^{2l-1} in Equation (65) and the second from the power of k^{2l-1} in Equation (66). Hence we arrive at

$$\begin{aligned} s_{2l,2l-1}(l) = \mathcal{C}_{0,2l-1} + \mathcal{C}_{1,2l-1} = \frac{(-1)^l}{2^l} s(2l, 2l-1) \\ + \frac{(-1)^k}{3 \cdot 2^{l-2}(l-2)!} s(2l-1, 2l-1). \end{aligned}$$

By introducing the results in Equation (6) or from Table 3 into the above equation, we find that

$$s_{2l,2l-1}(l) = \frac{(-1)^l}{2^l(l-1)!} \binom{2l+1}{3}. \tag{67}$$

Note the appearance of $(l-1)!$ in the denominator, which indicates that $s_{2l,2l-1}(l)$ vanishes for $l=0$.

So far, the results have been simple. However, in order to determine $\mathcal{C}_2$, we require two partitions with the next least number of ones and $k-l$ parts. That is, the least number of ones is now $k-2l+2$. In general, to determine $\mathcal{C}_j$, the number of ones in each partition will be $k-2l+j$. For $\mathcal{C}_2$, the two partitions with $k-2l+2$ ones and $k-l$ parts, are $\{1_{k-2l+2}, 2_{l-3}, 4\}$ and $\{1_{k-2l+2}, 2_{l-4}, 3_2\}$, which correspond to the two standard partitions summing to 2. That is, the partitions can be viewed as being homologous to $\{2\}$ and $\{1_2\}$ with the number of partitions given by $p(2)$, where, as before, $p(k)$ represents the partition function or the number of partitions summing to k.

To observe this homologous behaviour more clearly, let us consider the partitions required to evaluate $\mathcal{C}_5$. As indicated above, this means that all partitions will have $k-l$ parts with $k-2l+5$ ones. Since $p(5)=7$, we expect that there will be 7 distinct partitions, which are displayed in the second column of Table 5. These partitions have been arranged by beginning with the partition possessing the greatest number of twos and then in the order that the third program in the appendix generates them as the number of branches is incremented. The third column in the table lists the corresponding standard partitions summing to 5 in reverse lexicographic order as described in [14, page 127]. That is, they have been arranged beginning

	Partitions with $k-l$ parts and $k-2l+5$ ones	Partitions summing to 5
1	$\{1_{k-2l+5}, 2_{l-6}, 7\}$	$\{5\}$
2	$\{1_{k-2l+5}, 2_{l-7}, 3, 6\}$	$\{1,4\}$
3	$\{1_{k-2l+5}, 2_{l-7}, 4, 5\}$	$\{2,3\}$
4	$\{1_{k-2l+5}, 2_{l-8}, 3_2, 5\}$	$\{1_2, 3\}$
5	$\{1_{k-2l+5}, 2_{l-8}, 3, 4_2\}$	$\{1,2_2\}$
6	$\{1_{k-2l+5}, 2_{l-9}, 3_3, 4\}$	$\{1_3, 2\}$
7	$\{1_{k-2l+5}, 2_{l-10}, 3_5\}$	$\{1_5\}$

Table 5: Homology between partitions with $k-l$ parts and $k-2l+5$ ones and those summing to 5.

with the least number parts or branches in a partition tree and ending with the greatest number of parts or branches. If we examine the partitions in the columns, then we find that those summing to 5 match those partitions in the second column where the parts other than unity or two are incremented by 2. For example, the partition $\{1_2, 3\}$ in the third column corresponds to $\{1_{k-2l+5}, 2_{l-8}, 3_2, 5\}$ in the second column. Thus, parts 1 and 3 in the third column correspond to parts 3 and 5 in the second column.

Because of the homology between both classes of partitions, one does not need to create an entirely new program to generate the partitions for calculating each $\mathcal{C}_j$. That is, all one needs to do is modify program **numparts** again, which is discussed in the appendix before the listing of the third program, $\mathcal{C}_\mathbf{j}$ **Partitions**. This program prints out the partitions for any specified value of j denoted by the variable *tot* in the same manner as the second column of Table 5. Since the number of ones is fixed once j is specified, **numparts** has been modified so that it only considers parts greater than or equal to two. Even the number of twos possesses a constant value of $l-j-1$. Hence the program prints out the fifth partition in the second column of Table 5 as

5: (k-2l+5)(1) (l-8)(2) 1(3) 2(4).

From the foregoing analysis, Equation (63) can now be written as

$$s(k, k-l) = (-1)^l \sum_{j=0}^{l-1} (-1)^j \frac{\Gamma(-k+2l-j)}{\Gamma(-k)} L_R(l, j), \tag{68}$$

where Euler's reflection formula for the gamma function has been used to flip the quotient of gamma functions and $L_R(l, j)$ is defined in terms of the reduced partition operator as

$$L_R\big(l,j\big) = L^{(l-j)}_{P,(2l-j)/\{1\}}\left[\prod_{i=2}^{j+2} f(i,\lambda_i)\right],$$

with $f(i,\lambda_i) = 1/i^{\lambda_i}\lambda_i!$. Note that in accordance with Equation (43), the reduced partition operator only considers parts greater than unity up to $j+2$. Moreover, introducing Equation (1) into the above result yields

$$s(k,k-l) = (-1)^l \sum_{j=0}^{l-1}\sum_{i=0}^{2l-j} s(2l-j,i)\, k^i\, L_R(l,j). \tag{69}$$

The above double sum can be decomposed further by expressing it into two separate double sums as follows:

$$\begin{aligned} s(k,k-l) =&(-1)^l \sum_{i=0}^{l} k^i \sum_{j=0}^{l-1} s(2l-j,i)\, L_R(l,j) \\ &+ (-1)^l \sum_{i=l+1}^{2l} k^i \sum_{j=0}^{2l-i} s(2l-j,i)\, L_R(l,j). \end{aligned} \tag{70}$$

The first sum in Equation (70) possesses the lowest powers of k. In fact, the double sum is evaluated quickly in Mathematica once expressions for the reduced partition operator have been determined, which will be presented shortly. In addition, the $i=0$ term vanishes since we have seen from the introduction that $s(n,0)$ vanishes. Moreover, we shall see that the results obtained from the first term on the rhs of Equation (69) can be neglected when determining general expressions of $s(k,k-l)$ for large values of l.

The highest powers of k occur at the upper end of the summation over i in the second term on the rhs of Equation (70), which again requires expressions for the reduced partition operator, but in this instance, the results for the Stirling numbers of the first kind presented in Table 3 or given in Equation (6) are required. For example, the highest power of k is given by $i=2l$. From Equation (70) one finds that the coefficient of this term is given by

$$s_{2l,0}(l) = (-1)^l s(2l,2l)\, L_R(l,0) = (-1)^l s(2l,2l)\, L^{(l)}_{P,(2l)/\{1\}}\left[\prod_{m=2}^{2} \frac{1}{m^{\lambda_m}\lambda_m!}\right],$$

while the coefficient of the next leading order term becomes

$$s_{2l,1}(l) = (-1)^l\Big(s(2l,2l-1)\, L_R(l,0) + s(2l-1,2l-1)L_R(l,1)\Big).$$

or

$$s_{2l,1}(l) = (-1)^l \Biggl(s(2l, 2l-1)\, L^{(l)}_{P,(2l)/\{1\}} \Biggl[\prod_{m=2}^{2} \frac{1}{m^{\lambda_m}\lambda_m!} \Biggr] + s(2l-1, 2l-1) L^{(l)}_{P,(2l-1)/\{1\}} \Biggl[\prod_{m=2}^{3} \frac{1}{m^{\lambda_m}\lambda_m!} \Biggr] \Biggr).$$

To determine the coefficient of the leading order term, we require the contribution from the reduced partition operator for the partition with only twos summing to $2l$, i.e., l twos, multiplied by the value of $s(2l, 2l)$. This partition, denoted by $\{2_l\}$, contributes a value of $1/2^l l!$, while according to Equation (6) or Table 3, $s(2l, 2l)$ equals unity. Hence the coefficient of k^{2l} for the Stirling numbers of the first kind is $s_{2l,0}(l) = (-1)^l/2^l l!$, as we have already found.

The next leading order coefficient, viz. the coefficient of k^{2l-1}, is composed of the contributions from two partitions. In the first instance we require the partition with l twos summing to $2l$ as before, except that it is now multiplied by $s(2l, 2l-1)$, which also appears in both Equation (6) and Table 3. In the second instance we need the contributions from all the $l-1$-part partitions summing to $2l-1$ composed only of twos and threes. There is only one such partition, $\{2_{l-2}, 3\}$, whose contribution is multiplied by $s(2l-1, 2l-1)$ or unity. Hence the coefficient of k^{2l-1} in $s(k, k-l)$ reduces to

$$s_{2l,1}(l) = (-1)^l \Bigl(-\frac{l(2l-1)}{2^l\, l!} + \frac{1}{3 \cdot 2^{l-2}(l-2)!} \Bigr) = \frac{(-1)^{l-1}(2l+1)}{3 \cdot 2^l (l-1)!}.$$

More generally, for $j \le l-1$, the coefficient, $s_{2l,j}$, is given by

$$s_{2l,j}(l) = (-1)^l \sum_{i=0}^{j} s(2l-i, 2l-j)\, L_R(l, i). \tag{71}$$

For the remaining powers of k in the Stirling numbers of the first kind or $j > l-1$ in the above result, we need to include the first sum in Equation (70).

From the preceding analysis we have seen that in order to derive the coefficients in terms of l, we require a program that enables us to calculate the contributions from the reduced partition operator $L_R(l, j)$ for any value of j. Such a program appears as Program 4 in the appendix. Once again, there is no need to create an entirely new program because the partitions in the reduced partition operator are homologous to or correspond with standard integer partitions as demonstrated by Table 5. For example, if we consider $L_R(l, 5)$, then the required partitions will be composed of parts greater than unity up to 7 that sum to $2l-5$ with $l-i$ parts and i ranging from 0 to 5. Specifically, these partitions are $\{2_{l-6}, 7\}$, $\{2_{l-7}, 3, 6\}$, $\{2_{l-7}, 4, 5\}$, $\{2_{l-8}, 3_2, 5\}$, $\{2_{l-8}, 3, 4_2\}$, $\{2_{l-8}, 3_3, 4\}$ and $\{2_{l-8}, 3_5\}$, which, in turn, correspond with the partitions summing to 5, viz., $\{5\}$, $\{1,4\}$, $\{2,3\}$, $\{1_2, 3\}$, $\{1, 2_2\}$,

$\{1_3,2\}$ and $\{1_5\}$. Neglecting the twos in the partitions for the reduced partition operator for the time being, if one subtracts 2 from each remaining part, then one obtains the partitions summing to 5. Therefore, to arrive at the partitions for the reduced partition operator, all we need to do is increment all parts by 2 in an existing code that calculates the coefficients via the partition method for a power series expansion. This program **mathparv.cpp** appears in [14].

Another modification to the program is that it needs to determine the correct number of twos for each partition. By examining the partitions in the reduced partition operator, we see that each partition is accompanied by at least $2l-j-1$ twos. In fact, the precise number of twos is $2l-j-n$, where n is the number of parts in each partition summing to j. Therefore, before the other parts in each partition can be processed, the program must print out this number of twos, which is accomplished by the first and second print statements in **termgen** of Program 4 in the appendix. The final print statement appearing in the for loop prints out the values of the remaining parts and their multiplicities. Note that each part i in the code is incremented by 2 as discussed in the previous paragraph. Therefore, running the program with *tot* set equal to 4 generates the following output:

LR[l_,4]:= l!/(2^(l-5) (l-5)! 6^(1) 1!) + l!/(2^(l-6) (l-6)! 3^(1) 1! 5^(1) 1!) + l!/(2^(l-7) (l-7)! 3^(2) 2! 4^(1) 1!) + l!/(2^(l-8) (l-8)! 3^(4) 4!) + l!/(2^(l-6) (l-6)! 4^(2) 2!).

As can be seen in the above statement, there are five distinct contributions corresponding to the number of integer partitions summing to 4. Furthermore, the above output can be imported into Mathematica, whereupon by invoking FullSimplify and Expand routines, one arrives at

In[5]:= FullSimplify[Expand[LR[l, 4]]]
Out[5]= (2^(1 - l) (-195 + l (487 + 40 l (-9 + 2 l))) l!)/(1215 Gamma[-4 + l]).

Applying the Expand routine only to the polynomial in the above result, one obtains the $j=4$ result in Table 6, which also displays the first ten values of $L_R(l,j)$ obtained by this procedure.

In Table 6 the binomial factor of $\binom{l}{j+1}$ has been extracted for each value of $L_R(l,j)$. Consequently, we observe that each expression possesses a polynomial of degree $j-1$ in l with the highest order coefficient always positive, while the other coefficients alternate in sign. On this occasion, however, if one types the coefficients from the polynomials into the online encyclopedia of integer sequences, there is no sequence matching it.

Now that the $L_R(l,j)$ have been evaluated, we turn our attention to Equation (71) so that we can determine the Stirling numbers of the first kind as functions of both l and k. Previously, it was mentioned that the first sum on the rhs of Equation

j	$L_R(l,j)$
0	$\frac{1}{2^l l!}$
1	$\frac{2^{2-l}}{3(l-2)!}$
2	$\frac{2^{1-l}}{9(l-3)!}\big(4l-3\big)$
3	$\frac{2^{3-l}}{405(l-4)!}\big(20l^2-45l+22\big)$
4	$\frac{2^{1-l}}{1215(l-5)!}\big(80l^3-360l^2+487l-195\big)$
5	$\frac{2^{3-l}}{25515(l-6)!}\big(112l^4-840l^3+2177l^2-2289l+810\big)$
6	$\frac{2^{2-l}}{1148175(l-7)!}\big(2240l^5-25200l^4+105980l^3-206955l^2$ $+185729l-60228\big)$
7	$\frac{2^{4-l}}{3444525(l-8)!}\big(320l^6-5040l^5+31220l^4-96915l^3$ $+157919l^2-126708l+38448\big)$
8	$\frac{2^{1-l}}{10333575(l-9)!}\big(1280l^7-26880l^6+230048l^5-1036560l^4$ $+2642669l^3-3784806l^2+2789487l-802710\big)$
9	$\frac{2^{3-l}}{15345358875(l-10)!}\big(70400l^8-1900800l^7+21473760l^6$ $-132224400l^5+483575235l^4-1070117730l^3$ $+1389159277l^2-956995182l+263580120\big)$
10	$\frac{2^{2-l}}{322252536375(l-11)!}\big(394240l^9-13305600l^8+191748480l^7$ $-1545334560l^6+7653524340l^5-24064075035l^4$ $+47782284242l^3-57358923237l^2+37381321242l$ $-9920458152\big)$

Table 6: $L_R(l,j)$ for $j \leq 10$, where $f(i,\lambda_i) = 1/i^{\lambda_i}\lambda_i!$.

l	$s1(k,l)$
0	0
1	$\frac{1}{2}k$
2	$\frac{3}{8}k^2 - \frac{1}{12}k$
3	$\frac{17}{48}k^3 - \frac{1}{8}k^2$
4	$\frac{403}{1152}k^4 - \frac{5}{32}k^3 + \frac{1}{288}k^2 + \frac{1}{120}k$
5	$\frac{811}{2304}k^5 - \frac{53}{288}k^4 + \frac{13}{2880}k^3 + \frac{1}{48}k^2$
6	$\frac{16487}{46080}k^6 - \frac{14443}{69120}k^5 + \frac{13}{3840}k^4 + \frac{2029}{51840}k^3 - \frac{1}{1440}k^2 - \frac{1}{252}k$
7	$\frac{101093}{276480}k^7 - \frac{10673}{46080}k^6 - \frac{197}{207360}k^5 + \frac{6679}{103680}k^4 - \frac{1}{2240}k^3 - \frac{1}{72}k^2$
8	$\frac{34810009}{92897280}k^8 - \frac{418229}{1658880}k^7 - \frac{235583}{24883200}k^6 + \frac{47003}{483840}k^5 + \frac{33227}{17418240}k^4$ $- \frac{8513}{241920}k^3 + \frac{221}{604800}k^2 + \frac{1}{240}k$
9	$\frac{214300123}{557383680}k^9 - \frac{840907}{3110400}k^8 - \frac{7999633}{348364800}k^7 + \frac{4797361}{34836480}k^6 + \frac{5077}{552960}k^5$ $- \frac{181927}{2419200}k^4 - \frac{59}{134400}k^3 + \frac{3}{160}k^2$
10	$\frac{4398344897}{11147673600}k^{10} - \frac{4785174047}{16721510400}k^9 - \frac{8388953}{199065600}k^8 + \frac{388855811}{2090188800}k^7$ $+ \frac{5955563}{232243200}k^6 - \frac{149442551}{1045094400}k^5 - \frac{56549}{10886400}k^4 + \frac{434579}{7257600}k^3 - \frac{23}{60480}k^2$ $- \frac{1}{132}k$

Table 7: Values of the first sum in Equation (70) as functions of k for $l \leq 10$.

(70) can be evaluated once the values of $L_R(l,j)$ are known. Let us denote this sum as

$$s1(k,l) = (-1)^l \sum_{i=0}^{l} k^i \sum_{j=0}^{l-1} s(2l-j,i) L_R(l,j). \tag{72}$$

To evaluate these polynomials in Mathematica, first we need to introduce the values of $L_R(l,j)$ in Table 6 into a notebook. Then we insert the following command:

```
s1[k_, l_] := (-1)^l Sum[k^i Sum[StirlingS1[2l-j, i] LR[l, j], {j,0,l-1}], {i, 0, l}] .
```

For example, typing in s1[k,5] into the notebook generates the following output for $l = 4$:

Out[18]= $\frac{k^2}{48} + \frac{13k^3}{2880} - \frac{53k^4}{288} + \frac{811k^5}{2304}$.

This result appears as the $l=5$ value of $s1(k,l)$ in Table 7, which displays the values of $s1(k,l)$ generated by Mathematica up to $l=10$.

Again, it needs to be emphasized that the results in Table 7 represent the lowest order terms in k for the Stirling numbers of the first kind, while the highest order terms from $l+1$ to $2l$ are contained in the second sum on the rhs of Equation (70), which has yet to be studied in terms of l. Hence the results in Table 7 are incapable of providing a reasonable approximation to the Stirling numbers of the first kind when k is large. For example, $s1(k,1) = k/2$ provides the $k/2$ term for $s(k,k-1)$ in Table 7, but the dominant term of $-k^2/2$ is missing.

We have already determined the coefficients of the leading order and first order terms, i.e., $s_{2l,0}$ and $s_{2l,1}$, in the Stirling numbers of the first kind. In these cases there was only one partition that appeared in the analysis of the second sum on the rhs of Equation (70). To allow for more partitions to appear in the lower order coefficients, the second sum on the rhs of Equation (70) was developed further resulting in Equation (71), where the coefficients are now expressed in terms of the reduced partition operator displayed in Table 6. Therefore, we can now determine the coefficient of the next leading order term or the terms comprising k^{2l-2} term in the Stirling numbers of the first kind far more expediently. From Equation (71), we obtain

$$\begin{aligned} s_{2l,2}(l) = & (-1)^l \sum_{i=0}^{2} s(2l-i,2l-2) L_R(l,i) = (-1)^l s(2l,2l-2) L_R(l,0) \\ & + (-1)^l s(2l-1,2l-2) L_R(l,1) + (-1)^l s(2l-2,2l-2) L_R(l,2). \end{aligned}$$

The Stirling numbers of the first kind on the rhs of the above result can be obtained from Table 3. Thus, it was stated previously that to determine the l-dependence of in the coefficients of the powers of k in the Stirling numbers of the first kind, one requires them for the specific values of l in Table 3. Note also that the results in the table must have k replaced by $2l$ minus the value of summation index i. On the other hand, the values of $L_R(l,i)$ for $i \le 10$ are presented in Table 6. Consequently, $s_{2l,2}(l)$ can be expressed as

$$s_{2l,2}(l) = \frac{(-1)^l}{2^l l!}\Big(2l^4 - 10l^3/3 + 3l^2/2 - l/6 - 8l^4/3 + 20l^3/3 - 16l^2/3 + 4l/3 + 8l^4/9 - 10l^3/3 + 34l^2/9 - 4l/3\Big). \tag{73}$$

Applying the Factor command in Mathematica to the polynomial in the above result yields

In[12]:= Factor[-(l/6) - l^2/18 + (2 l^4)/9]

Out[12]= (1/18) (-1 + l) l (3 + 4 l + 4 l^2).

By cancelling $l(l-1)$ with the l factorial in the denominator of Equation (73), we arrive at the coefficient of k^{2l-2} in the Stirling numbers of the first kind. This is displayed on the fourth or $j = 3$ row of Table 8.

Table 8 displays the highest order coefficients in k up to $2l - 10$ for the two parameter version of the Stirling numbers of the kind or $s(k, k-l)$. As can be seen from the table, all the coefficients possess $(l-m)!$ in their denominators, where m is an integer including zero, but is not related to the power of k. In fact, m increments by unity when the leading power of l in the coefficient has increased by unity compared to its immediate predecessor and remains the same value as for the immediate predecessor when the leading power of l has increased by more than unity. For example, the leading power of l in $s_{2l,5}(l)$ is 6 compared with 5 in $s_{2l,4}(l)$. Thus, we see that the denominator in $s_{2l,5}(l)$ possesses $(l-4)!$ compared with $(l-3)!$ in $s_{2l,4}(l)$. On the other hand, the leading power of l in $s_{2l,6}$ is 8, but the denominator possesses a factor of $(l-4)!$, the same as $s_{2l,5}(l)$.

From the results in Table 8, we see that the coefficients, $s_{2l,j}(l)$, will only contribute if $l \ge m$. In addition, they possess a common factor of $(-1/2)^l$ in the numerator, while the numerical value in the denominator corresponds with the factor in the denominator of the results for the $L_R(l,j)$ in Table 6. Although they really apply for large values of l, they can be used in conjunction with the $s1(k,l)$ in Table 7 to determine the Stirling numbers of the first kind appearing in Table 3. However, due care must be exercised as explained below.

Consider $l = 1$. From Table 8, $s_{2,0}(1) = -1/2$. Hence this term yields the dominant term of $-k^2/2$ for $s(k, k-1)$. If we add the $l = 1$ result in Table 7 to the dominant term, then we obtain $s(k, k-1)$ or $-k^2/2 + k/2$ as displayed in Table 3. However, we can put $l = 1$ into $s_{2l,1}(l)$ appearing in Table 8. Then we obtain $k/2$. In this case we only accept the contribution from $s1(k,1)$ or from $s_{2,1}(1)$, but not both. This is because, according to Equation (70), none of the powers of k

j	$s_{2l,j}(l)$
0	$\frac{(-1/2)^l}{l!}$
1	$-\frac{(-1/2)^l}{3(l-1)!}\big(2l+1\big)$
2	$\frac{(-1/2)^l}{18(l-2)!}\big(4l^2+4l+3\big)$
3	$\frac{(-1/2)^{l+1}}{405(l-2)!}\big(40l^4-20l^3-22l^2-85l+48\big)$
4	$-\frac{(-1/2)^{l+3}}{1215(l-3)!}\big(80l^5-80l^4-56l^3-304l^2+405l-207\big)$
5	$\frac{(-1/2)^{l+3}}{25515(l-4)!}\big(224l^6-336l^5-112l^4-1176l^3+106790l^2$ $-935721l+2075040\big)$
6	$\frac{(-1/2)^{l+4}}{1148175(l-4)!}\big(2240l^8-13440l^7+18480l^6-15120l^5$ $+122676l^4-342432l^3+384569l^2-366513l+32940\big)$
7	$-\frac{(-1/2)^{l+4}}{3444525(l-4)!}\big(640l^{10}-7360l^9+28320l^8-44880l^7$ $+89832l^6-451812l^5+1233710l^4-1835975l^3$ $+1958373l^2-852588l+1270080\big)$
8	$\frac{(-1/2)^{l+7}}{10333575(l-5)!}\big(1280l^{11}-17920l^{10}+86016l^9-174336l^8$ $+326112l^7-1740288l^6+6139312l^5-11885264l^4$ $+16332453l^3-13062042l^2+15149673l-36384390\big)$
9	$\frac{(-1/2)^{l+7}}{15345358875(l-6)!}\big(140800l^{12}-2323200l^{11}+13418240l^{10}$ $-33749760l^9+63919680l^8-335348640l^7+1457426432l^6$ $-3538048272l^5+5957077510l^4-6831367755l^3$ $+4709121147l^2-33568759422l+37752946560\big)$
10	$\frac{(-1/2)^{l+8}}{322252536375(l-6)!}\big(394240l^{14}-9856000l^{13}+95701760l^{12}$ $-4590924800l^{11}+11213709091200l^{10}-383166378672000l^9$ $+5851111156004576l^8-52579582441143296l^7$ $+307925976912267796l^6-1228034255151640996l^5$ $+3377645217003820839l^4-6326617033681706214l^3$ $+7723529834025373749l^2-5549283333556239654l$ $+1781922994140980880\big)$

Table 8: Highest order coefficients in powers of k^{2l-j} for $s(k,k-l)$.

from the results in Table 8 should be the same as any of the powers of k in $s1(k,l)$. That is, double-counting of the coefficients of the lower powers must be avoided by ensuring that the sum over $s_{2l,j}(l)$ ranges from 0 to $l-1$. Hence the contribution from $s_{2,1}(1)$ is discarded, while $s1(k,1)$ can be added directly to yield $s(k,k-1)$. More generally, this can be written as

$$s(k,k-l) = \sum_{j=0}^{l-1} s_{2l,j}(l)\, k^{2l-j} + s1(k,l). \tag{74}$$

Since $s1(k,l)$ is a polynomial of degree l, we see that there is no double-counting of the coefficients because the lowest order of k from the first sum is $l+1$. Note also that each non-zero $s_{2l,j}(l)$ in $s(k,k-l)$ yields the coefficient of k^{2l+2-j} in $s(k,2l+2j-1)$, but evaluated with l replaced by $l+1$. Consequently, the results in Table 7 yield the right diagonal coefficients for all $s(k,k-l)$.

Let us confirm Equation (74) by considering $l=4$. Then we find that

$$\begin{aligned} s(k,k-4) =& s_{8,0}(4)k^8 + s_{8,1}(4)k^7 + s_{8,2}(4)k^6 + s_{8,3}(4)k^5 + \frac{403k^4}{1152} \\ & - \frac{5k^3}{32} + \frac{k^2}{288} + \frac{k}{120}. \end{aligned} \tag{75}$$

From Table 8 we find that

$$s_{8,0}(4) = (-1/2)^4/4! = 1/384, \quad s_{8,1}(4) = -(-1/2)^4 \times 9/(3\times 3!) = -1/32,$$
$$s_{8,2}(4) = (-1/2)^4(4\times 4^2 + 4\times 4 + 3)/(18\times 2!) = 83/576,$$

and

$$\begin{aligned} s_{8,3}(4) &= (-1/2)^5(40\times 4^4 - 20\times 4^3 - 22\times 4^2 - 85\times 4 + 48)/(415\times 2!) \\ &= -2079/66540. \end{aligned}$$

Therefore, Equation (75) becomes

$$s(k,k-4) = \frac{k^4}{384} - \frac{k^7}{32} + \frac{83k^6}{576} - \frac{77k^5}{240} + \frac{403k^4}{1152} - \frac{5k^3}{32} + \frac{k^2}{288} + \frac{k}{120},$$

which is identical to the $l=4$ result in Table 3.

We can also use Equation (74) to check the resulting expressions obtained for the $s_{2l,j}$, which is recommended in view of how cumbersome or unwieldy they become when $j \geq 5$. In fact, this was done for all the results in Table 8. To see this more clearly, suppose that we already know or have verified the results to $j=3$ in Table 8. Next we determine the $j=4$ result in the table by using the above method. At this stage we do not know if the new expression for $j=4$ is correct since it has not been verified. From Table 8, we observe that $s1(k,5)$ is a quintic in k. So if we wish to introduce it into Equation (74), then the lowest power in the first sum must be

k^6, which implies that $l = 5$. Since $l = 5$, we introduce $s1(k, 5)$ into Equation (74), which gives

$$s(k, k-5) = s_{2l,0}(5)k^{10} + s_{2l,1}(5)k^9 + s_{2l,2}(5)k^8 + s_{2l,3}(5)k^7 + s_{2l,4}(5)k^6 + \frac{811}{2304}k^5 - \frac{53}{288}k^4 + \frac{13}{2880}k^3 + \frac{1}{48}k^2.$$

Because the expressions for the four highest coefficients, i.e., $s_{2l,0}(l)$ to $s_{2l,4}(l)$ have already been verified, all we need to do is put $l = 5$ in them. Consequently, $s_{2l,0}(5) = -1/3840$, $s_{2l,1}(5) = 11/2304$, $s_{2l,2} = 41/1152$, and $s_{2l,3}(5) = 799/5760$. These not only represent the four highest order coefficients for the $l = 5$ result in Table 3, but also the coefficients from k^5 down also agree. Thus, we need to verify that the expression for $s_{2l,4}(5)$ agrees with the coefficient of k^6 in the $l = 5$ result in Table 3. Putting $l = 5$ into the $s_{2l,4}$ result of Table 8 yields

$$\begin{aligned} s_{2l,4}(5) &= \frac{(-1/2)^8}{2430}\Big(80\big(5^5\big) - 80\big(5^4\big) - 56\big(5^3\big) - 304(5^2) + 405 \times 5 - 207\Big) \\ &= \frac{3467}{11520}. \end{aligned}$$

This agrees with the value of the coefficient of k^6 in the $l = 5$ result of Table 3, which is given as $3467/(3 \times 3840)$. By adopting this approach, one can, therefore, verify all the results in Table 8.

4. Stirling Numbers of the Second Kind

In Section 2, we found that the main difference between the formulations of the Stirling numbers of both kinds via the partition method for a power series expansion was that in the case of the Stirling numbers of the second kind, each part i was assigned a value of $i!$, whereas for those of the first kind, each part i was assigned a value of i only. This is readily observed by comparing Equation (16) with Equation (22). In this section we apply the analysis of the previous section to determine the corresponding results for the Stirling numbers of the second kind.

Let us review how Equation (16) can be used to obtain the Stirling numbers of the second kind. It has already been stated that $S(n, j)$ represents the number of objects, n, which can be divided into j non-empty subsets or groups. From the partition tree in Figure 1, we see that j corresponds to the paths terminating with j branches. That is, if we fix a value of j, then the Stirling number of the second kind will be determined from those partitions with a terminating tuple after j branches. For example, $S(6, 3)$ will be determined using those tuples with a zero vertically after three branches, of which there are three. Specifically, these are the partitions, $\{1_2,4\}$, $\{1,2,3\}$ and $\{2_3\}$. Thus, the number of groups in the Stirling numbers of the second kind corresponds to using the same number of parts in the partitions summing to n.

To evaluate the Stirling numbers of the second kind, we introduce the partitions for that number of parts or j into Equation (16). Therefore, $S(6,3)$ is given by

$$S(6,3) = 6!\Big(\frac{1}{2!1!}\frac{1}{1!4!} + \frac{1}{1!1!}\frac{1}{2!1!}\frac{1}{3!1!} + \frac{1}{(2!)^3 3!}\Big) = 15 + 60 + 15 = 90.$$

It is interesting to observe that the single discrete partition makes the biggest contribution to $S(6,3)$, which is due to the fact that each part in a discrete partition has a multiplicity of unity. This does not apply to the Stirling numbers of the first kind because the values of the parts in the partitions are not factorials.

As was done for the Stirling numbers of the first kind, we can evaluate $S(k,k-l)$ for fixed values of l ranging from 0 to 10. In fact, all we need to do is replace the value of each part i by $i!$ instead of i. For $l=0$, which represents the case where k objects are divided into a maximum of k groups, the corresponding result for the Stirling numbers of the first kind appears in the paragraph above Equation (40). A maximum of k groups only occurs when each part represents a group, i.e., the parts are only equal to unity. In other words, we need to consider the partition, $\{1_k\}$. Then Equation (16) yields $S(k,k) = k!L^{(k)}_{P,k}[\prod_{i=1}^{k} 1/(i!)^{\lambda_i}\lambda_i!] = k!/(1!)^k k! = 1$, which is identical to $s(k,k)$. For the situation where we wish to divide k objects into $k-1$ groups, this means we must have $k-2$ single object groups and one group with 2 objects, corresponding to the partition, $\{1_{k-1},2\}$. Thus, we find that $S(k,k-1) = k!/(1!)^{k-1}(k-2)!(2!)1! = k(k-1)/2$ or $S(k,k-1) = \binom{k}{2}$. Therefore, except for a change of sign, we find that $s(k,k-1)$ and $S(k,k-1)$ agree. In other words, $S(k,k-1) = |s(k,k-1)|$. In addition, from Equations (17) and (18), we find that the special Worpitzky number with $n = k$ and $j = k-1$ is given by $W^{(k)}_{k-1,k-1} = (k-1)k!/2$.

If we wish to divide k objects into $k-2$ groups, the situation is no longer as simple as the first two examples since there is more than one method of creating the groups. In this instance, we can have either $k-3$ single object groups and one group with 3 objects or we can have $k-4$ single object groups and two groups with 2 objects in them. In short, these correspond or are homologous to the partitions, $\{1_{k-3},3\}$ and $\{1_{k-4},2_2\}$. Consequently, we arrive at

$$\begin{aligned} S(k,k-2) &= k!L^{(k)}_{P,k}\left[\prod_{i=1}^{k} 1/(i!)^{\lambda_i}\lambda_i!\right] = k!\Big(\frac{1}{(1!)^{k-3}(k-3)!(3!)1!} \\ &\quad + \frac{1}{(1!)^{k-4}(k-4)!(2!)^2 2!}\Big) = k!\Big(\frac{1}{8(k-4)!} + \frac{1}{(6(k-3)!}\Big) \\ &= \frac{1}{24}(3k-5)(k-2)(k-1)k. \end{aligned}$$

Hence from Equations (17) and (18), $W^{(k)}_{k-2,k-2} = (3k-5)(k-2)k!/24$.

Since the same partitions apply in the evaluation of $S(k, k-2)$ as those for $s(k, k-2)$, the same three partitions for evaluating $s(k, k-3)$ will apply to $S(k, k-3)$, which are $\{1_{k-4}, 4\}$, $\{1_{k-5},2,3\}$ and $\{1_{k-6}, 2_3\}$. Therefore, $S(k, k-3)$ is composed of $p(3)$ or three different methods of arranging k objects into $k-3$ groups. First, we have the case where there are $k-4$ single object groups and one group with 4 objects. Then we have the case of $k-5$ single object groups, and two remaining groups, one consisting of 2 objects and the other, 3 objects. Finally, there is the case where there are $k-6$ single object groups and three groups with 2 objects or paired groups. Each case makes a contribution to $S(k, k-3)$. For example, the contribution from the first case is obtained by taking the first term on the rhs of the intermediate member of Equation (41), replacing 4 in the term by 4! and multiplying by $k!$. This yields $\binom{k}{4}$. To obtain the contribution from the second case, we apply the same procedure to the second term of the intermediate member in Equation (41) except that 2 is replaced by 2! and 3 by 3!. Hence we find that arranging k objects into $k-5$ single object groups and two groups with 2 and 3 objects contributes $10\binom{k}{5}$ to $S(k, k-3)$, while the final term in the intermediate member of Equation (41) represents the case where there are $k-6$ single object groups and three paired groups. This contributes $15\binom{k}{6}$. Summing these contributions yields $(k-2)(k-3)\binom{k}{4}/2$ for $S(k, k-3)$, while from Equations (17) and (18), we find that $W^{(k)}_{k-3,k-3} = (k-3)^2(k-2)k!/48$.

From the above analysis, it is obvious that arranging k objects into $k-l$ groups, where l is fixed, is dependent upon determining the partitions summing to k with $k-l$ parts as was the case for $s(k, k-l)$ in the previous section. Moreover, there will be $p(l)$ ways of arranging the $k-l$ groups. For $l = 4$, $p(4) = 5$. However, in the analysis of $s(k, k-4)$, a new operator was devised, which enabled the exclusion of ones from appearing in the analysis. Thus, Equation (44) was used to evaluate $s(k, k-4)$. Since the same partitions are involved in the determination of $S(k, k-4)$, all that we need to do is adapt Equation (44) so that the parts i yield a value of $i!$ instead of i. Hence we arrive at

$$S(k, k-4) = \binom{k}{5} \sum_{j=1}^{4} L^{(j)}_{P,(j+4)/\{1\}} \left[\prod_{i=2}^{j+4} \frac{5!}{i!^{\lambda_i}\, \lambda_i!} \right] \prod_{i=1}^{j-1} (k-i-4). \tag{76}$$

By excluding unity in the partitions, much computation is avoided with this result. For $j = 1$, we need only consider the partitions summing to 5 with one part that must be greater than unity. This is the sole partition $\{5\}$. For $j = 2$, we consider the parts summing to 6 with two parts greater than unity, namely, $\{2,4\}$ and $\{3_2\}$. When $j = 3$, the three parts must sum to 7, resulting in the partition, $\{2_2, 3\}$. The upper limit of the sum over j in Equation (76) is 4, which means partitions summing to 8 with 4 parts greater than unity or $\{2_4\}$. Therefore, Equation (76) reduces to

$$S(k,k-4) = \binom{k}{5}\Big(\frac{1}{120} + \Big(\frac{1}{48}+\frac{1}{72}\Big)(k-5) + \frac{1}{48}(k-5)(k-6) + \frac{1}{384}(k-5) \\ \times\ (k-6)(k-7)\Big) = \frac{1}{48}\Big(15k^3 - 150k^2 + 485k - 502\Big)\binom{k}{5}. \quad (77)$$

For $k=167$, Equation (77) yields a value of 1 395 960 899 099 833, which agrees with the value obtained from StirlingS2(167,163) in Mathematica.

Another advantage of Equation (76) is that it can easily be generalized to the situation of arranging k objects into $k-l$ groups by replacing 4 with l as follows:

$$S(k,k-l) = \binom{k}{l+1}\sum_{j=1}^{l} L^{(j)}_{P,(j+l)/\{1\}}\left[\prod_{i=2}^{j+l}\frac{(l+1)!}{i!^{\lambda_i}\lambda_i!}\right]\prod_{i=1}^{j-1}(k-i-l). \quad (78)$$

Hence there are $p(l)$ ways of arranging k objects into $k-l$ groups. Table 9 displays $S(k,k-l)$ for fixed values of l up to and including $l=10$. These results were obtained by implementing Equation (78) in Mathematica. An interesting property of these results is that for odd values of l, the polynomials can be factored further, always yielding $(k-l+1)(k-l)$ times a polynomial of degree, $l-3$. This is different from the odd l values of $s(k,k-l)$, which always yielded $k(k-1)$ times a polynomial of degree, $l-3$. Finally, multiplying each $S(k,k-l)$ in the table by $(k-l)!$ yields $W^{(k)}_{k-l,k-l}$.

The results in Table 9 can be used in conjunction with Equation (29) to evaluate polynomial expressions of the Stirling polynomials with negative integer arguments. First, we replace $k+n$ and n in Equation (29) by k and $k-l$, respectively. This gives

$$S_l(l-k-1) = (-1)^l\,\frac{(k-l)!l!}{k!}\,S(k,k-l). \quad (79)$$

The above result simplifies drastically because the results in Table 9 can be expressed in the form of $S(k,k-l) = C_l f_l(k)\binom{k}{l+1}$, where $f_l(k)$ is the polynomial in powers of k with degree $l-1$. This was first noticed on p. 50 of [14], where the C_l are combined with $f_l(k)$ to form $R_l(k)$, which are, in turn, tabulated in Table 2.1 of the same reference. Consequently, we can write

$$S_l(l-k-1) = \binom{\frac{k-l}{l+1}}{}\,C_l f_l(k).$$

For $l=10$, where $C_l = 1/9216$, typing the following lines/instructions into Mathematica

```
SP[k_, l_] := (-1)^l (k - l) Const[l] F[k, l]/(l + 1)
Const[10] := 1/9216
F[k_, 10] := -10307425152 + 13175306672 k - 7220722828 k^2 + 2242194529
```

l	$S(k,k-l)$
0	1
1	$\binom{k}{2}$
2	$\frac{1}{4}\big(3k-5\big)\binom{k}{3}$
3	$\frac{1}{2}\big(k-2\big)\big(k-3\big)\binom{k}{4}$
4	$\frac{1}{48}\big(15k^3-150k^2+485k-502\big)\binom{k}{5}$
5	$\frac{1}{16}\big(3k^4-50k^3+305k^2-802k+760\big)\binom{k}{6}$
6	$\frac{1}{576}\big(63k^5-1575k^4+15435k^3-73801k^2+171150k$ $-152696\big)\binom{k}{7}$
7	$\frac{1}{144}\big(9k^6-315k^5+4515k^4-33817k^3+139020k^2$ $-295748k+252336\big)\binom{k}{8}$
8	$\frac{1}{3840}\big(135k^7-6300k^6+124110k^5-1334760k^4$ $+8437975k^3-31231500k^2+62333204k$ $-51360816\big)\binom{k}{9}$
9	$\frac{1}{768}\big(15k^8-900k^7+23310k^6-339752k^5+3040975k^4$ $-17065540k^3+58415444k^2-110941776k$ $+88864128\big)\binom{k}{10}$
10	$\frac{1}{9216}\big(99k^9-7425k^8+244530k^7-4634322k^6$ $+55598235k^5-436886945k^4+2242194592k^3$ $-7220722828k^2+13175306672k-10307425152\big)\binom{k}{11}$

Table 9: Stirling numbers of the second kind, $S(k, k-l)$, for fixed values of l.

k^3 - 436886945 k^4 + 5559823 k^5 - 4634322 k^6 + 244530 k^7 - 7425 k^8 + 99 k^9
Expand[SP[k, 10]]

generates the following output:

$$\frac{134211265}{132} - \frac{4439390371k}{3168} + \frac{107806231k^2}{128} - \frac{14821334059k^3}{50688} + \frac{2203687993k^4}{33792} - \frac{54720575k^5}{11264} + \frac{51903043k^6}{101376} - \frac{107267k^7}{1536} + \frac{805k^8}{256} - \frac{85k^9}{1024} + \frac{k^{10}}{1024}.$$

By replacing m in the expression for $S_{10}(m)$ above Equation (58) by $9-k$ in accordance with Equation (79), one obtains the same result as given above.

If one types the first few coefficients of the polynomials in Table 9 or the $R_l(k)$ in Table 2.1 of [14], into the online encyclopedia of integer sequences, viz., -5,3,6,-5,1,-502,485, then one is immediately directed to Sequence A075264 [11]. Here the unsigned versions of the coefficients are stated as being the triangle of numerators of coefficients, where the n-th row forms a polynomial in z, denoted by $P(n, z)$, that corresponds to the coefficient of x^n in the generating function of $(-\log(1-x)/x)^z$, for $n>0$. In addition, the denominators of the polynomials are represented by Sequence A053657, while the polynomials can be related to the generalized reciprocal logarithm numbers, $A_k(s)$, discussed in Chapter 2.3 of [14], but which were first studied extensively in [12]. Comparing Table 2.5 of [14] with Table 9 here, we see that each A_k is not only expressed as a monic polynomial of degree k divided by $2^k k!$, but that the polynomials do not always possess integer coefficients as those in Table 9 or Table 2.1 in [14]. To obtain the polynomials in the latter tables, the lowest common denominator must be extracted from the forms in Table 2.5 of [14]. When this is done, the resulting polynomials are identical to those in Tables 9 and 2.1 of [14], provided the latter are multiplied by k/l. This is similar to the situation for the $s(k, k-l)$, where we needed to extract k/l from the binomial factor to relate the $r_l(k)$ to the Nörlund polynomials. That is, the $R_l(k)$ need to be multiplied by k/l in order to give the generalized reciprocal logarithm numbers. Consequently, we arrive at

$$S(k, k-l) = \frac{(k-1)!}{(k-l-1)!} A_l(k). \tag{80}$$

In [12] it is shown that the generalized reciprocal logarithm numbers obey the finite sum given by

$$A_k(s+t) = \sum_{j=0}^{k} A_{k-j}(s) A_j(t). \tag{81}$$

This equation also appears as (2.166) in [14]. By replacing s, t and k in the above result by k_1, k_2 and l, respectively, and introducing Equation (80) into Equation

(81), one obtains

$$S(k_1+k_2,k_1+k_2-l) = \frac{(k_1+k_2-1)!}{(k_1-1)!(k_2-1)!} \sum_{j=0}^{l} \frac{(k_1-l+j-1)!(k_2-j-1)!}{(k_1+k_2-l-1)!} \times \quad S(k_1,k_1-l+j)S(k_2,k_2-j). \tag{82}$$

More compactly, Equation (82) can be expressed as

$$S(k_1+k_2,k_1+k_2-l) = \sum_{j=0}^{l} \frac{B(k_1-l+j,k_2-j)}{B(k_1,k_2)} S(k_1,k_1-l+j)S(k_2,k_2-j), \tag{83}$$

where $B(x,y)$ represents the beta function.

Let us verify that Equation (82) is indeed valid by setting $k_1 = 33$, $k_2 = 34$ and $l = 16$. Then by typing into Mathematica

StirlingS2[67,51],

we obtain the following output:

73667502745983700604456744062272831426.

On the other hand, the rhs of Equation (82) must be typed in as:

S[k1_, k2_, l_] := ((k1 + k2 -1)!/((k1 - 1)! (k2 - 1)! (k1 + k2 - l - 1)!)) Sum[StirlingS2[k1, k1 - l + j] StirlingS2[k2, k2 - j] ((k1 - l + j - 1)! (k2 - j - 1)!), j, 0, l].

By inserting the values of k_1, k_2 and l, one finds that

In[30]:= S[33, 34, 16]

Out[30]= 73667502745983700604456744062272831426.

Therefore, we have verified Equation (82) or its more elegant version, Equation (83).

As an interesting conjecture, let us consider replacing the Stirling numbers of the second kind by the Stirling numbers of the first kind in Equations (82) and (83). In this instance, we shall let $k1 = 34$, $k2 = 45$ and $l = 21$. Now typing into Mathematica the command StirlingS1[79,58] yields

-7012150967257561932831853219046166126571671650580282,

while the new instruction becomes

S1[k1_, k2_, l_] := ((k1 + k2 -1)!/((k1 - 1)! (k2 - 1)! (k1 + k2 - l - 1)!)) Sum[StirlingS1[k1, k1 - l + j] StirlingS1[k2, k2 - j] ((k1 - l + j - 1)! (k2 - j - 1)!), j, 0, l].

Introducing the values of $k1$, $k2$ and l given above, we arrive at

In[5]:=S1[34,45,21]

Out[5]:= -7012150967257561932831853219046166126571671650580282.

Therefore, S in Equations (82) and (83) can be either kind of Stirling number.

We can also consider small values of j, i.e., $j = 1, 2, 3, \dots$, in Equation (16). For $j = 1$, we expect $S(k, 1)$ to equal unity since there is only one group that can contain all k objects. In this instance Equation (16) reduces to

$$S(k,1) = k!\, L_{P,k}^{(1)} \Big[\prod_{i=1}^{k} \frac{1}{i!^{\lambda_i} \lambda_i!} \Big]. \tag{84}$$

According to the definition of the above operator in Equation (15), the sum of the multiplicities must equal unity in Equation (84), i.e., $\sum_{i=1}^{k} \lambda_i = 1$, together with the other constraint, $\sum_{i=1}^{k} i\lambda_i = k$. From the first constraint only one λ_i can equal unity with all the remaining multiplicities vanishing, while from the second constraint, this can only be λ_k. That is, $\lambda_k = 1$ with all the other multiplicities vanishing. Consequently, Equation (84) simplifies to $S(k, 1) = k!/k! = 1$, while from Equation (17), we have $W_{1,1}^{(k)} = 1$.

For $j = 2$, Equation (16) gives

$$S(k,2) = k!\, L_{P,k}^{(2)} \Big[\prod_{i=1}^{k} \frac{1}{i!^{\lambda_i} \lambda_i!} \Big].$$

In this case, the constraints in the restricted partition operator require that $\sum_{i=1}^{k} \lambda_i = 2$ and $\sum_{i=1}^{k} i\lambda_i = k$. There are two separate solutions. Either one $\lambda_i = 2$ and the other multiplicities are zero or there are two λ_i's, both equalling unity, while the other multiplicities equal zero. For the first solution, we also have $2i_1 = k$ or $i_1 = k/2$, which means that there is only a solution when k is an even integer. Hence the contribution from this solution in the partition operator must be multiplied by $(1 + (-1)^k)/2$. For the second solution, we also have $i_1 + i_2 = k$ or $i_2 = k - i_1$. In addition, $i_1 < i_2$ to avoid duplicating or repeating partitions. Furthermore, i_1 ranges from unity to $\lfloor k/2 \rfloor$ when k is an odd integer and ranges from unity to $k/2 - 1$ when k is an even integer. Thus, Equation (16) becomes

$$S(k,2) = k! \Big(\sum_{i=1}^{k_*} \frac{1}{i!(k-i)!} + \big(\frac{1 + (-1)^k}{2} \big) \frac{1}{2((k/2)!)^2} \Big),$$

where

$$k_* = \begin{cases} \lfloor k/2 \rfloor, & k \text{ odd}, \\ k/2 - 1, & k \text{ even}. \end{cases}$$

By writing 2^k as $(1+1)^k$ and applying the binomial theorem, we find that $S(k, 2) = 2^{k-1} - 1$, irrespective of whether k is an odd or even integer. This agrees with (2.29) in [14] and (6) in [34]. It can also be obtained by putting $j = 2$ in Equation (8). Furthermore, from Equation (17), we find that $W_{2,2}^{(k)} = 2^k - 2$.

To determine $S(k,3)$, we can use the above method, but it will be messy. However, we have already determined $L_{P,l+3/1}^{(3)}\Big[\prod_{i=2}^{l+1} 1/i^{\lambda_i}\lambda_i!\Big]$ for the Stirling numbers of the first kind, which is given by Equation (60). We can adapt this result by noting that: (1) we are no longer excluding parts with ones, (2) $l+3$ must be substituted by k and (3) all the quantities in the denominators need to be replaced by their factorial forms. Then we arrive at

$$\begin{aligned} S(k,3) =& k! L_{P,k}^{(3)}\Bigg[\prod_{i=1}^{k-2}\frac{1}{i!^{\lambda_i}\lambda_i!}\Bigg] = k!\Big(\sum_{i_1=1}^{\lfloor k/3\rfloor}\sum_{i_2=i_1}^{\lfloor (k-i_1)/2\rfloor}\frac{1}{i_1! i_2! (k-i_1-i_2)!} \\ & -\sum_{i_1=1}^{\lfloor k/3\rfloor}\frac{1}{2i_1!^2(k-2i_1)!}-\sum_{i_1=1}^{\lfloor k/3\rfloor}\frac{\big(1+(-1)^{k-i_1}\big)}{4i_1!\lfloor (k-i_1)/2\rfloor!^2} \\ & +\Big(\frac{1+2(-1)^k\cos(\pi k/3)}{18(k/3)!^3}\Big)\Big). \end{aligned} \tag{85}$$

More compactly, Equation (85) can be expressed as

$$\begin{aligned} S(k,3) =& \sum_{i_1=1}^{\lfloor k/3\rfloor}\sum_{i_2=i_1}^{\lfloor (k-i_1)/2\rfloor}\binom{k}{i_1}\binom{k-i_1}{i_2}-\frac{1}{2}\sum_{i_1=1}^{\lfloor k/3\rfloor}\binom{k}{i_1}\binom{k-i_1}{i_1} \\ & -\sum_{i_1=1}^{\lfloor k/3\rfloor}\Big(\frac{1+(-1)^{k-i_1}}{4}\Big)\binom{k}{i_1}\binom{k-i_1}{\lfloor (k-i_1)/2\rfloor}+\frac{k!}{18(k/3)!^3} \\ & \times\Big(1+2(-1)^k\cos(\pi k/3)\Big). \end{aligned} \tag{86}$$

Putting $k=57$ in Equation (86) yields a value of 261 673 816 441 622 674 568 827 825, which agrees with StirlingS2[57,3] in Mathematica. Moreover, multiplying Equation (86) by 3! yields $W_{3,3}^{(k)}$.

The first sum on the rhs of Equation (86) is not only the dominant contribution to $S(k,3)$, but also overestimates its value. The next two terms are of similar size to each other and reduce the contribution made by the first sum. As k increases, their relative sizes to the first sum decrease. The final sum, which is relatively small to the other sums, only yields a positive value when k is a multiple of three. Otherwise, it is zero.

It should be mentioned that the second sum on the rhs of Equation (86) can be evaluated in Mathematica and is given by

$$\sum_{i_1=1}^{\lfloor k/3 \rfloor} \binom{k}{i_1}\binom{k-i_1}{i_1} = {}_2F_1\Big(\frac{1-k}{2}, -\frac{k}{2}; 1; 4\Big) - 1 - \binom{k}{\lfloor k/3 \rfloor + 1}\binom{k-1-\lfloor k/3 \rfloor}{\lfloor k/3 \rfloor + 1} \times {}_3F_2\Big(1, 1-\frac{k}{2}+\lfloor\frac{k}{3}\rfloor, \frac{3-k}{2}+\lfloor\frac{k}{3}\rfloor; 2+\lfloor\frac{k}{3}\rfloor, 2+\lfloor\frac{k}{3}\rfloor; 4\Big).$$

Similarly, the inner sum of the first sum on the rhs of Equation (86) can be evaluated in Mathematica and yields

$$\sum_{i_2=i_1}^{\lfloor (k-i_1)/2 \rfloor} \binom{k-i_1}{i_2} = \binom{k-i_1}{i_1} {}_2F_1\big(1, 2i_1-k; i_1+1; -1\big) - \binom{k-i_1}{\lfloor (k-i_1)/2 \rfloor + 1} \times {}_2F_1\big(1, i_1-k+\lfloor (k-i_1)/2 \rfloor + 1; \lfloor (k-i_1)/2 \rfloor + 2; -1\big).$$

Consequently, Equation (86) can be reduced to 1-dimensional sums over i_1.

In spite of the preceding results a more compact result for $S(k,3)$ can be obtained from Equation (8) with $j=3$. Then one arrives at

$$S(k,3) = \frac{1}{2}\Big(3^{k-1} - 2^k + 1\Big).$$

In fact, we can use Equation (8) to express l^k, where l is a positive integer, in terms of the Stirling numbers of the second kind. For example, $2^k = S(k+1,2)+1$. Applying Equation (80) yields $(k-1)!A_{k-1}(k+1) = 2^k - 1$. More generally, it is found that

$$l^k = 1 + \sum_{i=0}^{l-2} \frac{(l-1)!}{i!} S(k+1, l-i).$$

Using Equation (80), we can express the above result in terms of the generalized reciprocal logarithm numbers as follows:

$$l^k = 1 + k! \sum_{i=0}^{l-2} \binom{l-1}{i} A_{k+1+i-l}(k+1).$$

As an aside, if we sum l from unity to n, then according to No. 4.1.1.3 of [24], more commonly known as Faulhaber's formula, we can express the Bernoulli polynomials with integer arguments as a two-dimensional finite sum involving the Stirling numbers of the second kind. This is given by

$$\frac{1}{k+1}\big(B_{k+1}(n+1) - B_{k+1}\big) = n + \sum_{j=2}^{n} S(k+1,j) \sum_{i=0}^{n-j} \frac{(i+j-1)!}{i!}. \qquad (87)$$

Note also that Quaintance and Gould express the Bernoulli numbers in terms of the Stirling numbers of the second kind in Chapter 15 of [25], specifically, by (15.2) and (15.10). Furthermore, a similar result to Equation (87) can be obtained for the Euler polynomials by summing powers of $(-l)^k$ via Nos. 4.1.1.4 and 4.1.1.5 in [24].

We can also evaluate $S(k,4)$ in a similar manner to $S(k,3)$ except the same changes must now be made to Equation (62). Bear in mind, that the denominators of $1/(l+4-i_1-i_2)^2$ in the third term and $1/(l+4-2i_1)^2$ in the fifth term on the rhs of Equation (62) become $1/((k-i_1-i_2)/2)!^2$ and $1/((k-2i_1)/2)!^2$, respectively. Consequently, we need to insert factors of 4 in these denominators to determine $S(k,4)$ or $k!L_{P,k}^{(4)}\Big[\prod_{i=1}^{k-2}\frac{1}{i!^{\lambda_i}\lambda_i!}\Big]$. Hence we arrive at

$$\begin{aligned}
S(k,4) =&k!\Big(\sum_{i_1=2}^{\lfloor k/4\rfloor}\sum_{i_2=i_1+1}^{\lfloor (k-i_1)/3\rfloor}\sum_{i_3=i_2}^{\lfloor (k-i_1-i_2)/2\rfloor}\frac{1}{i_1!i_2!i_3!(k-i_1-i_2-i_3)!}\\
&-\frac{1}{2}\sum_{i_1=2}^{\lfloor k/4\rfloor}\sum_{i_2=i_1+1}^{\lfloor (k-i_1)/3\rfloor}\frac{1}{i_1!i_2!^2(k-i_1-2i_2)!}\\
&-\frac{1}{4}\sum_{i_1=2}^{\lfloor k/4\rfloor}\sum_{i_2=i_1+1}^{\lfloor (k-i_1)/3\rfloor}\frac{(1+(-1)^{k-i_1-i_2})}{i_1!i_2!((k-i_1-i_2)/2)!^2}\\
&+\frac{1}{18}\sum_{i_1=2}^{\lfloor k/4\rfloor}\frac{(1+2(-1)^{k-i_1}\cos(\pi(k-i_1)/3))}{i_1!((k-i_1)/3)!^3}-\frac{1}{8}\sum_{i_1=2}^{\lfloor k/4\rfloor}\frac{(1+(-1)^{k-2i_1})}{i_1!^2((k-2i_1)/2)!^2}\\
&+\frac{1}{2}\sum_{i_1=2}^{\lfloor k/4\rfloor}\sum_{i_2=i_1+1}^{\lfloor (k-2i_1)/2\rfloor}\frac{1}{i_1!^2i_2!(k-2i_1-i_2)!}+\frac{1}{6}\sum_{i_1=2}^{\lfloor k/4\rfloor}\frac{1}{i_1!^3(k-3i_1)!}\\
&-\frac{(1+(-1)^k+2\cos(\pi k/2))}{4\times 4!(k/4)!^4}\Big). \qquad (88)
\end{aligned}$$

As a check, putting $k=12$ in the above result yields 611501, which agrees with StirlingS2[12,4] in Mathematica. As in the case of $S(k,3)$, a more compact result can be obtained from Euler's formula, viz., Equation (8). Therefore, we find that

$$S(k,4)=\frac{1}{24}\Big(4^k-4\times 3^k+6\times 2^k-4\Big).$$

Multiplying either of the two preceding results by 4! gives $W_{4,4}^{(k)}$. Moreover, from Equations (86) and (88), we see that 4^k and 3^k together with the more familiar 2^k, can be expressed as a series of combinatorial sums. However, this becomes increasingly cumbersome or laborious when one considers l^k for $l>4$.

As in the case of the Stirling numbers of the first kind, we now turn to calculating the Stirling numbers of the second kind, $S(k,k-l)$, when l is no longer fixed, but is a variable. In other words, we develop asymptotic results for large values of l and

k, specifically for $k \gg l \gg 1$. Consequently, Equation (16) becomes

$$S(k, k-l) = k!\, L_{P,k}^{(k-l)} \Big[\prod_{i=1}^{k} \frac{1}{i!^{\lambda_i} \lambda_i!}\Big]. \tag{89}$$

From Table 9, we observe that the highest order term of $S(k, k-l)$ in k is $2l$. This represents the situation where we have $k-2l$ single object groups and l paired groups. Altogether we have $k-l$ groups, whereas in the case of the Stirling numbers of the first kind, we had to deal with $k-2l$ ones and the maximum number of twos equal to l. The contribution from this partition in Equation (89) is given by

$$\mathcal{D}_0 = \frac{k!}{\Gamma(k-2l+1)} \frac{1}{2^l l!}. \tag{90}$$

As expected, except for the phase factor of $(-1)^l$, this is identical to Equation (64). This only occurs because 1! and 2! are equal to 1 and 2, which will not occur when other partitions or groups with more than two objects are considered. Applying the reflection formula for the gamma function and introducing Equation (1), one can express Equation (90) as

$$\mathcal{D}_0 = \frac{1}{2^l l!} \sum_{j=0}^{2l} s(2l, j) k^j. \tag{91}$$

Hence there is a contribution from the above result to each power of k in $S(k, k-l)$. The highest power in the above result yields the highest order term for $S(k, k-l)$, whose coefficient is equal to $1/2^l l!$, the same as $s(k, k-l)$. In fact, as we did for the Stirling numbers of the first kind in the previous section, we shall denote the coefficients of k^{2l-j} in $S(k, k-l)$ by $S_{2l,j}(l)$ and let j range from zero to $2l$. Therefore, $S_{2l,0}(l) = 1/2^l l!$.

On the other hand, the coefficient of k^{2l-1} will not only include the $j = 2l - 1$ contribution from Equation (91), but will also include a contribution from the partition with the next highest number of twos or paired groups $(l-2)$ that sums to $k-l$ groups. This case is represented by the partition, $\{1_{k-2l+1}, 2_{l-2}, 3\}$ and its contribution emanating from Equation (16) is given by

$$\mathcal{D}_1 = \frac{\Gamma(k+1)}{\Gamma(k-2l+2)} \frac{1}{2^{l-2}\,(l-2)!3!}. \tag{92}$$

Introducing Equation (1) into Equation (92) yields

$$\mathcal{D}_1 = \frac{1}{2^{l-2}\,(l-2)!\,3!} \sum_{j=0}^{2l-1} s(2l-1, j)\, k^j.$$

Thus, the leading order term of $\mathcal{D}_1$ is k^{2l-1}. By combining the coefficient of this term with the coefficient of the k^{2l-1} term in Equation (91), we find that the coefficient of k^{2l-1} in $S(k,k-l)$ is given by

$$S_{2l,1}(l) = \frac{l(l-1)}{2^{l-2}\,3!\,l!} - \frac{l(2l-1)}{2^l\,l!} = -\frac{(4l-1)}{3\,2^l\,(l-1)!}.$$

The negative term arises from the k^{2l-1} term in Equation (60). The above result has a similar structure to Equation (67) including the property that it vanishes when $l=1$.

From here on, the calculations begin to become more complicated. However, we can exploit the homology in Table 5 to determine the successive coefficients. The only difference is that the left column in Table 5 is now representing groups instead of partitions. For example, the left column represents all the cases of determining how k objects can be arranged into $k-l$ groups with $k-2l+5$ unit groups. This is, of course, 7 since $p(5)=7$. Consequently, we can use the program, $\mathcal{C}_{\mathbf{j}}$ **Partitions**, in the appendix again, while Equation (89) can be expressed as

$$S(k,k-l) = \sum_{j=0}^{l-1} (-1)^j \frac{\Gamma(-k+2l-j)}{\Gamma(-k)} L_{R_2}\big(l,j\big).$$

Aside from the phase factor of $(-1)^l$ occurring in the definition of the Stirling numbers of the first kind, this is identical to Equation (68) except that the restricted partition operator is given by

$$L_{R_2}\big(l,j\big) = L^{(l-j)}_{P,(2l-j)/\{1\}}\left[\prod_{i=2}^{j+2} f_2(i,\lambda_i)\right], \tag{93}$$

where $f_2(i,\lambda_i) = 1/i!^{\lambda_i}\,\lambda_i!$. Specifically, the argument inside the operator is different since each part, i, now has a value of $i!$. Consequently, we still need to evaluate L_{R_2} in the same manner as Table 6 so that general expressions for the coefficients of $S(k,k-l)$ in terms of k and l can be derived. Therefore, Program 4 in the appendix has had to be adapted, which as explained in the previous section, is a modified version of **mathparv.cpp** in [14]. Table 10 displays the values of $L_{R_2}(l,j)$ for j ranging from 0 to 10. For each value of j, they vanish whenever $l \le (j+1)$, while the numerator possesses a polynomial in l of degree $j-1$.

Since the $L_{R_2}(l,j)$ have been determined for the Stirling numbers of the second kind, we can now turn our attention to determining the equivalent polynomials of those given by Equation (72), which, as we have seen, not only yielded the lowest order terms in k for the Stirling numbers of the first kind, but were also required to show how the results in Table 8 when combined with them yielded the results for the specific Stirling numbers of the first kind displayed in Table 3. We shall denote these polynomials by $s2(k,l)$. Note that the phase factor of $(-1)^l$ in Equations (71) and (72) must be neglected because it arises from the definition of the Stirling

j	$L_{R_2}(l,j)$
0	$\frac{1}{2^l l!}$
1	$\frac{2^{1-l}}{3(l-2)!}$
2	$\frac{2^{-l}}{9(l-3)!}\big(2l-3\big)$
3	$\frac{2^{1-l}}{405(l-4)!}\big(10l^2-45l+47\big)$
4	$\frac{2^{-1-l}}{1215(l-5)!}\big(20l^3-180l^2+511l-447\big)$
5	$\frac{2^{-l}}{25515(l-6)!}\big(28l^4-420l^3+2261l^2-5103l+3978\big)$
6	$\frac{2^{-1-l}}{1148175(l-7)!}\big(280l^5-6300l^4+54670l^3-226485l^2+441337l$ $-315774\big)$
7	$\frac{2^{-l}}{3444525(l-8)!}\big(40l^6-1260l^5+16030l^4-104685l^3+366412l^2$ $-641847l+428598\big)$
8	$\frac{2^{-3-l}}{10333575(l-9)!}\big(80l^7-3360l^6+58856l^5-554400l^4+3010901l^3$ $-9330414l^2+15036975l-9469710\big)$
9	$\frac{2^{-2-l}}{15345358875(l-10)!}\big(4400l^8-237600l^7+5479320l^6-70187040l^5$ $+543204915l^4-2581604190l^3+7279178401l^2$ $-10952824806l+6551022600\big)$
10	$\frac{2^{-3-l}}{322252536375(l-11)!}\big(12320l^9-831600l^8+24412080l^7$ $-407678040l^6+4249907970l^5-28518471135l^4$ $+122236785748l^3-319157419581l^2+452863114182l$ $-258521202504\big)$

Table 10: $L_{R_2}(l,j)$ for $j \leq 10$, where $f_2(i,\lambda_i)=1/i!^{\lambda_i}\lambda_i!$.

numbers of the first kind. As an aside, it is interesting to note that the Stirling numbers of the first kind are required for evaluating the coefficients in the Stirling numbers of the second kind in terms of k and l. This indicates that the Stirling numbers of the first kind are more fundamental than their second kind counterparts.

Table 11 displays the polynomials, $s2(k,l)$, for the Stirling numbers of the second kind. Unlike the $s1(k,l)$ polynomials displayed in Table 7, the coefficients oscillate in sign. Since the $s1(k,l)$ polynomials give the lowest order terms in k for $s(k,k-l)$, i.e., for powers of $k \le l$, we expect the same to occur with the $s2(k,l)$ polynomials and $S(k,k-l)$, which will be confirmed shortly.

We have also seen that the highest order terms in powers of k occur in the second sum on the rhs of Equation (70), which for the Stirling numbers of the second kind is expressed as

$$S(k,k-l) = \sum_{i=0}^{l} k^i \sum_{j=0}^{l-1} s(2l-j,i)\, L_{R_2}(l,j) + \sum_{i=l+1}^{2l} k^i \sum_{j=0}^{2l-i} s(2l-j,i)\, L_{R_2}(l,j). \tag{94}$$

Here, $L_{R_2}(l,j)$ is given by Equation (93).

As for the Stirling numbers of the first kind, the coefficient of the leading order term is found by evaluating the contribution from the reduced partition operator for the partition with only twos summing to $2l$, i.e., l twos, multiplied by the value of $s(2l,2l)$. This partition, denoted by $\{2_l\}$, contributes a value of $1/2!^l l!$, the same as the Stirling numbers of the first kind. From Equation (6), $s(2l,2l)$ equals unity. Therefore, if we denote the coefficients of k^{2l-j} as $S_{2l,j}$ in $S(k,k-l)$, i.e., $S(k,k-l) = \sum_{j=0}^{2l} S_{2l,j}(l)\, k^{2l-j}$, then $S_{2l,0}(l) = 2^{-l}/l!$, while $S_{2l,1} = -2^{-l}(4l-1)/3(l-1)!$.

The next highest order term in $S(k,k-l)$ is obtained by putting $i = 2l-2$ in Equation (94). Then one finds that

$$S_{2l,2}(l) = \sum_{j=0}^{2} s(2l-j, 2l-2) L_{R_2}(l,j). \tag{95}$$

Introducing the relevant results from Tables 3 and 10 yields

$$\begin{aligned} S_{2l,2} = {} & \frac{2^{-l}(-3+2l)}{9(-3+l)!} + \frac{2^{-l}(-1+2l-(-1+2l)^2)}{3(-2+l)!)} \\ + & \quad \frac{2^{-3-l}(-((4l)/3)+12l^2-(80l^3)/3+16l^4)}{l!}. \end{aligned}$$

This is the initial form produced by Mathematica for $S_{2l,2}$. To simplify the result further, one should take out a factor of $l!$ in the denominator, which means that the first and second terms on the rhs of Equation (95) need to be multiplied by $l(l-1)(l-2)$, and $l(l-1)$, respectively. Then by introducing the resulting material

l	$s2(k,l)$
0	0
1	$\frac{1}{2}k$
2	$\frac{7}{8}k^2 - \frac{5}{12}k$
3	$-\frac{97}{48}k^3 + 2k^2 - \frac{3}{4}k$
4	$\frac{5971}{1152}k^4 - \frac{757}{96}k^3 + \frac{1837}{288}k^2 - \frac{251}{120}k$
5	$-\frac{10817}{768}k^5 + \frac{1859}{64}k^4 - \frac{106837}{2880}k^3 + \frac{1903}{72}k^2 - \frac{95}{12}k$
6	$\frac{610009}{15360}k^6 - \frac{7160267}{69120}k^5 + \frac{2109037}{11520}k^4 - \frac{10768279}{51840}k^3 + \frac{97367}{720}k^2 - \frac{19087}{504}k$
7	$-\frac{31768357}{276480}k^7 + \frac{25078799}{69120}k^6 - \frac{42739139}{51840}k^5 + \frac{6766943}{5184}k^4 - \frac{164149417}{120960}k^3 + \frac{296881}{360}k^2 - \frac{5257}{24}k$
8	$\frac{31475819161}{92897280}k^8 - \frac{2084719939}{1658880}k^7 + \frac{87172693177}{24883200}k^6 - \frac{10431979727}{1451520}k^5 + \frac{182268754763}{17418240}k^4 - \frac{7373396321}{725760}k^3 + \frac{3533679881}{604800}k^2 - \frac{1070017}{720}k$
9	$-\frac{565095881387}{557383680}k^9 + \frac{1505173070029}{348364800}k^8 - \frac{199327529047}{13934592}k^7 + \frac{6314626388839}{174182400}k^6 - \frac{37965486053}{552960}k^5 + \frac{3534223753}{37800}k^4 - \frac{11540264461}{134400}k^3 + \frac{132101511}{2800}k^2 - \frac{231417}{20}k$
10	$\frac{1629168499097}{530841600}k^{10} - \frac{247120507816003}{16721510400}k^9 + \frac{79067876871539}{1393459200}k^8 - \frac{359199693359141}{2090188800}k^7 - \frac{93737943001543}{232243200}k^6 - \frac{748648816815259}{1045094400}k^5 + \frac{401983316754 79}{43545600}k^4 - \frac{5873080029689}{7257600}k^3 + \frac{6467854987}{15120}k^2 - \frac{26842253}{264}k$

Table 11: Values of the first sum in Equation (70) or Equation (94) using the results from Table 10.

into the FullSimplify routine in Mathematica, one obtains

$$S_{2l,2} = \frac{2^{-1-l}(3+2l(-1+8l))}{9\Gamma[-1+l]}.$$

Next one applies the Expand routine to the resulting polynomial, i.e.,

Expand[(3 + 2 l (-1 + 8 l))]=3 - 2 l + 16 l^2.

For low values of l, one can obtain compact results directly by applying the FullSimplify routine in Mathematica. However, for higher values of $l > 3$, one not only must follow the above method, but also one must apply the Factor routine to expand and factor the polynomial. For example, factoring the polynomial for the $l = 5$ coefficient yields

Factor[8640 + 3330 l - 9105 l^2 + 22622 l^3 - 116415 l^4 + 224292 l^5 - 245700 l^6 + 158928 l^7 - 53760 l^8 + 7168 l^9]= (-2 + l) (-1 + l) (4320 + 8145 l + 5505 l^2 + 15496 l^3-37716 l^4 + 47824 l^5 - 32256 l^6 + 7168 l^7).

Hence the first two factors and l, which appears from applying the Simplify routine earlier, can be cancelled with $l!$ in the denominator, thereby yielding $(l-3)!$ in the final form of the result displayed in Table 12.

Table 12 lists all the coefficients for the powers of k in $S(k, k-l)$ up to $j = 10$ after applying the above method to the output resulting from the first term on the rhs of Equation (94). As can be seen, they have similar forms to the corresponding coefficients for the Stirling numbers of the first kind given in Table 8. Despite the fact that the leading terms for both kinds of Stirling numbers are identical and that the coefficients change sign for odd values of j, the main differences between the tabulated results occur in the powers of two outside the polynomials and in the coefficients of the final polynomials. The $j = 9$ case is the only result where the factorial in the denominator is different. In this case the Stirling numbers of the first kind possess a factor of $(l-6)!$ as opposed to $(l-5)!$ for $S_{2l,9}$. This means that the resulting polynomial in the Stirling numbers of the second kind is one order higher than the $j = 9$ result in Table 8.

We can check the results in Table 12, by combining them with the results for $s2(k, l)$ in Table 11 and observing if they yield the actual values for the Stirling numbers of the second kind from mathematical software packages such as Mathematica. Alternatively, they can be checked by observing if they produce the same results appearing in Table 9. To see this more clearly, let us consider the $j = 4$ result in Table 12, which gives the coefficient of k^{2l-4} in the Stirling numbers of the second kind. In order to obtain the entire expression for the Stirling numbers, we need to add $s2(k, l)$, but only for $l = 4$, because the $j = 4$ result will yield the coefficient of k^5, while $s2(k, 4)$ yields the coefficients from k^4 to k or the lowest four

j	$S_{2l,j}(l)$
0	$\frac{2^{-l}}{l!}$
1	$-\frac{2^{-l}}{3(l-1)!}\big(4l-1\big)$
2	$\frac{2^{-l}}{18(l-2)!}\big(16l^2-2l+3\big)$
3	$-\frac{2^{-l-1}}{405(l-2)!}\big(320l^4-520l^3+202l^2-185l-48\big)$
4	$\frac{2^{-l-3}}{1215(l-3)!}\big(1280l^5-2240l^4+1372l^3-1060l^2-459l-207\big)$
5	$-\frac{2^{-l-3}}{25515(l-3)!}\big(7168l^7-32256l^6+47824l^5-37716l^4+15496l^3$ $+5505l^2+8145l+4320\big)$
6	$\frac{2^{-l-4}}{1148175(l-4)!}\big(143360l^8-698880l^7+1202880l^6-1086120l^5$ $+417156l^4+49158l^3+259039l^2+281487l+32940\big)$
7	$-\frac{2^{-l-4}}{3444525(l-4)!}\big(81920l^{10}-727040l^9+2415360l^8-3940320l^7$ $+3427224l^6-1258908l^5+325870l^4-299455l^3-995499l^2$ $+259668l+1270080\big)$
8	$\frac{2^{-l-7}}{10333575(l-5)!}\big(327680l^{11}-3112960l^{10}+11347968l^9-20617728l^8$ $+19537872l^7-8029920l^6+2674216l^5-706208l^4-6185331l^3$ $+8390214l^2+38193687l+36384390\big)$
9	$-\frac{2^{-l-7}}{15345358875(l-5)!}\big(72089600l^{13}-1063321600l^{12}+6284861440l^{11}$ $-19271352320l^{10}+32937051840l^9-30477804720l^8$ $+13520307328l^7-4342455128l^6-1480934740l^5+16485376565l^4$ $+7869687882l^3-85934170017l^2-226649809530l-188764732800\big)$
10	$\frac{2^{-l-8}}{322252536375(l-6)!}\big(403701760l^{14}-6307840000l^{13}+39979089920l^{12}$ $-132803686400l^{11}+246743481600l^{10}-247059970080l^9$ $+117411003248l^8-35870826992l^7-9085023320l^6+192883224734l^5$ $+242794805847l^4-811462211208l^3-4118925793455l^2$ $-7215470135334l-4630998219120\big)$

Table 12: Highest order coefficients in powers of k^{2l-j} for $S(k,k-l)$.

orders. Therefore, putting $l = 4$ yields

$$S(2k,4) = \sum_{j=0}^{3} S_{8,j}(l)k^{2l-j} + s2(k,4),$$

where from the first four results in Table 12, we have

$$S_{8,0} = \frac{1}{384}, \quad S_{8,1} = -\frac{5}{96}, \quad S_{8,2} = \frac{251}{576}, \quad \text{and} \quad \mathrm{S}_{8,3} = -\frac{473}{240}.$$

Introducing $s2(k,4)$ from Table 11 into the above yields

$$S(2k,4) = \frac{1}{384}k^8 - \frac{5}{96}k^7 + \frac{251}{576}k^6 - \frac{473}{240}k^5 + \frac{5971}{1152}k^4 - \frac{757}{96}k^3 + \frac{1837}{288}k^2 - \frac{251}{120}k.$$

This agrees with the $l=4$ result in Table 9 when the latter is expanded in powers of k. Moreover, if we put $l=101$ in the above result, then we find that Mathematica prints out a value of 23057744878245, while the same value is obtained when one types in StirlingS2[101,97] into the mathematical software package. As was found for the Stirling numbers of the first kind, even though the coefficients are often improper fractions, the final results for the Stirling numbers of the second kind are always positive integers, irrespective of the value of k. This behaviour only applies to the results for $S(k, k-l)$ listed in Table 9 since they are complete for each value of l. However, the coefficients in Table 12 represent the coefficients of k^{2l-j} in $S(k, k-l)$ for any value of l, provided $j < l$.

5. Conclusion

This paper has presented an extensive analysis based on the partition method for a power series expansion into the structure and properties of both kinds of Stirling numbers, $s(k,n)$ and $S(k,n)$, with reference to related topics such as the Worpitzky numbers and Stirling polynomials. Exact polynomial expressions in the primary variable, k, for these numbers have been determined for two cases: (1) where the secondary variable is fixed for low positive integers, e.g., for $s(k,l)$ and l ranging from 0 to 10, and (2) where the secondary variable is set equal to $k-l$ and l becomes a variable such as $S(k, k-l)$. In the first case, despite the fact that the coefficients are often improper fractions, the polynomials always yield integers when the primary variable is a positive integer. They also agree with specific values for both kinds of Stirling numbers when they are evaluated in Mathematica. An interesting question is whether these results have a meaning or application when k is extended to complex or even just real values.

In the second case, the coefficient of the highest power of the polynomials, viz., k^{2l}, was found to be identical for both kinds of Stirling numbers, but the coefficients of k^{2l-j}, where j ranges from 1 to 10, though yielding higher order polynomials in l as j increases, are different. For both kinds of Stirling numbers, one requires the results for the first kind of Stirling numbers determined in the first case, which are given in Table 3. The general results obtained in the second case are expected to yield accurate approximations when $k \gg l \gg 1$.

It was found that the particular partitions required for applying the partition method for a power series expansion to the Stirling numbers, were homologous to standard integer partitions. This meant that the code, **numparts**, which is discussed in detail in [14], only needed to be modified slightly to evaluate the Stirling numbers. This code has been presented for the first time in its entirety in the appendix in order for readers to understand how the brcp algorithm is applied to partitions with a specific number of parts. As a consequence, this enables one to observe how the required modifications for determining the results for the Stirling numbers can be implemented and processed in Mathematica.

Acknowledgement. The author thanks the referee for their insightful comments, especially making him aware of Sequences A100655 and A075264 from the Online Encyclopedia of Integer Sequences, and for alerting him to typographical errors, all of which helped improve the manuscript.

References

[1] M. Abramowitz and I.A. Stegun, (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing*, Dover, New York, 1972, Chapter 24.

[2] L. Carlitz, Note on Nörlund's polynomial $B_n^{(z)}$, *Proc. Amer. Math. Soc.* **11** (1960), 452-455.

[3] C.A. Charalambides, *Enumerative Combinatorics*, Chapman & Hall/CCRC, Boca Raton, 2002, Chapter 8.

[4] J. Choi and H.M. Srivastava, Some summation formulas involving harmonic numbers and generalized harmonic numbers, *Math. Comput. Model.* **54** (2011), 2220-2234.

[5] L. Comtet, *Advanced Combinatorics*, D.Reidel, Dordrecht, 1974.

[6] C.M. da Fonseca, M.L. Glasser and V. Kowalenko, Generalized cosecant numbers and trigonometric inverse power sums, *Appl. Anal. Disc. Math.* **12** (2018), 70-109.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions, Vol. 3*, Krieger, New York, 1981, 257.

[8] P. Flajolet and B. Salvy, Euler sums and contour integral representations, *Experimental Mathematics* **7** (1998), No. 1.

[9] H.W. Gould, *Topics in Combinatorics, 2nd Ed.*, Published by author, Morgantown, Va, 1987.

[10] I.S. Gradshteyn/I.M. Ryzhik, Alan Jeffrey (Ed.), *Table of Integrals, Series and Products, Fifth Ed.*, Academic Press, London, 1994.

[11] P.D. Hanna, The On-line Encyclopedia of Integer Sequences (2005), Sequence A075264.

[12] V. Kowalenko, Generalizing the reciprocal logarithm numbers by adapting the partition method for a power series expansion, *Acta Appl. Math.* **106** (2009), 369-420.

[13] V. Kowalenko, Properties and applications of the reciprocal logarithm numbers, *Acta. Appl. Math.* **109** (2010), 413-437.

[14] V. Kowalenko, *The Partition Method for a Power Series Expansion: Theory and Applications*, Academic Press/Elsevier Oxford, Oxford, 2017.

[15] V. Kowalenko, Developments from programming the partition method for a power series expansion, preprint, `arXiv:1203.4967v1`.

[16] V. Kowalenko, Euler and divergent series, *Eur. J. Pure Appl. Math.* **4** (2011), 370-423.

[17] V. Kowalenko, Towards a theory of divergent series and its importance to asymptotics, in *Recent Research Developments in Physics*, Vol. 2, Transworld Research Network, Trivandrum, India, 2001, 17-68.

[18] V. Kowalenko, *The Stokes Phenomenon, Borel Summation and Mellin-Barnes Regularisation*, Bentham ebooks, http://www.bentham.org, 2009.

[19] V. Kowalenko, Applications of the cosecant and related numbers, *Acta Appl. Math.* **114** (2011), 15-134, DOI:10.1007/s10440-011-9604-z.

[20] D. Lim and F. Qi, Some symmetric identities involving the Stirling polynomials under the finite symmetric group, *Mathematics*, **6** (2018), 332-341.

[21] N.E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer, Berlin, 1924.

[22] F.W.J. Olver et al, (Eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010, Chapter 26.

[23] T. Piesk, The On-line Encyclopedia of Integer Sequences, 2012, Sequence A181897.

[24] A.P. Prudnikov, Yu. A, Brychkov and O.I. Marichev, *Integral and Series Vol. 1: Elementary Functions*, Gordon and Breach, New York, 1986.

[25] J. Quaintance and H.W. Gould, *Combinatorial Identities for Stirling Numbers: The Unpublished Notes of H.W. Gould*, World Scientific, Singapore, 2016.

[26] J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1980, 35-38.

[27] S. Roman, *The Umbral Calculus*, Academic Press, New York, 1984, 128-129.

[28] N.J.A. Sloane, The On-line Encyclopedia of Integer Sequences, 2004, Sequence A100655.

[29] J. Sondow and E.W. Weisstein, Harmonic Number, From MathWorld-A Wolfram Web Resource, http://mathworld.wolfram.com/HarmonicNumber.html, Nov. 4, 2018.

[30] J. Spanier and K.B. Oldham, *An Atlas of Functions*, Hemisphere Publishing, New York, 1987, Chapter 18.

[31] E.W. Weisstein, Bell Polynomial, From MathWorld-A Wolfram Web Resource, http://mathworld.wolfram.com/BellPolynomial.html, Nov. 4, 2018.

[32] E.W. Weisstein, Wolstenholme Number, From MathWorld-A Wolfram Web Resource, http://mathworld.wolfram.com/WolstenholmeNumber.html, Nov. 29, 2018.

[33] E.W. Weisstein, Stirling Polynomial, From MathWorld-A Wolfram Web Resource, http://mathworld.wolfram.com/StirlingPolynomial.html, Aug. 12, 2019.

[34] E.W. Weisstein, Stirling Number of the Second Kind, From MathWorld–A Wolfram Web Resource.http://mathworld.wolfram.com/StirlingNumberoftheSecondKind.html, Feb. 15, 2016.

[35] J. Worpitzky, Studien uber die Bernoullischen und Eulerschen zahlen, *Jour. Reine und Angewandte Math.* **94** (1883), 203-232.

Appendix

This appendix presents the various codes that have been developed in the course of deriving the general formulas for the Stirling numbers of the first and second kinds. These codes have been created by modifying the code **numparts**, which is described in detail in [14], though not presented in its entirety. The code is presented here to enable the reader to understand the modifications that are necessary for obtaining the general formulas for both kinds of Stirling numbers. However, before we can proceed, we need to introduce the concept of a partition tree and how they are implemented in the codes.

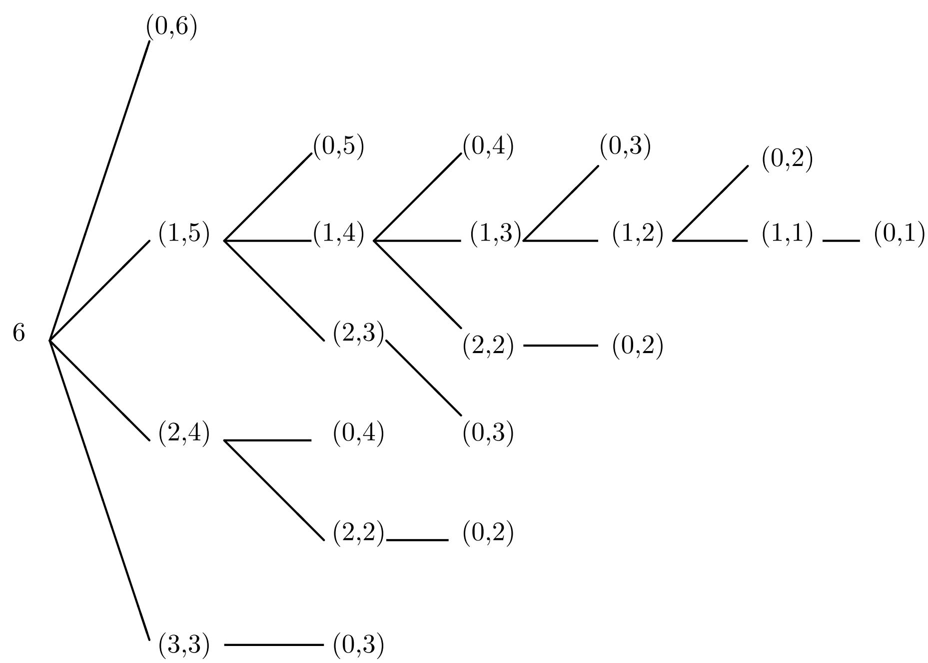

Figure 1: Partition tree for partitions summing to 6.

Figure 1 presents the partition tree when the seed number is or the partitions sum to 6. To construct a partition tree, one draws branches or lines to all pairs of numbers that can be summed to the seed number k, where the first member in the tuple is an integer less than or equal to $\lfloor k/2 \rfloor$ or the greatest integer less than or equal to $k/2$. For example, to obtain Figure 1, we draw branches to (0,6), (1,5), (2,4) and (3,3). Whenever a zero appears as the first member of a tuple, the path terminates, as can be seen by (0,6) or (0,3) in Figure 1. For the remaining pairs, one draws another set of branch lines to all pairs with integers that sum to the second member of the preceding pair, but once again according to the prescription that the first member of each new pair is less than or equal to half of the second member of the previous tuple. For example, in the case of (1,5) one draws paths branching out to (0,5), (1,4) and (2,3). This recursive approach is continued until all paths in the tree are terminated by a tuple possessing a zero.

Unfortunately, this, is not all that is required to create a partition tree. There is a possibility that duplicated paths or repeated partitions can occur. That is, each partition should only appear once in the tree. To avoid this problem, we note that the entire tree emanating from (1,5) in Figure 1 represents the same partition tree as if the seed number were 5 instead of 6. Similarly, the partition tree emanating from (1,4) is the same partition tree as if the seed number were equal to 4. This recursive behaviour continues all the way down to the last partition whose parts are only composed of ones, or the partition with the greatest length or longest path. Next we note that only partitions with unity in them emanate from (1,5), whereas the partitions emanating from (2,4) only possess parts that are greater than or equal to 2. Similarly, the partitions emanating from (3,3) only possess parts that are greater than or equal to 3. In fact, in the last instance since 3 is one half of 6, there will only be threes involved along the path from (3,3). In summary, we observe that for branches emanating from each tuple the left member of the new tuple must be greater than or equal to the left member of the previous tuple. Furthermore, we see that the last path represents the central partition $\{3, 3\}$, which would have been $\{3, 4\}$ had we constructed a partition with the seed number equal to 7. Therefore, the second member in a tuple decreases by unity with each rightward or horizontal movement, while the first member of each tuple increments with each downward or vertical movement.

By counting the number of terminating tuples we find that there are eleven partitions summming to 6, while extracting the first member of each tuple along a path and the second member of the terminating tuple yields the partition itself. As a result, the partitions summing to 6 are generated in the following order in the partition tree: $\{6\}$, $\{1,5\}$, $\{1_2,4\}$, $\{1_3,3\}$, $\{1_4,2\}$, $\{1_6\}$, $\{1_2,2_2\}$, $\{1,2,3\}$, $\{2,4\}$, $\{2_3\}$, and $\{3_2\}$. This also represents the order in which the partitions are printed out.

The program **numparts**, which is described in [14], but is not presented in its entirety as it is here, determines all the partitions summing to a value *tot* with a specific number of parts, which is represented by the global variable *numparts* in the code. This is accomplished by the bivariate recursive central partition algorithm, which is implemented in the function subprogram called **brcp**. In terms of a partition tree the partitions generated by the code below are those with a terminating tuple after *numparts* branches from the seed number. For example, the partitions summing to 6 with three parts in Figure 1 are those with the terminating tuples of (0,4) and (0,3) in the vertical column three branches from the seed number. These tuples correspond to the partitions, $\{1_2,4\}$ and $\{1,2,3\}$.

Program 1 **numparts**

```
/* This program determines partitions that sum to tot
   with numparts parts. */

#include <stdio.h>
#include <memory.h>
#include <stdlib.h>

int tot, numparts, *part;
/* numparts is the number of parts specified by the
   user */
long unsigned int term=1;

void termgen()
{
int freq,i,sumparts=0;
/* sumparts is the number of parts in a partition */
for(i=0;i<tot;i++){
      sumparts= sumparts+part[i];
     \hline
537 \             }
if(sumparts != numparts) goto end;
else
    printf("%ld:␣",term++);
    for(i=0;i<tot;i++){
                freq=part[i];
                if(freq) printf("%i(%i)␣",freq,i+1);
                      }
printf("\n");
end:        ;
}
```

```
void brcp(int p,int q)
{
part[p-1]++;
termgen();
part[p-1]--;
p -= q;
while(p >= q){
        part[q-1]++;
        brcp(p--,q);
        part[q++ -1]--;
              }
}

int main(int argc,char *argv[])
{
int i;
if(argc != 3) printf( "usage:␣numparts␣<#partitions>
␣␣␣␣␣␣␣␣<#parts>␣\n" );
else{
        tot=atoi(argv[1]);
        numparts=atoi(argv[2]);
        part=(int *) malloc(tot*sizeof(int));
        if(part == NULL) printf("unable␣to␣allocate
␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣array\n\n");
        else{
                for(i=0;i<tot;i++) part[i]=0;
                brcp(tot, 1);
                free(part);
            }
     }
printf("\n");
return(0);
```

Now that we have presented program **numparts**, we turn our attention to modifying the code so that it can determine the refined rencontres numbers. To understand these modifications, we need to study the partition tree. From Figure 1, we observe: (1) $\{6\}$ is the sole one-part partition, (2) $\{1,5\}$, $\{2,4\}$ and $\{3_2\}$ are the two-part or paired partitions, (3) $\{1_2,4\}$, $\{1,2,3\}$ and $\{2_3\}$ are the three-part partitions, (4) $\{1_3,3\}$ and $\{1_2,2_2\}$ are the four-part partitions, (5) $\{1_4,2\}$ is the only five-part partition, while (6) $\{1_6\}$ is the sole 6-part partition. If we begin with the partition possessing the most parts and move backwards to the partition possessing only one part, i.e. *numparts*=1, then as we decrement the length or number of parts and list the partitions with terminating tuples vertically from top to bottom, we obtain the partitions as they are listed in Table 1. For example, by using this approach we begin with $\{1_6\}$ and $\{1_4,2\}$ and then the two four-part partitions, $\{1_3,3\}$ and $\{1_2,2_2\}$ and so on. In other words, by decrementing the length of the partitions or the number of branches and scanning downwards for terminating tuples, one obtains the correct order of the refined rencontres numbers as described in [23].

Basically, we need to make two modifications to **numparts** in order to obtain the refined rencontres numbers. The first of these is that we need to consider different lengths in *numparts*. This means that the variable, *numparts* is no longer fixed or specified by the user as input, but begins with *tot* and continues to decrement to unity. This is accomplished by introducing a for loop in **main** and no longer typing in *numparts* as a fixed value. The second change for determining the refined rencontres number for each partition requires the introduction of another for loop in **termgen**. This loop will be responsible for printing out the relevant partitions. In this loop the rencontres number for each partition will also be generated in symbolic form, which can be imported into Mathematica to yield the actual value. For example, according to Table 1, the refined rencontres numbers for the partitions $\{1_3,3\}$ and $2_3\}$ are 40 and 15, respectively. In the case of program **refined_rencontres** listed below, the code would simply print out 6!/((3! 1^(3))(1! 3^(1))) and 6!/((3! 2^(3))).

Program 2 **refined_rencontres**

```
/* This program determines the forms of the refined rencont
   res numbers for partitions summing to a value tot. These
   forms can be imported into Mathematica to determine the
   actual sequence of numbers for tot. */

#include <stdio.h>
#include <memory.h>
#include <stdlib.h>

int tot, numparts, *part;
/* numparts represents the no. of parts in a partition */
long unsigned int term=1;

void termgen()
{
int freq,i,sumparts=0;
/* sumparts is the no. of parts for a partition */
for(i=0;i<tot;i++){
      sumparts= sumparts+part[i];
                  }
if(sumparts != numparts) goto end;
else
```

```
    printf("%ld:␣",term++);
    for(i=0;i<tot;i++){
                freq=part[i];
                if(freq) printf("%i(%i)␣",freq,i+1);
                      }
    printf("␣and␣the␣refined␣rencontres␣number␣is:
␣␣␣␣␣␣␣␣␣␣␣␣%i!/(",tot);
    for(i=0;i<tot;i++){
                freq=part[i];
                if(freq) printf("(%i!␣%i^(%i))",freq,i+1,
                        freq);
                      }
    printf(")");
    printf("\n");
end:        ;
}

void brcp(int p,int q)
{prod_{i=2}^{j+2}
\frac{1}{{i}^{\lambda_i}\lambda_i!}
part[p-1]++;
termgen();
part[p-1]--;
p -= q;
while(p >= q){
        part[q-1]++;
        brcp(p--,q);
        part[q++ -1]--;
             }
}
int main(int argc,char *argv[])
{
int i;
if(argc != 2) printf("usage:␣refined␣rencontres
␣␣␣␣␣␣␣␣<sum␣of␣partitions>␣\n");
else{
        tot=atoi(argv[1]);
        for(numparts=tot;numparts>=1;numparts--){
        part=(int *) malloc(tot*sizeof(int));
        if(part == NULL) printf("unable␣to␣allocate
␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣array\n\n");
        else{
                for(i=0;i<tot;i++) part[i]=0;
                brcp(tot,1);
```

```
                    free(part);
               }
          }
      }
printf("\n");
return(0);
}
```

In Section 3 the homology between partitions summing to a value j and those possessing $k-l$ parts and $k-2l+j$ ones, was discussed. The latter class of partitions are necessary to evaluate the contributions to $\mathcal{C}_j$, which are, in turn, required to determine the coefficients of k^{2l-j} in the Stirling numbers of the first kind given by $s(k,k-l)$. Because of the homology between both classes of partitions, we only need to modify **numparts** to generate all the partitions for determining the coefficients of the powers of k in $s(k,k-l)$. As in the case of the **refined_rencontres** numbers, we include a second for loop in **main** so that we can generate all partitions with $k-2l+j$ ones in reverse lexicographic order, which follows when the partitions with terminating tuples are written vertically downwards as the number of branches is increased from unity to j or the most number of branches. Note also that the standard lexicographic order emerges when the process is reversed by considering the most number of branches first and then moving down to unity.

The next modification occurs also at the input stage. In **numparts** one is required to state the value to which the parts in each partition are summed in addition to the number of parts in each partition. This no longer applies when dealing with $k-l$ parts summing to k. Because both k and l are fixed, the only input value is j, which, as stated earlier, determines the total number of partitions via the partition function, $p(j)$. Thus, j takes the role of *tot* in the original **numparts**, while there is no need to provide the number of parts via the variable *numparts*. In addition, once j or the value of *tot* is specified, each partition will possess $k-2l+j$ ones, which becomes the first quantity to be printed out by the printf statement in **termgen** below.

Since the number of ones is fixed, the partitions vary by decrementing the number of twos, instead of 1 as in **numparts**. In fact, the first partition becomes the partition with the most number of twos, viz. $\{1_{k-2l+j}, 2_{l-j-1}, j+2\}$, which corresponds to the single-part partition $\{j\}$ for the partitions summing to j as indicated in Table 5. The other partitions possess parts greater than 2 as the number of twos, represented by the variable *numtwos* is decremented.

Program 3 $\mathcal{C}_{\mathrm{j}}$ **Partitions**

```
/* This program determines partitions possessing k-l
   parts and summing to k. Both k and l are fixed
   algebraic quantities. The input variable tot is
```

```
   responsible for generating the partitions which
   all possess (k-2l+tot) ones.*/

#include <stdio.h>
#include <memory.h>
#include <stdlib.h>

int tot,numparts,numtwos=0,*part;
/* numparts is the number of parts excluding ones
   and twos, while numtwos is used to determine
   the number of twos in the final partition */
long unsigned int term=1;

void termgen()
{
int freq,i,sumparts=0;
/* sumparts is the number of parts or elements
   in a partition excluding ones and twos */
for(i=0;i<tot;i++){
      sumparts= sumparts+part[i];
                  }
if(sumparts != numparts) goto end;
else
    numtwos=tot+numparts;
    printf("%ld:␣(k-2l+%i)(1)␣(l-%i)(2)␣",term++,
            tot,numtwos);
    for(i=0;i<tot;i++){
                freq=part[i];
                if(freq) printf("%i(%i)␣",freq,i+3);
                      }
printf("\n");
end:        ;
}

void brcp(int p,int q)
{
part[p-1]++;
termgen();
part[p-1]--;
p -= q;
while(p >= q){
        part[q-1]++;
        brcp(p--,q);
        part[q++ -1]--;
```

```
                }
}
int main(int argc,char *argv[])
{
int i,j;
if(argc != 2) printf( "usage:␣numparts␣<#partitions>
␣␣␣␣␣␣␣␣\n" );
else{
        tot=atoi(argv[1]);
        part=(int *) malloc(tot*sizeof(int));
        for (j=0;j<tot;j++){
          numparts=j+1;
          if(part == NULL) printf("cannot␣allocate
␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣array␣\n\n");
          else{
             for(i=0;i<tot;i++) part[i]=0;
             brcp(tot,1);
             }
        }
        free(part);
     }
printf("\n");
return(0);
}
```

In order to determine the coefficients of the powers of k in $s(k,k-l)$ as functions of l, it is necessary to determine the contributions from partitions in the reduced partition operator $L_R(l,j)$. For each power of k^i, these partitions are composed of $2l-j-l(\lambda_j)$ twos, where $l(\lambda_j)$ is the number of partitions or length of the corresponding partition summing to j, and all the parts that appear in the partitions summing to j are to be incremented by 2. In the program displayed below, the variable *num_parts* in **termgen** represents $l(\lambda_j)$, while j is represented by *tot*, which is in turn the input value for executing the program.

The first print statement in **termgen** prints out LR[l_,6]:= when *tot* is set equal to 6. That is, the code begins with all contributions to $L_R(l,j)$ when $j=6$. Thus, partitions summing to 6 are considered first in **termgen**. The next print statement prints out a plus sign every time three partitions have been parsed by evaluating modulo 3 of the variable *termcnt*. This is necessary so that the output can be interpreted properly in Mathematica, thereby avoiding the insertion of control characters. The first for loop in **termgen** determines the total number of parts in a partition or its length. Using this value, **termgen** proceeds to print out the number of twos in the partition. Then it completes the processing of the partition by printing out $(i+2)^{\lambda_{i+2}}\lambda_{i+2}!$ for each part $i+2$. In this instance the multiplicity,

λ_{i+2}, of each part $i+2$ is given by the value stored in the array, *part[i]*.

Program 4 **Contributions to** $\mathbf{L_R[l, j, f(i, \lambda_i)]}$

```
#include <stdio.h>
#include <memory.h>
#include <stdlib.h>
#include <time.h>

int tot, *part;
long unsigned int termcnt=1;


void termgen(int p)
{
int f=0,i,num_parts=0,l;
/* num_parts is the total # of parts in the partition.
*/

if(p==tot) printf("LR[l_,%i]:=",p--);
else {
      if (termcnt>=1) printf("+");
      if (termcnt % 3 == 0) printf("\n");
      termcnt++;
      }
for (i=1;i<=tot; i++){
         f=part[i];
         if(f>0){
                  num_parts += l = f;
                  }
         }
f=0;
if( num_parts >= l ){
         printf("␣l!/(2^(l-%i)␣(l-%i)!",tot+num_parts,
                  tot+num_parts);
         for (i=1; i<=tot; i++){
                 f=part[i];
                 if(f>0){
                         printf("␣%i^(%i)␣%i!", i+2,f,f);
                                  }
                         }
                  }
         printf(")␣");
}
```

```
void brcp(int p1, int q)
{
part[p1]++;
termgen(p1);
part[p1]--;
p1 -= q;
while(p1 >= q){
        part[q]++;
        brcp(p1--,q);
        part[q++]--;
        }
}

int main( int argc, char *argv[] )
{
int i;
if(argc != 2) printf("execution:␣reduced␣part␣op
␣␣␣␣␣␣␣␣<sum␣of␣partitions>\n" );
else{
        tot=atoi(argv[1]);
/* tot is the sum of the partitions required by the
   brcp function subprogram  */
        part=(int *) malloc((tot+1) *sizeof(int));
        if(part == NULL) printf("unable␣to␣allocate
␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣array\n\n");
        else{
                for (i=0;i<tot;i++) part[i]=0;
                brcp(tot,1);
                free(part);
            }
        }
printf("\n");
return(0);
```