ON NATURAL LEAPING CONVERGENTS OF REGULAR CONTINUED FRACTIONS AND AN APPLICATION TO LINEAR FRACTIONAL TRANSFORMATIONS

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Abstract

Let $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be the coefficient matrix with nonzero determinant $\Delta := \det\sigma$ of the linear fractional transformation $\sigma(x) = (ax+b)/(cx+d)$, where $a,b,c,d \in \mathbb{Z}$. In this paper we introduce a new concept called natural leaping convergents. Such convergents are defined in terms of combinatorial pairings between the rational convergents of an irrational number ξ and those of its transformation $\sigma(\xi)$. The structural properties of natural leaping convergents are then studied and sufficient conditions are given for determining whether $\sigma(p_n/q_n) \in \mathcal{C}(\sigma(\xi)) \cap \sigma(\mathcal{C}(\xi))$, where $\mathcal{C}(\eta)$ denotes the set of convergents of the number η. We then present a theorem for expressing the convergents for all quadratic irrationals in closed form and establish for the quadratic irrationals, as well as for transcendental numbers, that given a suitable value for Δ, there are at most finitely many convergents p_n/q_n of ξ satisfying $\sigma(p_n/q_n) \in \mathcal{C}(\sigma(\xi)) \cap \sigma^*(\mathcal{C}(\xi))$, where * indicates the added property that $\gcd(ap_n + bq_n, cp_n + dq_n) = 1$.

1. Introduction

It has long been observed that certain Hurwitz-type continued fractions, for example $e = \left[\, 2, \overline{1, 2k, 1}\, \right]_{k=1}^{\infty}$, possess partitions among their convergents according to certain structural properties. The investigation into such structural properties led to the study of *leaping convergents* [1], [6], [7]. Later, in [4] *non-linear leaping convergents* were found after transforming Hurwitz-type continued fractions using a certain ra-

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tional function, also known as a linear fractional transformation (lft)

$$\sigma(x) = \frac{ax+b}{cx+d},$$

with integer coefficients a, b, c, d and a nonzero determinant $\Delta := ad - bc$. Without loss of generality, we may assume that $\gcd(a, b, c, d) = 1$. In the case $\mu := \sigma(\xi)$ with an lft satisfying $\Delta = \pm 1$, the two numbers ξ and μ are called *equivalent*. A well-known result of Serret [9, 8, §17], from the 19th century states:

> The necessary and sufficient condition for the regular continued fractions of two irrational numbers ξ, μ to agree from a certain partial denominator consists in the equivalence of the numbers ξ and μ.

From the proof of this statement in [8] it also follows that there is a certain index ν such that all convergents p_n/q_n of ξ with $n \geq \nu$ become convergents of $\sigma(\xi)$ by the mapping $\sigma\big(p_n/q_n\big)$. In this way one obtains all convergents of $\sigma(\xi)$ up to at most finitely many exceptions.

In this paper, we shift our focus to lfts with determinants $|\Delta| \geq 2$. In Section 2, the linear fractional transformations $\sigma(x)$ are at first very generally assumed with four rational coefficients, from which a suitable equivalent lft with integer coefficients (and nonzero determinant) is formed. Again we want to ask the question of transferring convergents of ξ to convergents of $\sigma(\xi)$. For this purpose, we define *natural leaping convergents* in terms of combinatorial pairings between $p/q \in \mathcal{C}(\xi)$ and $p'/q' \in \mathcal{C}(\sigma(\xi))$, where $\mathcal{C}(\alpha)$ denotes the set of convergents of α. The set of such pairings is called a *σ-relation*, R_σ. We thus generalize the notion of leaping convergents in such a way that σ-relations between the convergents of both ξ and $\sigma(\xi)$ can be formulated.

The extent to which the sets of convergents of ξ and $\sigma(\xi)$ are σ-related depends mainly on approximation properties with rationals and thus on the irrationality measure of ξ. We then establish conditions for

$$\sigma\left(\frac{p_n}{q_n}\right) \in \mathcal{C}(\sigma(\xi)) \cap \sigma(\mathcal{C}(\xi))\,,$$

based on the arithmetical properties of the convergents, and the proofs of these results are then provided in Section 4.

Finally in Section 5, auxiliary results including new closed form formulae for all convergents of the class of quadratic irrational numbers are established. These results are applied to examples around natural leaping convergents and minor convergents.

2. Definitions

Let $\xi = [a_0, a_1, a_2, \dots]$ be a real irrational number. We denote the convergents of ξ by p_n/q_n and the convergents of the transformed number $\sigma(\xi)$ by p'_n/q'_n. Let

$$\begin{aligned}
\mathcal{C}(\xi) &:= \left\{ \frac{p_n}{q_n} \,:\, n \geqslant 0 \right\}, \\
\mathcal{C}\big(\sigma(\xi)\big) &:= \left\{ \frac{p'_n}{q'_n} \,:\, n \geqslant 0 \right\}, \\
\sigma\big(\mathcal{C}(\xi)\big) &:= \left\{ \frac{ap_n + bq_n}{cp_n + dq_n} \,:\, \frac{p_n}{q_n} \in \mathcal{C}(\xi) \right\}, \\
\sigma^*\big(\mathcal{C}(\xi)\big) &:= \left\{ \frac{ap_n + bq_n}{cp_n + dq_n} \,:\, \gcd(ap_n + bq_n, cp_n + dq_n) = 1,\ \frac{p_n}{q_n} \in \mathcal{C}(\xi) \right\}, \quad (1) \\
\mathcal{C}_{f(n)}(\xi) &:= \left\{ \frac{p_{f(n)}}{q_{f(n)}} \in \mathcal{C}(\xi) \,:\, n \geq 0 \right\}.
\end{aligned}$$

It is clear that $\sigma^*\big(\mathcal{C}(\xi)\big) \subseteq \sigma\big(\mathcal{C}(\xi)\big)$ and for the indexed sets, $\mathcal{C}_{f(n)}(\xi) \subseteq \mathcal{C}_n(\xi) = \mathcal{C}(\xi)$. The starting point is a quite general approach for introducing a suitable set of lfts with rational coefficients that do not even have to be in truncated form. Let

$$\sigma_1(x) := \frac{q_1 x + q_2}{q_3 x + q_4} \qquad \text{and} \qquad \sigma_2(x) := \frac{q'_1 x + q'_2}{q'_3 x + q'_4},$$

where $q_\nu, q'_\nu \in \mathbb{Q}$ $(\nu = 1, 2, 3, 4)$, $\Delta(\sigma_1) = q_1 q_4 - q_2 q_3 \neq 0$ and $\Delta(\sigma_2) = q'_1 q'_4 - q'_2 q'_3 \neq 0$. We now define a relation $\sim$, on the set $\mathcal{M}$ of all such lfts, by

$$\sigma_1 \sim \sigma_2 \quad \text{if and only if} \quad \big(q_1 q'_2 = q'_1 q_2 \quad \text{and} \quad q_2 q'_3 = q'_2 q_3 \quad \text{and} \quad q_3 q'_4 = q'_3 q_4\big).$$

The relation $\sim$ is an equivalence relation on $\mathcal{M}$. Next, for an lft

$$\sigma_1(x) = \frac{q_1 x + q_2}{q_3 x + q_4}$$

we want to find an uniquely determined integer $z \neq 0$ such that σ_1 takes the form

$$\frac{(q_1 z)x + q_2 z}{(q_3 z)x + q_4 z}$$

of an lft in x with coprime integer coefficients. For this purpose, let

$$\begin{aligned}
r &:= \operatorname{lcm}\big(\operatorname{denom}(q_1), \operatorname{denom}(q_2), \operatorname{denom}(q_3), \operatorname{denom}(q_4)\big), \\
s &:= \gcd\big(rq_1, rq_2, rq_3, rq_4\big),
\end{aligned}$$

where $\operatorname{denom}(0) := 1$. Moreover, let

$$a := \pm\frac{rq_1}{s}, \quad b := \pm\frac{rq_2}{s}, \quad c := \pm\frac{rq_3}{s}, \quad d := \pm\frac{rq_4}{s},$$

where the upper signs $+$ are used if

$$\big(\, q_1 \;=\; q_2 \quad \text{and} \quad q_3 \;>\; q_4 \,\big) \quad \text{or} \quad \big(\, q_1 \;>\; q_2 \,\big)\,,$$

and the lower signs $-$ are used otherwise. Note that $q_1 = q_2$ and $q_3 = q_4$ is impossible by our assumption $q_1q_4 - q_2q_3 \neq 0$. By this construction, the integers a, b, c and d are uniquely determined. Now put $z := \pm r/s$, and we have reached our goal by setting

$$\sigma(x) \;:=\; \frac{ax+b}{cx+d} \;\in\; \mathcal{M}\,. \tag{2}$$

The lft σ is uniquely defined, and we have

$$\gcd\big(\,a, b, c, d\,\big) \;=\; 1\,, \qquad \Delta(\sigma) \;=\; ad - bc \;=\; \Big(\frac{r}{s}\Big)^2 \big(q_1q_4 - q_2q_3\big) \;\neq\; 0\,,$$

and

$$aq_2 = \pm\frac{rq_1q_2}{s} = q_1b\,, \qquad bq_3 = \pm\frac{rq_2q_3}{s} = q_2c\,, \qquad cq_4 = \pm\frac{rq_3q_4}{s} = q_3d\,,$$

so that $\sigma \sim \sigma_1$. Of course, for all real numbers x from the domain of σ we have the identity $\sigma(x) = \sigma_1(x)$. If $[\sigma]_\sim$ is an equivalence class of $\sim$ containing σ, we have $\sigma_1 \in [\sigma]_\sim$.

Let an arbitrary lft $\sigma' \in \mathcal{M}$ with $\sigma \sim \sigma'$ be given. Then we will agree to work exclusively with the lft σ from Equation (2). We write $\sigma = \operatorname{proj}(\sigma')$. If σ has coprime integer entries and a non-zero determinant, then $\sigma = \operatorname{proj}(\sigma)$ if and only if ($a = b$ and $c > d$) or $a > b$. The inverse lft σ^{-1} corresponding to σ from Equation (2) is uniquely given by

$$\sigma^{-1}(x) \;=\; \begin{cases} \dfrac{dx-b}{-cx+a}, & \text{if } (d = b \text{ and } -c < a) \text{ or } (d > -b) \\[2ex] \dfrac{-\,dx+b}{cx-a}, & \text{otherwise.} \end{cases}$$

In any case we have $\sigma^{-1} = \operatorname{proj}(\sigma^{-1})$. Let

$$\mathcal{L} \;:=\; \Big\{\sigma(x) = \frac{ax+b}{cx+d} \;:\; \sigma \;=\; \operatorname{proj}(\sigma') \quad \text{and} \quad \sigma' \in \mathcal{M}\Big\}\,.$$

$\mathcal{L}$ consists of exactly those lfts σ that are of interest to us in the context of leaping convergents, namely lfts satisfying $\sigma = \operatorname{proj}(\sigma)$. We write $\sigma_1 \circ \sigma_2$ to denote the usual composition of two lfts from $\mathcal{M}$. We now define a binary operation $\circ_{\mathcal{L}}$ on $\mathcal{L}$. For two lfts $\sigma_1, \sigma_2 \in \mathcal{L}$, let

$$\sigma_1 \circ_{\mathcal{L}} \sigma_2 \;:=\; \operatorname{proj}\big(\sigma_1 \circ \sigma_2\big)\,.$$

Then, $(\mathcal{L}, \circ_{\mathcal{L}})$ is a non-Abelian group. The neutral element is the lft $x \in \mathcal{L}$, because for every lft $\sigma \in \mathcal{L}$ we have

$$\sigma \circ_{\mathcal{L}} x \;=\; \operatorname{proj}\big(\sigma(x) \circ x\big) \;=\; \operatorname{proj}\big(\sigma\big) \;=\; \sigma\,.$$

Any inverse element σ^{-1} of an lft σ is uniquely determined satisfying $\sigma \circ_{\mathcal{L}} \sigma^{-1} = x = \sigma^{-1} \circ_{\mathcal{L}} \sigma$. Note that the set of all rational functions with real coefficients provided with the usual composition $\circ$ of two functions has no group structure.

The definitions of the generalized terms of leaping convergents and natural leaping convergents now follow, with the latter explained in the context of an lft. For all remaining considerations, we will only consider lfts σ from the set $\mathcal{L}$.

Definition 1. Any subset $\mathcal{C}_{f(n)}(\xi)$ of $\mathcal{C}(\xi)$ is said to be *a set of leaping convergents of* ξ, and as well, any subset $\mathcal{C}_{g(n)}\big(\sigma(\xi)\big)$ of $\mathcal{C}\big(\sigma(\xi)\big)$ is said to be *a set of leaping convergents of* $\sigma(\xi)$, for some indexing functions $f, g : \mathbb{N}_0 \to \mathbb{N}_0$.

In the special cases

$$\big|\mathcal{C}(\xi) \setminus \mathcal{C}_{f(n)}(\xi)\big| < \infty \qquad \text{or} \qquad \big|\mathcal{C}\big(\sigma(\xi)\big) \setminus \mathcal{C}_{g(n)}\big(\sigma(\xi)\big)\big| < \infty\,,$$

the sets $\mathcal{C}_{f(n)}(\xi)$ or $\mathcal{C}_{g(n)}\big(\sigma(\xi)\big)$ are said to be *sets of conformal leaping convergents of* ξ *or* $\sigma(\xi)$*, respectively.*

Definition 2. Let

$$\frac{p}{q} \in \mathcal{C}(\xi) \qquad \text{and} \qquad \frac{p'}{q'} \in \mathcal{C}\big(\sigma(\xi)\big)\,.$$

A relation $R_\sigma \subseteq \mathcal{C}(\xi) \times \mathcal{C}\big(\sigma(\xi)\big)$ is given by

$$\left(\frac{p}{q}, \frac{p'}{q'}\right) \in R_\sigma \quad \text{if and only if} \quad \sigma\!\left(\frac{p}{q}\right) \;=\; \frac{p'}{q'}\,.$$

Since $\Delta \neq 0$, the inverse linear fractional transformation σ^{-1} exists. Therefore the inverse relation of R_σ is given by the relation $R_{\sigma^{-1}}$:

$$\begin{aligned}\left(\frac{p}{q}, \frac{p'}{q'}\right) \in R_\sigma \quad &\text{if and only if} \quad \sigma\!\left(\frac{p}{q}\right) \;=\; \frac{p'}{q'}\\ &\text{if and only if} \quad \sigma^{-1}\!\left(\frac{p'}{q'}\right) \;=\; \frac{p}{q}\\ &\text{if and only if} \quad \left(\frac{p'}{q'}, \frac{p}{q}\right) \in R_{\sigma^{-1}}\,.\end{aligned}$$

This motivates the following definition.

Definition 3. Let
$$\left(\frac{p}{q}, \frac{p'}{q'}\right) \in R_\sigma .$$
Then the two convergents p/q and p'/q' are said to be *σ-related.*

The *range* of the relation R_σ is given by
$$\operatorname{Ran}\big(R_\sigma\big) := \left\{ \frac{p'}{q'} \in \mathcal{C}\big(\sigma(\xi)\big) \,\middle|\, \text{there exists}\, \frac{p}{q} \in \mathcal{C}(\xi) \,:\, \left(\frac{p}{q}, \frac{p'}{q'}\right) \in R_\sigma \right\}.$$
The *domain* of the relation R_σ is given by
$$\operatorname{Dom}\big(R_\sigma\big) := \left\{ \frac{p}{q} \in \mathcal{C}(\xi) \,\middle|\, \text{there exists}\, \frac{p'}{q'} \in \mathcal{C}\big(\sigma(\xi)\big) \,:\, \left(\frac{p}{q}, \frac{p'}{q'}\right) \in R_\sigma \right\}.$$
Note that
$$\operatorname{Dom}\big(R_{\sigma^{-1}}\big) = \operatorname{Ran}\big(R_\sigma\big) \quad \text{and} \quad \operatorname{Ran}\big(R_{\sigma^{-1}}\big) = \operatorname{Dom}\big(R_\sigma\big).$$

Definition 4. Let $f : A \to B$ and $g : C \to D$ be two arbitrary indexing functions with $A, B, C, D \subseteq \mathbb{N}_0$. Any subset $\mathcal{C}_{f(n)}(\xi)$ of $\operatorname{Dom}\big(R_\sigma\big)$ is said to be *a set of natural leaping convergents with respect to the set $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big) \subseteq Ran\big(R_\sigma\big)$ of σ-related convergents of $\sigma(\xi)$.* Any subset $\mathcal{C}_{g(n)}\big(\sigma(\xi)\big)$ of $\operatorname{Ran}\big(R_\sigma\big)$ is said to be *a set of natural leaping convergents with respect to the set $\sigma^{-1}\big(\mathcal{C}_{g(n)}\big(\sigma(\xi)\big)\big) \subseteq Dom\big(R_\sigma\big)$ of σ-related convergents of ξ.*

Remark 1. In the definition above, we point out that since $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big) \subseteq \operatorname{Ran}\big(R_\sigma\big)$ and $\sigma^{-1}\big(\mathcal{C}_{g(n)}\big(\sigma(\xi)\big)\big) \subseteq \operatorname{Dom}\big(R_\sigma\big)$, then
$$\sigma\big(\mathcal{C}_{f(n)}(\xi)\big) = \mathcal{C}_{g(n)}\big(\sigma(\xi)\big) \quad \text{and} \quad \sigma^{-1}\big(\mathcal{C}_{g(n)}\big(\sigma(\xi)\big)\big) = \mathcal{C}_{f(n)}(\xi),$$
respectively, for some functions f and g.

Definition 5. Let either $\mathcal{N} \subseteq \mathcal{C}(\xi)$ or $\mathcal{N} \subseteq \mathcal{C}\big(\sigma(\xi)\big)$, and let $i, r \in \mathbb{Z}$ where $r \geqslant 2$ and $0 \leqslant i < r$. $\mathcal{N}$ is said to be *a set of arithmetical ordered leaping convergents with respect to the residue class* $i \bmod r$ if and only if $p_m/q_m \in \mathcal{N}$ implies that $m \equiv i \pmod{r}$.

In the papers [1], [6], and [7], only sets of arithmetical ordered leaping convergents are studied, whereas sets of nonlinear leaping convergents occur in [4].

Definition 6. If there is a subset $\mathcal{C}_{f(n)}(\xi) \subseteq \mathcal{C}(\xi)$ such that
$$\big|\mathcal{C}(\xi) \setminus \mathcal{C}_{f(n)}(\xi)\big| < \infty \;\text{ and }\; \big|\mathcal{C}\big(\sigma(\xi)\big) \setminus \sigma\big(\mathcal{C}_{f(n)}(\xi)\big)\big| < \infty,$$
then the pattern of the natural leaping convergents, $\mathcal{C}_{g(n)}\big(\sigma(\xi)\big)$, of $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big)$ (with respect to the set $\mathcal{C}_{f(n)}(\xi)$ of σ-related convergents of ξ) is said to *σ-conform.* If, on the other hand, the two statements
$$\big|\mathcal{C}(\xi) \setminus \mathcal{C}_{f(n)}(\xi)\big| < \infty \;\text{ and }\; \big|\mathcal{C}\big(\sigma(\xi)\big) \cap \sigma\big(\mathcal{C}_{f(n)}(\xi)\big)\big| < \infty$$

are true, then the pattern is said to be σ-*adverse*. Moreover, if $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big)$ is replaced by $\sigma^*\big(\mathcal{C}_{f(n)}(\xi)\big)$, we speak of σ^*-*conform* or σ^*-*adverse*, respectively.

Recall that if for some $n \in \mathbb{N}_0$ we have $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big) \in \mathcal{C}\big(\sigma(\xi)\big)$, then $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big) \in \mathcal{C}_{g(n)}\big(\sigma(\xi)\big)$ for some $g : \mathbb{N}_0 \to \mathbb{N}_0$. As well, if for all $n \in \mathbb{N}_0$ we have $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big) \in \mathcal{C}\big(\sigma(\xi)\big)$, then $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big) = \mathcal{C}_{g(n)}\big(\sigma(\xi)\big)$. Intuitively, Definition 6 states that for a subset $\mathcal{C}_{f(n)}(\xi) \subseteq \mathcal{C}(\xi)$, the natural leaping convergents of $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big)$ have a leaping pattern $g(n)$, that is said to σ-conform if at most finitely many of the convergents from $\mathcal{C}_{f(n)}(\xi)$ and $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big)$ are not σ-related. Similarly, if at most a finite amount of convergents from $\mathcal{C}_{f(n)}(\xi)$ and $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big)$ are σ-related, then the pattern is σ-adverse. As mentioned in the introduction of Section 1, in the case $|\Delta| = 1$ there is always a set $\mathcal{C}_{f(n)}(\xi) \subseteq \mathcal{C}(\xi)$, so that the pattern of natural leaping convergents of $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big)$ σ-conforms.

3. Theorems

We introduce some notation and quantities where $n \geqslant 0$ is a fixed integer:

$$\begin{aligned} \xi &:= [\,a_0, a_1, a_2, \ldots\,] \in \mathbb{R} \setminus \mathbb{Q}\,, \\ \frac{p_n}{q_n} &:= [\,a_0, a_1, a_2, \ldots\, a_n]\,, \\ G_n &:= \gcd\big(ap_n + bq_n, cp_n + dq_n\big)\,, \\ u_n &:= \frac{ap_n + bq_n}{G_n}\,, \\ v_n &:= \frac{cp_n + dq_n}{G_n}\,. \end{aligned}$$

We recall that

$$\sigma\Big(\frac{p_n}{q_n}\Big) = \frac{ap_n + bq_n}{cp_n + dq_n} = \frac{u_n}{v_n}$$

and

$$\frac{1}{q_n(q_{n+1} + q_n)} < \Big|\,\xi - \frac{p_n}{q_n}\Big| < \frac{1}{q_n q_{n+1}} \leqslant \frac{1}{q_n^2}\,.$$

Theorem 1. *Let ξ be a real irrational number, and let $0 < \varepsilon < 1$. Then there is a positive integer n_0 depending at most on ε so that for $n \geqslant n_0$ the following three* *statements hold:*

$$G_n^2 a_{n+1} \geqslant 2(1+\varepsilon)|\Delta| \quad \textit{implies} \quad \frac{u_n}{v_n} \in \mathcal{C}\big(\sigma(\xi)\big) \cap \sigma\big(\mathcal{C}(\xi)\big)\,; \tag{3}$$

$$G_n^2\big(2+a_{n+1}\big) \leqslant (1-\varepsilon)|\Delta| \quad \textit{implies} \quad \frac{u_n}{v_n} \notin \mathcal{C}\big(\sigma(\xi)\big)\,; \tag{4}$$

$$G_n = 1 \textit{ and } a_{n+1} \geqslant 2(1+\varepsilon)|\Delta| \quad \textit{implies} \quad \frac{u_n}{v_n} \in \mathcal{C}\big(\sigma(\xi)\big) \cap \sigma^*\big(\mathcal{C}(\xi)\big). \tag{5}$$

The right-hand side in Equation (3) is equivalent to $\big(p_n/q_n, u_n/v_n\big) \in R_\sigma$, so that by Definition 4 the number p_n/q_n in the set $\mathcal{A} := \{p_n/q_n\}$ is a natural leaping convergent with respect to the set $\sigma(\mathcal{A})$, and, vice versa, the number u_n/v_n in the set $\mathcal{B} := \{u_n/v_n\}$ is a natural leaping convergent with respect to the set $\sigma^{-1}(\mathcal{B})$. Using the definition of $\sigma^*\big(\mathcal{C}(\xi)\big)$, the statement in Equation (5) follows from the statement in Equation (3). Setting $\varepsilon = |\Delta|^{-1}$, we obtain a corollary.

Corollary 1. *Let ξ be a real irrational number, and $|\Delta| \geqslant 2$. Then there is a positive integer n_0 depending at most on Δ so that for $n \geqslant n_0$ the following three statements hold.*

$$G_n^2 a_{n+1} \geqslant 2(\,|\Delta|+1\,) \quad \textit{implies} \quad \frac{u_n}{v_n} \in \mathcal{C}\big(\sigma(\xi)\big) \cap \sigma\big(\mathcal{C}(\xi)\big)\,; \tag{6}$$

$$G_n^2\big(2+a_{n+1}\big) \leqslant |\Delta|-1 \quad \textit{implies} \quad \frac{u_n}{v_n} \notin \mathcal{C}\big(\sigma(\xi)\big)\,; \tag{7}$$

$$G_n = 1 \textit{ and } a_{n+1} \geqslant 2(\,|\Delta|+1\,) \quad \textit{implies} \quad \frac{u_n}{v_n} \in \mathcal{C}\big(\sigma(\xi)\big) \cap \sigma^*\big(\mathcal{C}(\xi)\big).$$

Example 1. Let

$$\sigma(x) := \frac{ax+b}{cx+d}\,,$$

such that $|\Delta| \geqslant 4$ and $2(\,|\Delta|+1\,) \geqslant 10$. Since

$$e = \exp(1) = \big[\,2, \overline{1, 2m, 1}\,\big]_{m=1}^{\infty}\,,$$

we know by Equation (6) that for all sufficiently large integers k, the fractions

$$\frac{u_{3k+1}}{v_{3k+1}} = \frac{ap_{3k+1}+bq_{3k+1}}{cp_{3k+1}+dq_{3k+1}}$$

are convergents of $\sigma(e)$. Indeed, for all large k we have $G_{3k+2}^2 a_{3k+2} \geqslant a_{3k+2} = 2(k+1) \geqslant \Delta$.

Conversely, let $\gcd(ap_n + bq_n, cp_n + dq_n) = G_n = 1$ for some sufficiently large integer n with either $n \equiv 0 \pmod 3$ or $n \equiv 2 \pmod 3$. Applying Equation (7) and the fact that $G_n^2(2 + a_{n+1}) = 3 \leqslant |\Delta| - 1$, we conclude that u_n/v_n is not a convergent of $\sigma(e)$.

For some applications, a variant of Theorem 1 proves more practical. We need some new quantities. We denote by p_n^*/q_n^* the convergents of ξ satisfying the inequality

$$\left| \xi - \frac{p_n^*}{q_n^*} \right| < \frac{1}{\sqrt{5}{q_n^*}^2} \,.$$

It is well-known that for any three consecutive convergents to ξ, at least one satisfies this inequality, see [5, Theorem 195]. Furthermore, let

$$\begin{aligned} G_n^* &:= \gcd(ap_n^* + bq_n^*, cp_n^* + dq_n^*)\,, \\ u_n^* &:= \frac{ap_n^* + bq_n^*}{G_n^*}\,, \\ v_n^* &:= \frac{cp_n^* + dq_n^*}{G_n^*}\,. \end{aligned}$$

Theorem 2. *Let ξ be a real irrational number, and let $0 < \varepsilon < 1$. Then there is a positive integer n_0 depending at most on ε so that for $n \geqslant n_0$ the following four statements hold:*

$$\begin{aligned} G_n^2 \frac{q_{n+1}}{q_n} \geqslant 2(1+\varepsilon)|\Delta| \qquad &\textit{implies} \quad \frac{u_n}{v_n} \in \mathcal{C}\big(\sigma(\xi)\big) \cap \sigma\big(\mathcal{C}(\xi)\big)\,; \\ G_n^2 \big(1 + \frac{q_{n+1}}{q_n}\big) \leqslant (1-\varepsilon)|\Delta| \qquad &\textit{implies} \quad \frac{u_n}{v_n} \notin \mathcal{C}\big(\sigma(\xi)\big)\,; \qquad (8) \\ \sqrt{5}\big(G_n^*\big)^2 \geqslant 2(1+\varepsilon)|\Delta| \qquad &\textit{implies} \quad \frac{u_n^*}{v_n^*} \in \mathcal{C}\big(\sigma(\xi)\big) \cap \sigma\big(\mathcal{C}(\xi)\big)\,; \\ G_n = 1 \textit{ and } \frac{q_{n+1}}{q_n} \geqslant 2(1+\varepsilon)|\Delta| \qquad &\textit{implies} \quad \frac{u_n}{v_n} \in \mathcal{C}\big(\sigma(\xi)\big) \cap \sigma^*\big(\mathcal{C}(\xi)\big)\,. \end{aligned}$$

Because a proof of the statements in Theorem 2 is analogous to the proof of Theorem 1, we state Theorem 2 here without proof.

Example 2. Let

$$\xi := \frac{1+\sqrt{5}}{2} = [\,1,1,1,\ldots] = 1.61803\ldots\,.$$

Then we have

$$\frac{p_n}{q_n} = \frac{q_{n+1}}{q_n} = \frac{F_{n+2}}{F_{n+1}} \overset{n\to\infty}{\longrightarrow} \xi \,,$$

where $F_n = F_{n-1} + F_{n-2}$ is the nth Fibonacci number for $n \geqslant 2$, with $F_0 = 0,\ F_1 = 1$. Let $0 < \varepsilon \leqslant (2-\xi)/3$ be a real number, and σ a linear fractional transformation with $|\Delta| \geqslant 3$. When $n \geqslant n_0$ is some sufficiently large number and $G_n = 1$, then u_n/v_n is not a convergent of $\sigma(\xi)$, since the inequality on the left-hand side in Equation (8) holds:

$$G_n^2\big(1 + \frac{q_{n+1}}{q_n}\big) = 1 + \frac{F_{n+2}}{F_{n+1}} < 1 + \xi < 3(1-\varepsilon) \leqslant (1-\varepsilon)|\Delta| \,.$$

Let ξ be a real irrational number. The irrationality measure of ξ is the uniquely determined positive real number $w = w(\xi)$, which allows the following two inequalities in the rational approximation of ξ:

$$\left.\begin{array}{l} \text{for all } \varepsilon > 0 \text{ there exist infinitely many } \dfrac{p}{q} \in \mathbb{Q} \text{ such that} \\ \qquad\qquad \left|\xi - \dfrac{p}{q}\right| < \dfrac{1}{q^{w-\varepsilon}}\,, \\ \text{and for all } \varepsilon > 0 \text{ there exist at most finitely many } \dfrac{p}{q} \in \mathbb{Q} \text{ such that} \\ \qquad\qquad \left|\xi - \dfrac{p}{q}\right| < \dfrac{1}{q^{w+\varepsilon}}\,. \end{array}\right\} \quad (9)$$

The second statement can be formulated differently: For all $\varepsilon > 0$ there exists a positive integer q_0 such that we have, for all fractions $p/q \in \mathbb{Q}$ with $q \geq q_0$, the inequality

$$\left|\xi - \frac{p}{q}\right| \geqslant \frac{1}{q^{w+\varepsilon}}\,.$$

The irrationality measure of any real number ξ is greater than or equal to 2.

Theorem 3. *Let*

$$\sigma(x) := \frac{ax+b}{cx+d}$$

be an lft from $\mathcal{L}$. Then we have for every real irrational number ξ the identity

$$w\big(\xi\big) = w\big(\sigma(\xi)\big)\,.$$

In the proof of this theorem, natural leaping convergents are used. For simplicity, let $w - \varepsilon > 0$. Then, the infinite (truncated) fractions p/q from the upper statement in Equation (9) are convergents of ξ. Let $\mathcal{C}_{f(n)}(\xi)$ be the set of these convergents p/q. The main argument in the proof of Theorem 3 is that $\sigma\big(\mathcal{C}_{f(n)}(\xi)\big)$ is an infinite set of σ-related convergents of $\sigma(\xi)$, and vice versa (see Definition 4).

The following theorem deals with quadratic irrational numbers ξ. It shows that each such ξ has only at most finitely many convergents, which are immediately mapped from any lft with sufficiently large determinant to truncated convergents of $\mathcal{C}(\xi)$. The theorem builds mainly on Corollary 2 in [1] and on the second statement in the above Corollary 1.

Theorem 4. *Let ξ be a real quadratic irrational number. Then there is a positive integer D, depending at most on ξ, such that for every linear fractional transformation $\sigma \in \mathcal{L}$ and determinant $\Delta \geqslant D$, there are at most finitely many convergents p/q of ξ satisfying*

$$\sigma\Big(\frac{p}{q}\Big) \in \mathcal{C}\big(\sigma(\xi)\big) \cap \sigma^*\big(\mathcal{C}(\xi)\big)\,.$$

In the sense of Definition 6 with $\mathcal{C}_{f(n)}(\xi) = \mathcal{C}(\xi)$, the statement of Theorem 4 is:

> The pattern of the natural leaping convergents of $\sigma\big(\mathcal{C}(\xi)\big)$ with respect to $\mathcal{C}(\xi)$ is σ^*-adverse.

A statement analogous to Theorem 4 can also be made for transcendental numbers.

Theorem 5. *There are uncountably many transcendental numbers ξ with the following property. For every linear fractional transformation $\sigma \in \mathcal{L}$ and determinant $\Delta \geqslant 5$, there are at most finitely many convergents p/q of ξ satisfying*

$$\sigma\Big(\frac{p}{q}\Big) \in \mathcal{C}\big(\sigma(\xi)\big) \cap \sigma^*\big(\mathcal{C}(\xi)\big)\,.$$

Our last theorem in this section shows that the transition from an untruncated transformed approximate fraction $\sigma(p_n/q_n)$ to its truncated form u_n/v_n requires only dividing by an integer which depends on σ but not on n. Thus, for fixed $\sigma \in \mathcal{L}$, the truncation factor is absolutely bounded.

Theorem 6. *Let a, b, c, d be the integer coefficients of an lft $\sigma \in \mathcal{L}$. Then there is a constant C, depending at most on a, b, c, d, such that for any real irrational number ξ and its convergents p_n/q_n the inequality*

$$\gcd\big(ap_n + bq_n,\, cp_n + dq_n\big) \leqslant C$$

holds for all integers $n \geqslant 0$.

4. Proof of the Theorems

Proof of Theorem 1. We apply the Intermediate Value Theorem to the function $\sigma(x)$: For all nonnegative integers n, there exists a real number η_n such that the inequality

$|\xi - \eta_n| \leqslant |\xi - p_n/q_n|$ hold as well as the identity

$$\left|\,\sigma(\xi) - \sigma\Big(\frac{p_n}{q_n}\Big)\,\right| \;=\; \big|\sigma'(\eta_n)\big| \cdot \left|\,\xi - \frac{p_n}{q_n}\,\right|, \tag{10}$$

where

$$\sigma'(x) \;=\; \frac{\Delta}{\left(cx+d\right)^2}\,.$$

Since $\xi \notin \mathbb{Q}$, we have $c\xi + d \neq 0$. Moreover, η_n tends to ξ for increasing n; therefore $c\eta_n + d \neq 0$ holds for all sufficiently large n. Thus,

$$\lim_{n\to\infty} \big|\sigma'(\eta_n)\big| \;=\; \lim_{n\to\infty} \frac{\big|\Delta\big|}{\left(c\eta_n + d\right)^2} \;=\; \frac{\big|\Delta\big|}{\left(c\xi + d\right)^2} \;>\; 0\,.$$

Hence, for all real numbers ε between 0 and 1 there is an integer $n_0 > 0$ such that for all integers $n \geqslant n_0$ we have

$$\omega := \frac{|\Delta|\sqrt{1-\varepsilon}}{\left(c\xi + d\right)^2} \;<\; \big|\sigma'(\eta_n)\big| \;<\; \frac{|\Delta|\sqrt{1+\varepsilon}}{\left(c\xi + d\right)^2} =: \Omega\,. \tag{11}$$

To prove Equation (3), we now assume $G_n^2 a_{n+1} \geqslant 2(1+\varepsilon)|\Delta|$ for some sufficiently large integer n. Now, for $\varepsilon > 0$ and large n, we get

$$\begin{aligned}
\Big(\frac{cp_n}{q_n} + d\Big)^2 \;&=\; \left|\,c\Big(\frac{p_n}{q_n} - \xi\Big) + (c\xi + d)\,\right|^2 \\
&\leqslant\; c^2\left|\,\xi - \frac{p_n}{q_n}\,\right|^2 + 2|c|\left|\,\xi - \frac{p_n}{q_n}\,\right|\big|c\xi + d\big| + \big|c\xi + d\big|^2 \\
&<\; \frac{c^2}{q_n^4} + \frac{2|c|\,|\,c\xi + d\,|}{q_n^2} + \big(c\xi + d\big)^2 \\
&<\; \big(c\xi + d\big)^2\sqrt{1+\varepsilon}\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
2\Omega\Big(\frac{cp_n}{q_n} + d\Big)^2 \;&<\; \frac{2|\Delta|\sqrt{1+\varepsilon}}{\left(c\xi + d\right)^2}\,\big(c\xi + d\big)^2\sqrt{1+\varepsilon} \\
&=\; 2(1+\varepsilon)\,|\Delta| \\
&\leqslant\; G_n^2 a_{n+1} \;<\; G_n^2\Big(\,a_{n+1} + \frac{q_{n-1}}{q_n}\,\Big)\,;
\end{aligned} \tag{12}$$

the penultimate inequality is based on our assumption in Equation (3). Multiplying Inequality (12) with q_n^2, we obtain

$$2\Omega\big(cp_n + dq_n\big)^2 \;<\; q_n(a_{n+1}q_n + q_{n-1})G_n^2 \;=\; q_n q_{n+1} G_n^2\,. \tag{13}$$

Hence, using Equation (10), the right-hand part of Inequality (11), and Inequality (13),

$$\left|\,\sigma(\xi) - \frac{u_n}{v_n}\,\right| \;=\; \left|\,\sigma(\xi) - \sigma\Big(\,\frac{p_n}{q_n}\,\Big)\,\right| < \Omega\left|\,\xi - \frac{p_n}{q_n}\,\right| \;<\; \frac{\Omega}{q_n q_{n+1}} \;<\; \frac{G_n^2}{2\left(cp_n + dq_n\right)^2} \;=\; \frac{1}{2v_n^2}\,.$$

We know from [5, Theorem 184] that u_n/v_n is a convergent of $\sigma(\xi)$, as stated in Equation (3).

To prove Equation (4), we next assume $G_n^2\big(2+a_{n+1}\big) \leqslant (1-\varepsilon)|\Delta|$ for some sufficiently large integer n. Here, we obtain the following for $0<\varepsilon<1$ and large n:

$$\begin{aligned} \Big(\frac{cp_n}{q_n}+d\Big)^2 &\geqslant \big|c\xi+d\big|^2 - c^2\Big|\,\xi-\frac{p_n}{q_n}\,\Big|^2 - 2|c|\Big|\,\xi-\frac{p_n}{q_n}\,\Big|\big|c\xi+d\big| \\ &> \big(c\xi+d\big)^2 - \frac{c^2}{q_n^4} - \frac{2|c||\,c\xi+d\,|}{q_n^2} \\ &> \big(c\xi+d\big)^2\sqrt{1-\varepsilon}\,. \end{aligned}$$

Hence,

$$\begin{aligned} \omega\Big(\frac{cp_n}{q_n}+d\Big)^2 &> \frac{|\Delta|\sqrt{1-\varepsilon}}{\big(c\xi+d\big)^2}\,\big(c\xi+d\big)^2\sqrt{1-\varepsilon} \\ &= (1-\varepsilon)\,|\Delta| \\ &\geqslant G_n^2(2+a_{n+1})\,. \end{aligned}$$

From this, after multiplication by q_n^2, we obtain

$$\begin{aligned} \omega\big(cp_n+dq_n\big)^2 &> q_n(a_{n+1}q_n+2q_n)G_n^2 \\ &\geqslant q_n(a_{n+1}q_n+q_{n-1}+q_n)G_n^2 \\ &= q_n(q_{n+1}+q_n)G_n^2\,. \end{aligned} \tag{14}$$

Again using Equation (10), and additionally the left-hand part of Inequality (11), and Inequality (14), we obtain

$$\Big|\sigma(\xi)-\frac{u_n}{v_n}\,\Big| = \Big|\,\sigma(\xi)-\sigma\Big(\frac{p_n}{q_n}\Big)\,\Big| > \omega\Big|\,\xi-\frac{p_n}{q_n}\,\Big| > \frac{\omega}{q_n(q_{n+1}+q_n)} > \frac{G_n^2}{\big(cp_n+dq_n\big)^2} = \frac{1}{v_n^2}.$$

By construction, $\gcd(u_n,v_n)=1$. We conclude from [5, Theorem 171] that u_n/v_n is not a convergent of $\sigma(\xi)$. This proves Equation (4). □

Proof of Theorem 3. For brevity, we write w instead of $w(\xi)$. There is an infinite sequence $(p_n/q_n)_{n\geqslant 0}$ of leaping convergents with positive denominators q_n so that for any $\varepsilon>0$ the inequality

$$\Big|\,\xi-\frac{p_n}{q_n}\,\Big| < \frac{1}{q_n^{w-\varepsilon/2}} \tag{15}$$

holds. We apply the Intermediate Value Theorem to the function $\sigma(x)$: For all non-negative integers n, there exists a real number η_n, such that we have the inequality $|\xi-\eta_n| \leqslant |\,\xi-p_n/q_n|$ and the identity

$$\Big|\,\sigma(\xi)-\sigma\Big(\frac{p_n}{q_n}\Big)\,\Big| = \frac{|\Delta|}{\big(c\eta_n+d\big)^2}\cdot\Big|\,\xi-\frac{p_n}{q_n}\,\Big|\,. \tag{16}$$

Therefore, we obtain by Inequality (15) and by $|\xi - \eta_n| \leqslant |\,\xi - p_n/q_n|$:

$$\big|\xi - \eta_n\big| \;<\; \frac{1}{q_n^{w-\varepsilon/2}}\,. \tag{17}$$

`Case 1.` $c \neq 0$. Then,

$$\begin{aligned} \big|c\eta_n + d\big| &= |c|\cdot\Big|\eta_n + \frac{d}{c}\Big| \;=\; |c|\cdot\Big|\Big(\xi + \frac{d}{c}\Big) + \big(\eta_n - \xi\big)\Big| \\ &\geqslant |c|\cdot\Big(\Big|\xi + \frac{d}{c}\Big| - \big|\eta_n - \xi\big|\Big) \\ &> |c|\cdot\Big(\Big|\xi + \frac{d}{c}\Big| - \frac{1}{q_n^{w-\varepsilon/2}}\Big)\,. \end{aligned}$$

The last estimate is based on Inequality (17). Thus, for all sufficiently large integers n, the inequality

$$\big|c\eta_n + d\big| \;>\; \alpha|c| \tag{18}$$

is guaranteed, where $\alpha > 0$ is a constant depending only on ξ and on the fraction d/c, but not on n. Note that $\xi \neq -d/c$ by the irrationality of ξ. From Equation (16), using Inequality (18), we claim the existence of a positive integer n_0 such that we have for all integers $n \geqslant n_0$,

$$\Big|\,\sigma(\xi) - \sigma\Big(\frac{p_n}{q_n}\Big)\,\Big| \;<\; \frac{|\Delta|}{c^2\alpha^2}\cdot\Big|\,\xi - \frac{p_n}{q_n}\,\Big|\,. \tag{19}$$

`Case 2.` $c = 0$. We have $d \neq 0$ by $ad - bc \neq 0$. Following directly from Equation (16), we have for all integers $n \geqslant 0$:

$$\Big|\,\sigma(\xi) - \sigma\Big(\frac{p_n}{q_n}\Big)\,\Big| \;=\; \frac{|\Delta|}{d^2}\cdot\Big|\,\xi - \frac{p_n}{q_n}\,\Big|\,. \tag{20}$$

We define the number β by

$$\beta \;:=\; \big|\Delta\big|\cdot\begin{cases} c^{-2}\alpha^{-2} \quad , & \text{when } c \neq 0\,, \\ \\ \;\;d^{-2} \quad , & \text{when } c = 0\,. \end{cases}$$

Above, we see that $\beta > 0$ and it depends at most on ξ and on a, b, c and d. Now we can combine Inequality (19) and Equation (20) from the two cases, still using Inequality (15): There is a positive integer n_0 such that we obtain, for all integers $n \geqslant n_0$:

$$\Big|\,\sigma(\xi) - \sigma\Big(\frac{p_n}{q_n}\Big)\,\Big| \;\leqslant\; \beta\cdot\Big|\,\xi - \frac{p_n}{q_n}\,\Big| \;<\; \frac{\beta}{q_n^{w-\varepsilon/2}}\,. \tag{21}$$

Now we have

$$\sigma\Big(\,\frac{p_n}{q_n}\,\Big) \;=\; \frac{(ap_n + bq_n)D_n^{-1}}{(cp_n + dq_n)D_n^{-1}} \;=:\; \frac{u_n}{v_n}\,,$$

where $D_n := \gcd(ap_n + bq_n,\, cp_n + dq_n)$ and

$$u_n \;:=\; \frac{ap_n + bq_n}{D_n}\,, \qquad v_n \;:=\; \frac{cp_n + dq_n}{D_n}\,.$$

Set $T := \max\{|c|,\, |d|\,\}$. Then,

$$\begin{aligned} |v_n| \;&=\; \frac{1}{D_n}\big|cp_n + dq_n\big| \;\leqslant\; \frac{1}{D_n}\big(|c|\cdot|p_n| + |d|\cdot q_n\big) \\ &\leqslant\; \frac{T}{D_n}\big(|p_n| + q_n\big) \;=\; \frac{Tq_n}{D_n}\Big(\,1 + \frac{|p_n|}{q_n}\,\Big) \\ &\leqslant\; \frac{Tq_n}{D_n}(2 + |\xi|\,)\,, \end{aligned} \tag{22}$$

because

$$\Big|\,\xi - \frac{p_n}{q_n}\,\Big| \leqslant 1 \quad \text{implies} \quad \frac{|p_n|}{q_n} = \Big|\frac{p_n}{q_n}\Big| \leqslant 1 + |\xi|\,.$$

We solve Inequality (22) for q_n and thus find another upper bound for the right-hand side of Inequality (21). There is a positive integer n_0 such that we obtain, for all integers $n \geqslant n_0$, the inequalities

$$\Big|\,\sigma(\xi) - \sigma\Big(\frac{p_n}{q_n}\Big)\,\Big| \;\leqslant\; \frac{\beta\big(T(2+|\xi|\,)\big)^{w-\varepsilon/2}}{D_n^{w-\varepsilon/2}|v_n|^{w-\varepsilon/2}} \;\leqslant\; \frac{\beta\big(T(2+|\xi|\,)\big)^{w}}{|v_n|^{w-\varepsilon/2}}\,; \tag{23}$$

note that $|T| \geqslant 1$ and $D_n \geqslant 1$. We must now convince ourselves that $|v_n|$ grows unboundedly as n takes larger and larger values. This is not self-evident and can be justified by Theorem 6: by this theorem, there is a constant C depending at most on a, b, c, d, so that $D_n \leqslant C$ holds. Then,

$$|v_n| \;=\; \frac{|cp_n + dq_n|}{D_n} \;\geqslant\; \frac{|cp_n + dq_n|}{C} \;\to\; \infty\,,$$

if either

$$\big(c \;=\; 0 \quad \text{and} \quad d \;\neq\; 0\big) \qquad \text{or} \qquad \big(c \;\neq\; 0 \quad \text{and} \quad d \;=\; 0\big)\,.$$

By the fact that $\Delta \neq 0$ it remains to consider the case when $c \neq 0$ and $d \neq 0$. Because ξ as an irrational number is different from $-d/c$, the sequence of numbers $cp_n/q_n + d$ converges to the nonzero number $c\xi + d$. This means $|c\xi + d| > 0$, and so

$$|v_n| \;\geqslant\; \frac{|cp_n + dq_n|}{C} \;=\; \frac{q_n}{C}\Big|\frac{cp_n}{q_n} + d\Big| \;>\; \frac{q_n}{C}\cdot\frac{|c\xi + d|}{2} \;\to\; \infty$$

for n tending to infinity. It follows that

$$\frac{\beta\big(T(2+|\xi|\,)\big)^{w}}{|v_n|^{\varepsilon/2}} \;<\; 1$$

for all large n, since β, T, w and $2+|\xi|$ depend at most on a, b, c, d and ξ (and not on n). Therefore, we can further simplify Inequality (23) for $n \geqslant n_0$ into

$$\left|\sigma(\xi) - \sigma\Big(\frac{p_n}{q_n}\Big)\right| \leqslant \frac{\beta\big(T(2+|\xi|)\big)^w}{|v_n|^{\varepsilon/2}} \cdot \frac{1}{|v_n|^{w-\varepsilon}} < \frac{1}{|v_n|^{w-\varepsilon}}\,. \tag{24}$$

Finally, set

$$r_n := u_n \quad \text{and} \quad s_n := v_n \qquad (\text{if } v_n > 0)\,,$$

and

$$r_n := -u_n \quad \text{and} \quad s_n := -v_n \qquad (\text{if } v_n < 0)\,.$$

This gives $\sigma(p_n/q_n) = u_n/v_n = r_n/s_n$ with $s_n > 0$ and $\gcd(r_n, s_n) = 1$. So far we have proven by Equation (24) that for every positive real number ε there exists a positive integer n_0 such that we have for all integers $n \geqslant n_0$ the inequality

$$\left|\sigma(\xi) - \frac{r_n}{s_n}\right| < \frac{1}{s_n^{w-\varepsilon}}\,. \tag{25}$$

It is well known that $w = w(\xi) \geqslant 2$, and from Inequality (25) we have by definition of the irrationality measure:

$$w(\xi) \leqslant w\big(\sigma(\xi)\big)\,. \tag{26}$$

`Case 1.` $w = 2$. In this case, Inequality (25) is trivial, and the sequence of approximation fractions r_n/s_n constructed for Inequality (25) can be replaced by *the sequence of all convergents* of the irrational number $\sigma(\xi)$; Inequality (25) then still holds. The set of all convergents may be considered as a *conformal set of leaping convergents* by Definition 1.

`Case 2.` $w > 2$. We choose ε in Inequality (25) to be small enough that $w - \varepsilon > 2$. For large n, we then have

$$\frac{1}{s_n^{w-\varepsilon}} < \frac{1}{2s_n^2}\,.$$

We know from [5, Theorem 184] that r_n/s_n is a convergent of $\sigma(\xi)$. Hence, the truncated fractions r_n/s_n form *a set of leaping convergents* of $\sigma(\xi)$. So, in any case, in Inequality (25) there are underlying leaping convergents r_n/s_n, and therefore we have the same initial situation as in Inequality (15). Therefore, we now repeat the previous construction for the number $\sigma(\xi)$, but we apply it with the linear fractional transformation σ^{-1} inverse to σ, which, as is well known, is also represented by a fraction of the shape

$$\sigma^{-1}(x) = \frac{a'x + b'}{c'x + d'}$$

with integer coefficients a', b', c' and d' and with a nonzero determinant. Instead of Inequality (25) we then obtain for every positive real number ε a positive integer n_0 such that we have for all integers $n \geqslant n_0$ the inequality

$$\left| \xi - \frac{r'_n}{s'_n} \right| = \left| \sigma^{-1}\big(\sigma(\xi)\big) - \frac{r'_n}{s'_n} \right| < \frac{1}{(s'_n)^{w-\varepsilon}}\,.$$

Now we get analogously to Inequality (26):

$$w\big(\sigma(\xi)\big) \leqslant w(\xi)\,. \tag{27}$$

Inequalities (26) and (27) prove Theorem 3. □

Proof of Theorem 4. Let

$$\xi = \big[\,a_0, a_1, \ldots, a_\rho, \overline{T_1, \ldots, T_w}\,\big]$$

be the continued fraction expansion of ξ, where $\rho \geqslant 0$ and $w \geqslant 1$, and

$$A := \max\{|a_0|, a_1, \ldots, a_\rho, T_1, \ldots, T_w\}\,.$$

Moreover, we assume that $p/q \in \mathcal{C}(\xi)$ and $\sigma(p/q) \in \sigma^*\big(\mathcal{C}(\xi)\big)$ with $p = p_m$ and $q = q_m$, say, for some $p_m/q_m \in \mathcal{C}(\xi)$. In particular, we have by Equation (1):

$$G_m := \gcd\big(ap_m + bq_m,\, cp_m + dq_m\big) = 1\,.$$

For every lft σ with $|\Delta| \geqslant A + 3$, the inequality on the left-hand side of Equation (7) in Corollary 1 is fulfilled, since

$$G_m(2 + a_{m+1}) = a_{m+1} + 2 \leqslant A + 2 \leqslant |\Delta| - 1\,.$$

From Corollary 1 we conclude that

$$\sigma\Big(\frac{p}{q}\Big) = \sigma\Big(\frac{p_m}{q_m}\Big) \notin \mathcal{C}\big(\sigma(\xi)\big)\,.$$

This proves the theorem. □

Proof of Theorem 5. As is known, there are uncountably many infinite number sequences whose elements consist only of the numbers 1 and 2. As a sequence of partial denominators, a real irrational number is thus uniquely assigned to each number sequence via the continued fraction expansion. Since the set of the real-algebraic numbers is countable, we have thus an uncountable set of transcendental numbers ξ with partial denominators 1 and 2 in their continued fraction expansion. For any such ξ, the statement of Theorem 5 follows from Equation (1) and from Equation (4) in Theorem 1 by setting $G_n = 1$ and $\varepsilon = 1/10$. Then we have:

$$(1-\varepsilon)|\Delta| \geqslant \frac{9}{10}\cdot 5 > 4 \geqslant G_n(2 + a_{n+1}) \qquad (n \geqslant 1)\,,$$

so that

$$\sigma\Big(\frac{p_n}{q_n}\Big) \notin \mathcal{C}\big(\sigma(\xi)\big) \cap \sigma^*\big(\mathcal{C}(\xi)\big)$$

for all sufficiently large integers n. This completes the proof of Theorem 5. □

Note that almost all real numbers (in the sense of the Lebesgue measure) have an unbounded continued fraction expansion.

Proof of Theorem 6. Let p be a prime and k a positive integer such that

$$\gcd\big(ap_n + bq_n,\, cp_n + dq_n\big) \equiv 0 \pmod{p^k}. \tag{28}$$

Let us consider the identity

$$a\big(cp_n + dq_n\big) - c\big(ap_n + bq_n\big) = (ad - bc)q_n = \Delta q_n\,. \tag{29}$$

From Equation (28) and Equation (29) we have simultaneously,

$$ap_n + bq_n \equiv 0 \pmod{p^k}, \quad cp_n + dq_n \equiv 0 \pmod{p^k}, \quad \Delta q_n \equiv 0 \pmod{p^k}. \tag{30}$$

Let $k = k_1 + k_2$ with non-negative integers k_1 and k_2 satisfying $\Delta \equiv 0 \pmod{p^{k_1}}$ and $q_n \equiv 0 \pmod{p^{k_2}}$.

`Case 1.` $k_2 = 0$, i.e., $\Delta \equiv 0 \pmod{p^k}$. This gives

$$p^k \leqslant |\Delta|\,.$$

`Case 2.` $k_2 > 0$. From $\Delta \equiv 0 \pmod{p^{k_1}}$ we have $p^{k_1} \leqslant |\Delta|$, and moreover, $q_n \equiv 0 \pmod{p^{k_2}}$ with $k_2 \geqslant 1$. Since p_n and q_n are coprime, the first and second congruences in Equation (30) yield

$$a \equiv 0 \pmod{p^{k_2}} \quad \text{and} \quad c \equiv 0 \pmod{p^{k_2}};$$

note that $k_2 < k$. Thus, we obtain

$$p^{k_2} \leqslant \min\big\{|a|,\,|c|\big\}\,.$$

If we summarize the results, we obtain the inequality

$$p^k = p^{k_1}p^{k_2} \leqslant \big|\Delta\big| \cdot \min\big\{|a|,\,|c|\,\big\}\,. \tag{31}$$

If we assume the identity

$$d\big(ap_n + bq_n\big) - b\big(cp_n + dq_n\big) = (ad - bc)p_n = \Delta p_n\,,$$

instead of Equation (29), we obtain in an analogous way the inequality

$$p^k \leqslant \big|\Delta\big| \cdot \min\big\{|b|,\,|d|\,\big\}\,. \tag{32}$$

If we combine the results from Inequalities (31) and (32), we get

$$p^k \leqslant \big|\Delta\big| \cdot \min\big\{|a|,\, |b|,\, |c|,\, |d|\,\big\} \tag{33}$$

for all primes p and all positive integers k satisfying Equation (28). With respect to Equation (28), the special case for $k = 1$ means that there are at most finitely many prime divisors of $\gcd\big(ap_n + bq_n,\, cp_n + dq_n\big)$ for all integers $n \geqslant 0$. And if we apply Inequality (33) for a second time with $k \geqslant 1$, we obtain the statement of the theorem. □

We give an explicit bound C for the greatest common divisor in Theorem 6. It has been shown that every prime power p^k dividing the gcd of $ap_n + bq_n$ and $cp_n + dq_n$ is bounded by $D := |\Delta| \min\{|a|, |b|, |c|, |d|\}$ (see Equation (31)). Now, let $\pi(x)$ be the prime counting function over the interval $[2, x]$. Let

$$\gcd\big(ap_n + bq_n, cp_n + dq_n\big) \;=\; p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

be the prime factorization of the greatest common divisor. Then we have

$$\gcd\big(ap_n + bq_n, cp_n + dq_n\big) \;\leqslant\; D^r \;\leqslant\; D^{\pi(D)} =: C\,.$$

This number C can be chosen for the constant C in Theorem 6. Applying the prime number theorem, we have

$$e^{(1-\varepsilon)D} \;\ll\; C \;\ll\; e^{(1+\varepsilon)D}$$

for every real number $0 < \varepsilon < 1$ and implicit constants depending at most on ε. □

5. Supplementary Results

Lemma 1. *Let $n \geqslant 2$ be an integer, and let $c_1, \ldots, c_n$ be real numbers. We consider the sum of two determinants of order n and $n-2$, respectively,*

$$U(c_1, \ldots, c_n) := \begin{vmatrix} c_1 & -1 & & & \\ 1 & c_2 & -1 & 0 & \\ & & \ddots & & \\ & 0 & & c_{n-1} & -1 \\ & & & 1 & c_n \end{vmatrix} + \begin{vmatrix} c_2 & -1 & & & \\ 1 & c_3 & -1 & 0 & \\ & & \ddots & & \\ & 0 & & c_{n-2} & -1 \\ & & & 1 & c_{n-1} \end{vmatrix}.$$

For $n = 2$ the value of the second determinant is set to 1. In this case, we have $U(c_1, \ldots, c_n) = U(c_n, c_1, c_2, \ldots, c_{n-1})$.

Proof. Let

$$A(c_1, \ldots, c_n) \;:=\; \begin{vmatrix} c_1 & -1 & & & \\ 1 & c_2 & -1 & 0 & \\ & & \ddots & & \\ & 0 & & c_{n-1} & -1 \\ & & & 1 & c_n \end{vmatrix}.$$

We shall prove the following two identities,

$$\begin{aligned}\det\big(A(c_n,c_1,\dots,c_{n-1})\big)&-\det\big(A(c_2,c_3,\dots,c_{n-1})\big)\\ &= c_n\det\big(A(c_1,c_2,\dots,c_{n-1})\big)\\ &= \det\big(A(c_1,c_2,\dots,c_n)\big)\ -\ \det\big(A(c_1,c_2,\dots,c_{n-2})\big).\end{aligned}\tag{34}$$

This can in fact be changed to

$$\begin{aligned}\det\big(A(c_1,c_2,\dots,c_n)\big)&+\det\big(A(c_2,c_3,\dots,c_{n-1})\big)\\ &=\det\big(A(c_n,c_1,\dots,c_{n-1})\big)+\det\big(A(c_1,c_2,\dots,c_{n-2})\big),\end{aligned}$$

which is equivalent to the assertion of the lemma.

To prove the first identity in Equation (34), we expand the determinant

$$A\big(c_n,c_1,\dots,c_{n-1}\big)\ =\ \begin{vmatrix} c_n & -1 & & & \\ 1 & c_1 & -1 & 0 & \\ & & \ddots & & \\ & 0 & & c_{n-2} & -1 \\ & & & 1 & c_{n-1}\end{vmatrix}$$

according to its first line,

$$\begin{aligned}\det\big(A(c_n,c_1,\dots,c_{n-1})\big)&\\ &= c_n\det\big(A(c_1,\dots,c_{n-1})\big)-(-1)\begin{vmatrix} 1 & -1 & & & & \\ 0 & c_2 & -1 & & 0 & \\ 0 & 1 & c_3 & -1 & & \\ & & \ddots & & & \\ & 0 & & & c_{n-2} & -1 \\ & & & & 1 & c_{n-1}\end{vmatrix}\\ &= c_n\det\big(A(c_1,\dots,c_{n-1})\big)+\det\big(A(c_2,c_3,\dots,c_{n-1})\big),\end{aligned}$$

where the last determinant resulted from an expansion according to the first column. In an analogous way, the second identity in Equation (34) is proved. Here, $A(c_1,c_2,\dots,c_n)$ is expanded first according to the last column, and the remaining second determinant then according to its last line:

$$\begin{aligned}\det\big(A(c_1,c_2,\dots,c_n)\big)&\\ &= c_n\det\big(A(c_1,\dots,c_{n-1})\big)-(-1)\begin{vmatrix} c_1 & -1 & & & & \\ 1 & c_2 & -1 & & 0 & \\ & & \ddots & & & \\ & & & c_{n-3} & -1 & \\ & 0 & & 1 & c_{n-2} & -1 \\ & & & 0 & 0 & 1\end{vmatrix}\\ &= c_n\det\big(A(c_1,\dots,c_{n-1})\big)+\det\big(A(c_1,c_2,\dots,c_{n-2})\big).\end{aligned}$$

The two identities in Equation (34) are shown, and thus the lemma is proved. □

Proposition 1. *Let*

$$\xi := \left[a_0, a_1, \ldots, a_\rho, \overline{T_1, \ldots, T_w} \right] \tag{35}$$

be the continued fraction expansion of a real quadratic irrational number ξ, where $\rho \geqslant 0$ and $w \geqslant 2$. The convergents of ξ are denoted by p_n/q_n. Let

$$\begin{aligned} U \quad := & \begin{vmatrix} -T_1 & -1 & & & \\ 1 & -T_2 & -1 & 0 & \\ & & \ddots & & \\ & 0 & & -T_{w-1} & -1 \\ & & & 1 & -T_w \end{vmatrix} + \ldots \\ + & \begin{vmatrix} -T_2 & -1 & & & \\ 1 & -T_3 & -1 & 0 & \\ & & \ddots & & \\ & 0 & & -T_{w-2} & -1 \\ & & & 1 & -T_{w-1} \end{vmatrix} \end{aligned} \tag{36}$$

and

$$\left.\begin{aligned} \psi_1 &:= \frac{1}{2}\Big((-1)^w U + \sqrt{U^2 + 4(-1)^{w-1}} \Big), \\ \psi_2 &:= \frac{1}{2}\Big((-1)^w U - \sqrt{U^2 + 4(-1)^{w-1}} \Big), \end{aligned}\right\} \tag{37}$$

$$\left.\begin{aligned} C_1^{(i)} &:= \frac{p_{i+w} - p_i \psi_2}{\psi_1 - \psi_2}, \\ C_2^{(i)} &:= \frac{p_i \psi_1 - p_{i+w}}{\psi_1 - \psi_2}, \\ C_3^{(i)} &:= \frac{q_{i+w} - q_i \psi_2}{\psi_1 - \psi_2}, \\ C_4^{(i)} &:= \frac{q_i \psi_1 - q_{i+w}}{\psi_1 - \psi_2}. \end{aligned}\right\} \quad (\rho \leqslant i \leqslant \rho + w - 1) \tag{38}$$

Then, $U \neq 0$, $\psi_1 \neq \psi_2$, and for $n \geqslant 0$ and $\rho \leqslant i \leqslant \rho + w - 1$, we have

$$p_{nw+i} = C_1^{(i)} \psi_1^n + C_2^{(i)} \psi_2^n, \tag{39}$$

$$q_{nw+i} = C_3^{(i)} \psi_1^n + C_4^{(i)} \psi_2^n, \tag{40}$$

$$\frac{p_{nw+i}}{q_{nw+i}} = \frac{p_{i+w}\big(\psi_1^n - \psi_2^n\big) + (-1)^{w-1} p_i \big(\psi_1^{n-1} - \psi_2^{n-1}\big)}{q_{i+w}\big(\psi_1^n - \psi_2^n\big) + (-1)^{w-1} q_i \big(\psi_1^{n-1} - \psi_2^{n-1}\big)}. \tag{41}$$

Remark 2. For $w = 2$ the value of the second determinant in Equation (36) is set to 1.

Proof. We apply Corollary 1 from [2] in the particular cases where the Hurwitz-type continued fraction $\alpha = \xi$ is a real quadratic irrational number, and with $r = w$. Let

$$T(a) \,:=\, T_{w\{(a-\rho-1)/w\}+1}\,,$$

where $\{.\}$ denotes the fractional part function, and

$$D_l(a) \,:=\, \begin{vmatrix} -T(a) & -1 & & & \\ 1 & -T(a+1) & -1 & 0 & \\ & & \ddots & & \\ & 0 & & -T(a+l-2) & -1 \\ & & & 1 & -T(a+l-1) \end{vmatrix}$$

for integers a. Then, for $w \geqslant 2$ and $0 \leqslant \rho \leqslant i \leqslant \rho + w - 1$, the identity

$$\left.\begin{aligned} 0 \;=\;& (-1)^{w-1} D_{w-1}(M-w)\cdot z_n \\ &+ \big(D_{r-1}(M)D_w(M-w) + D_{w-1}(M-w)D_{w-2}(M+1)\big)\cdot z_{n-1} \\ &- D_{w-1}(M)\cdot z_{n-2} \end{aligned}\right\} \tag{42}$$

holds for $z_n = p_{wn+i}$ and $z_n = q_{wn+i}$ with $M := (n-1)w + i + 2$ and $n \geqslant 2$. Now, we observe that

$$T(M+t) \,=\, T_{w\{((n-1)w+i+t+2-\rho-1)/w\}+1} \,=\, T_{w\{(i+t-\rho+1)/w\}+1}\,,$$

and

$$T(M-w+t) \,=\, T_{w\{(i+t-\rho+1)/w\}+1} \,=\, T(M+t)$$

hold for $0 \leqslant t \leqslant w-2$. Therefore, the two integers $D_{w-1}(M)$ and $D_{w-1}(M-w)$ in Equation (42) coincide, and because of the fact that $D_{w-1}(M) \neq 0$, they do not both disappear. Thus, Equation (42) can be simplified as follows:

$$0 \,=\, (-1)^{w-1} z_n + \big(D_w(M-w) + D_{w-2}(M+1)\big) z_{n-1} - z_{n-2}\,; \tag{43}$$

see Theorem 2 in [3]: $D_l(M) =: (-1)^l K_l(M) \neq 0$ with $U(M+1) = U(M+2) = \cdots = U(M+l-1) = 1$. In Equation (43), the determinant $D_w(M-w)$ is given by

$$\begin{vmatrix} -T_{w\{(i+1-\rho)/w\}+1} & -1 & & & \\ 1 & -T_{w\{(i+2-\rho)/w\}+1} & -1 & & \\ & & \ddots & & \\ & 0 & & -1 & \\ & & & -T_{w\{(i+w-\rho)/w\}+1} & \end{vmatrix}, \tag{44}$$

whereas the determinant $D_{w-2}(M+1)$ is

$$\begin{vmatrix} -T_{w\{(i+2-\rho)/w\}+1} & -1 & & & \\ 1 & -T_{w\{(i+3-\rho)/w\}+1} & -1 & & \\ & & \ddots & & \\ & 0 & & & -1 \\ & & & & -T_{w\{(i+w-1-\rho)/w\}+1} \end{vmatrix} . \tag{45}$$

The sum $D_w(M-w)+D_{w-2}(M+1)$ of the determinants from Equation (44) and Equation (45) does not depend on i, which follows from Lemma 1. We specifically choose $i = w-1+\rho$ such that $\rho+1 \leqslant i < \rho+w$ holds because $w \geqslant 2$. Following from Equation (36), we have

$$U \;=\; D_w(M-w)+D_{w-2}(M+1)\,,$$

so that Equation (43) takes the form

$$0 \;=\; z_n + (-1)^{w-1} U z_{n-1} + (-1)^w z_{n-2} \qquad (n \geqslant 2)\,. \tag{46}$$

Assuming that $U = 0$, Equation (46) would then simplify to $z_n = \pm z_{n-2}$, or $q_{nw+i} = \pm q_{(n-2)w+i}$, which is impossible. Therefore, we have $U \neq 0$. The characteristic polynomial $P(X)$ of Equation (46) is

$$P(X) \;=\; X^2 + (-1)^{w-1} UX + (-1)^w \tag{47}$$

with the roots

$$\left.\begin{aligned} \psi_1 &= \frac{1}{2}\Big((-1)^w U + \sqrt{U^2 + 4(-1)^{w-1}}\Big)\,, \\ \psi_2 &= \frac{1}{2}\Big((-1)^w U - \sqrt{U^2 + 4(-1)^{w-1}}\Big)\,. \end{aligned}\right\} \tag{48}$$

Next, let us assume that $\psi := \psi_1 = \psi_2 \neq 0$. In particular, ψ is a rational number. Now, it is well known from combinatorics that a sequence $(z_n)_{n\geqslant 0}$ of numbers satisfying the recurrence formula in Equation (46), whose characteristic polynomial of degree two has a non-vanishing double zero ψ, is given by

$$z_n \;=\; \big(C_1 n + C_2\big)\psi^n\,.$$

We know that C_1 and C_2 are two real constants depending only on $z_0 = p_i$ and $z_1 = p_{w+i}$ (for $z_n = p_{nw+i}$) as well as on $z_0 = q_i$ and $z_1 = q_{w+i}$ (for $z_n = q_{nw+i}$). They are given by $z_0 = p_i = C_2^{(i)}$ and

$$z_1 \;=\; p_{w+i} \;=\; (C_1^{(i)} + C_2^{(i)})\psi \;=\; (C_1^{(i)} + p_i)\psi\,,$$

such that

$$C_1^{(i)} = \frac{p_{w+i} - p_i\psi}{\psi}\,.$$

Similarly, we calculate constants $C_3^{(i)}$ and $C_4^{(i)} = q_i$ for the numbers $z_n = q_{nw+i}$. In particular,

$$C_3^{(i)} = \frac{q_{w+i} - q_i\psi}{\psi}\,.$$

Therefore, we get

$$\frac{p_{nw+i}}{q_{nw+i}} = \frac{C_1^{(i)}n + C_2^{(i)}}{C_3^{(i)}n + C_4^{(i)}} \quad \overset{n\to\infty}{\longrightarrow} \quad \xi = \frac{C_1^{(i)}}{C_3^{(i)}} \in \mathbb{Q}\,.$$

For the rational number $C_1^{(i)}/C_3^{(i)}$, we know that neither the cases $C_1^{(i)} \neq 0$ and $C_3^{(i)} = 0$ nor the case $C_1^{(i)} = 0 = C_3^{(i)}$ occurs. In the latter case we would obtain

$$\frac{p_{w+i}}{p_i} = \psi = \frac{q_{w+i}}{q_i}\,, \quad \text{or} \quad \frac{p_{w+i}}{q_{w+i}} = \psi = \frac{p_i}{q_i}\,,$$

a contradiction. But ξ was assumed to be irrational, again a contradiction. This shows that $\psi_1 \neq \psi_2$.

It is well known from combinatorics that a sequence $(z_n)_{n\geqslant 0}$ of numbers satisfying the recurrence formula in Equation (46) can be represented by an explicit formula of the form

$$z_n = C_1\psi_1^n + C_2\psi_2^n \qquad (n \geqslant 0)\,, \tag{49}$$

where ψ_1 and ψ_2 with $\psi_1 \neq \psi_2$ are the roots of the characteristic polynomial of the recurrence formula in Equation (46), and C_1, C_2 are two real numbers depending only on $z_0 = p_i$ and $z_1 = p_{w+i}$ (for $z_n = p_{nw+i}$) as well as on $z_0 = q_i$ and $z_1 = q_{w+i}$ (for $z_n = q_{nw+i}$). In order to compute C_1 and C_2 we write down Equation (49) for $n = 0$ and $n = 1$. It suffices to show the argument for $z_n = p_{nw+i}$, as the arguments for $z_n = q_{nw+i}$ are the same:

$$\begin{aligned} p_i &= C_1^{(i)} + C_2^{(i)}\,, \\ p_{w+i} &= C_1^{(i)}\psi_1 + C_2^{(i)}\psi_2\,. \end{aligned}$$

We solve this quadratic, inhomogeneous and linear system of equations with unknowns $C_1^{(i)}$ and $C_2^{(i)}$ by Cramer's rule. Since $\psi_1 \neq \psi_2$ holds by the hypothesis of Theorem 1, we have

$$C_1^{(i)} = \frac{\begin{vmatrix} p_i & 1 \\ p_{w+i} & \psi_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \psi_1 & \psi_2 \end{vmatrix}} = \frac{p_{i+w} - p_i\psi_2}{\psi_1 - \psi_2}\,, \tag{50}$$

$$C_2^{(i)} = \frac{\begin{vmatrix} 1 & p_i \\ \psi_1 & p_{w+i} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \psi_1 & \psi_2 \end{vmatrix}} = \frac{p_i\psi_1 - p_{i+w}}{\psi_1 - \psi_2}\,. \tag{51}$$

The statements in Equations (37) to (40) follow from Equations (48), (49), (50) and (51). It remains to prove Equation (41), which follows from Equations (38) to (40) as shown below:

$$\begin{aligned}
\frac{p_{nw+i}}{q_{nw+i}} &= \frac{(p_{i+w} - p_i\psi_2)\psi_1^n + (p_i\psi_1 - p_{i+w})\psi_2^n}{(q_{i+w} - q_i\psi_2)\psi_1^n + (q_i\psi_1 - q_{i+w})\psi_2^n} \\
&= \frac{p_{i+w}\left(\psi_1^n - \psi_2^n\right) - \psi_1\psi_2 p_i\left(\psi_1^{n-1} - \psi_2^{n-1}\right)}{q_{i+w}\left(\psi_1^n - \psi_2^n\right) - \psi_1\psi_2 q_i\left(\psi_1^{n-1} - \psi_2^{n-1}\right)} \\
&= \frac{p_{i+w}\left(\psi_1^n - \psi_2^n\right) + (-1)^{w-1} p_i\left(\psi_1^{n-1} - \psi_2^{n-1}\right)}{q_{i+w}\left(\psi_1^n - \psi_2^n\right) + (-1)^{w-1} q_i\left(\psi_1^{n-1} - \psi_2^{n-1}\right)}\,;
\end{aligned}$$

the last equality is based on the identity $\psi_1\psi_2 = (-1)^w$. This completes the proof of the proposition. □

We still need a supplementary theorem for real square numbers with a period length of $w = 1$.

Proposition 2. *Let*

$$\xi := \left[\, a_0, a_1, \ldots, a_\rho, \overline{T}\,\right]$$

be the continued fraction expansion of a real quadratic irrational number ξ*, where* $\rho \geqslant 0$*. The convergents of* ξ *are denoted by* p_n/q_n*. Let*

$$\left.\begin{aligned}
\psi_1 &:= \frac{1}{2}\left(T + \sqrt{T^2 + 4}\right), \\
\psi_2 &:= \frac{1}{2}\left(T - \sqrt{T^2 + 4}\right),
\end{aligned}\right\} \tag{52}$$

$$\left.\begin{aligned}
C_1 &:= \frac{\psi_2 p_{\rho+1} - p_{\rho+2}}{\psi_1^{\rho+1}\psi_2 - \psi_1^{\rho+2}}\,, \\
C_2 &:= \frac{\psi_1 p_{\rho+1} - p_{\rho+2}}{\psi_1\psi_2^{\rho+1} - \psi_2^{\rho+2}}\,, \\
C_3 &:= \frac{\psi_2 q_{\rho+1} - q_{\rho+2}}{\psi_1^{\rho+1}\psi_2 - \psi_1^{\rho+2}}\,, \\
C_4 &:= \frac{\psi_1 q_{\rho+1} - q_{\rho+2}}{\psi_1\psi_2^{\rho+1} - \psi_2^{\rho+2}}\,.
\end{aligned}\right\} \tag{53}$$

Then we have for $n \geqslant \rho + 1$:

$$p_n = C_1\psi_1^n + C_2\psi_2^n\,, \tag{54}$$
$$q_n = C_3\psi_1^n + C_4\psi_2^n\,, \tag{55}$$

$$\frac{p_n}{q_n} = \frac{p_{\rho+2}\big(\psi_1^{n-\rho-1} - \psi_2^{n-\rho-1}\big) + p_{\rho+1}\big(\psi_1^{n-\rho-2} - \psi_2^{n-\rho-2}\big)}{q_{\rho+2}\big(\psi_1^{n-\rho-1} - \psi_2^{n-\rho-1}\big) + q_{\rho+1}\big(\psi_1^{n-\rho-2} - \psi_2^{n-\rho-2}\big)}\,. \tag{56}$$

Proof. For $z_n = p_n$ and $z_n = q_n$, we have the recurrence formula

$$z_n = Tz_{n-1} + z_{n-2} \qquad (n \geqslant \rho + 1)\,.$$

Its characteristic polynomial is

$$P(X) := X^2 - TX - 1$$

with two real roots given by ψ_1 and ψ_2 in Equation (52). Since $\psi_1 \neq \psi_2$, we know the explicit formulas in Equations (54) and (55) for $z_n = p_n$ and $z_n = q_n$, respectively, where the constants C_j $(j = 1, 2, 3, 4)$ can again be determined from initial conditions using Cramer's rule:

$$p_{\rho+1} = C_1\psi_1^{\rho+1} + C_2\psi_2^{\rho+1}\,,$$

$$p_{\rho+2} = C_1\psi_1^{\rho+2} + C_2\psi_2^{\rho+2}\,.$$

Then,

$$C_1 = \frac{\begin{vmatrix} p_{\rho+1} & \psi_2^{\rho+1} \\ p_{\rho+2} & \psi_2^{\rho+2} \end{vmatrix}}{\begin{vmatrix} \psi_1^{\rho+1} & \psi_2^{\rho+1} \\ \psi_1^{\rho+2} & \psi_2^{\rho+2} \end{vmatrix}} = \frac{\psi_2 p_{\rho+1} - p_{\rho+2}}{\psi_1^{\rho+1}\psi_2 - \psi_1^{\rho+2}}\,,$$

and

$$C_2 = \frac{\begin{vmatrix} \psi_1^{\rho+1} & p_{\rho+1} \\ \psi_1^{\rho+2} & p_{\rho+2} \end{vmatrix}}{\begin{vmatrix} \psi_1^{\rho+1} & \psi_2^{\rho+1} \\ \psi_1^{\rho+2} & \psi_2^{\rho+2} \end{vmatrix}} = \frac{\psi_1 p_{\rho+1} - p_{\rho+2}}{\psi_1\psi_2^{\rho+1} - \psi_2^{\rho+2}}\,.$$

The formulas for C_3 and C_4 in Equation (53) result accordingly. By directly recalculating with the formulas from Equations (53) to (55), and using the identity $\psi_1\psi_2 = -1$, one verifies the identity in Equation (56). □

Example 3. Let

$$\xi := \big[\,0, 1, 2, \overline{3, 4, 5}\,\big] = \frac{85 + 2\sqrt{1297}}{3\big(39 + \sqrt{1297}\big)} = 0.69777201\ldots\,.$$

By Proposition 1 we have:

$$\rho = 2\,, \quad w = 3\,, \quad a_0 = 0\,, \quad a_1 = 1\,, \quad a_2 = 2\,, \quad T_1 = 3\,, \quad T_2 = 4\,, \quad T_3 = 5\,.$$

The convergents p_n/q_n for $n = 0, \dots, 9$ are

$$1\,, \quad \frac{2}{3}\,, \quad \frac{7}{10}\,, \quad \frac{30}{43}\,, \quad \frac{157}{225}\,, \quad \frac{501}{718}\,, \quad \frac{2161}{3097}\,, \quad \frac{11306}{16203}\,, \quad \frac{36079}{51706}\,, \quad \frac{155622}{223027}\,.$$

We compute

$$U \;=\; \begin{vmatrix} -3 & -1 & 0 \\ 1 & -4 & -1 \\ 0 & 1 & -5 \end{vmatrix} + (-4) \;=\; -72\,,$$

$$\begin{aligned} \psi_1 &= 36 + \sqrt{1297}\,, \\ \psi_2 &= 36 - \sqrt{1297}\,. \end{aligned}$$

We now write down Equation (41) for $2 \leqslant i \leqslant 4$:

$$\begin{aligned} \frac{p_{3n+2}}{q_{3n+2}} &= \frac{157\big(\psi_1^n - \psi_2^n\big) + 2\big(\psi_1^{n-1} - \psi_2^{n-1}\big)}{225\big(\psi_1^n - \psi_2^n\big) + 3\big(\psi_1^{n-1} - \psi_2^{n-1}\big)}\,, \\ \frac{p_{3n+3}}{q_{3n+3}} &= \frac{501\big(\psi_1^n - \psi_2^n\big) + 7\big(\psi_1^{n-1} - \psi_2^{n-1}\big)}{718\big(\psi_1^n - \psi_2^n\big) + 10\big(\psi_1^{n-1} - \psi_2^{n-1}\big)}\,, \\ \frac{p_{3n+4}}{q_{3n+4}} &= \frac{2161\big(\psi_1^n - \psi_2^n\big) + 30\big(\psi_1^{n-1} - \psi_2^{n-1}\big)}{3097\big(\psi_1^n - \psi_2^n\big) + 43\big(\psi_1^{n-1} - \psi_2^{n-1}\big)}\,. \end{aligned}$$

These three formulas can be applied for all integers $n \geqslant 0$.

Example 4. Let $a \geqslant 1$ be an integer, and

$$\xi \;:=\; \big[\,\overline{a}\,\big] \;=\; \frac{1}{2}\big(\,a + \sqrt{a^2+4}\,\big)\,.$$

Again, by Proposition 2 we have:

$$\rho = 0\,, \quad a_0 = a\,, \quad T = a\,.$$

The convergents p_n/q_n for n=0,1,2 are

$$a\,, \quad \frac{a^2+1}{a}\,, \quad \frac{a^3+2a}{a^2+1}\,.$$

We compute the numbers from Equations (52) and (53) in Proposition 2:

$$\begin{aligned} \psi_1 &= \frac{1}{2}\big(\,a + \sqrt{a^2+4}\,\big) \;=\; \xi\,, \\ \psi_2 &= \frac{1}{2}\big(\,a - \sqrt{a^2+4}\,\big)\,, \end{aligned}$$

$$\begin{aligned}
C_1 &= \frac{(a^2+1)\psi_2 - a^3 - 2a}{\psi_1\psi_2 - \psi_1^2} = \frac{\psi_1^2}{\sqrt{a^2+1}}\,,\\
C_2 &= \frac{(a^2+1)\psi_1 - a^3 - 2a}{\psi_1\psi_2 - \psi_2^2} = -\frac{\psi_2^2}{\sqrt{a^2+1}}\,,\\
C_3 &= \frac{a\psi_2 - a^2 - 1}{\psi_1\psi_2 - \psi_1^2} = \frac{\psi_1}{\sqrt{a^2+1}}\,,\\
C_4 &= \frac{a\psi_1 - a^2 - 1}{\psi_1\psi_2 - \psi_2^2} = -\frac{\psi_2}{\sqrt{a^2+1}}\,.
\end{aligned}$$

Then we obtain from Equations (54) and (55):

$$\begin{aligned}
p_n &= \frac{\psi_1^{n+2} - \psi_2^{n+2}}{\sqrt{a^2+4}}\,,\\
q_n &= \frac{\psi_1^{n+1} - \psi_2^{n+1}}{\sqrt{a^2+4}}\,.
\end{aligned}$$

In the above Proposition 2 we can also use Equation (56):

$$\frac{p_n}{q_n} = \frac{(a^3+2a)\big(\psi_1^{n-1} - \psi_2^{n-1}\big) + (a^2+1)\big(\psi_1^{n-2} - \psi_2^{n-2}\big)}{(a^2+1)\big(\psi_1^{n-1} - \psi_2^{n-1}\big) + a\big(\psi_1^{n-2} - \psi_2^{n-2}\big)} = \frac{\psi_1^{n+2} - \psi_2^{n+2}}{\psi_1^{n+1} - \psi_2^{n+1}}\,.$$

Example 5. Let $r \geqslant 1$ be an integer, and

$$\xi := \big[\, r, \overline{2r}\,\big] = \sqrt{r^2+1}\,.$$

Again, by Proposition 2 we have

$$\rho = 0\,, \quad a_0 = r\,, \quad T = 2r\,.$$

We compute the numbers from Equations (52) and (53) in Proposition 2:

$$\begin{aligned}
\psi_1 &= r + \sqrt{r^2+1}\,,\\
\psi_2 &= r - \sqrt{r^2+1}\,,
\end{aligned}$$

$$\begin{aligned}
C_1 &= \frac{(2r^2+1)\psi_2 - 4r^3 - 3r}{\psi_1\psi_2 - \psi_1^2} = \frac{r\psi_1 + 1}{2\sqrt{r^2+1}}\,,\\
C_2 &= \frac{(2r^2+1)\psi_1 - 4r^3 - 3r}{\psi_1\psi_2 - \psi_2^2} = -\frac{r\psi_2 + 1}{2\sqrt{r^2+1}}\,,\\
C_3 &= \frac{2r\psi_2 - 4r^2 - 1}{\psi_1\psi_2 - \psi_1^2} = \frac{\psi_1}{2\sqrt{r^2+1}}\,,\\
C_4 &= \frac{2r\psi_1 - 4r^2 - 1}{\psi_1\psi_2 - \psi_2^2} = -\frac{\psi_2}{2\sqrt{r^2+1}}\,.
\end{aligned}$$

Then we obtain from Equations (54) and (55):

$$p_n = \frac{(r\psi_1+1)\psi_1^n - (r\psi_2+1)\psi_2^n}{2\sqrt{r^2+1}},$$

$$q_n = \frac{\psi_1^{n+1} - \psi_2^{n+1}}{2\sqrt{r^2+1}}.$$

If for all $n \geqslant 0$ we have indexing functions $f(n)$ and $g(n)$ such that

$$\left(\frac{p_{f(n)}}{q_{f(n)}}, \frac{p'_{g(n)}}{q'_{g(n)}}\right) \in R_\sigma ,$$

then we say that $g(n)$ is the leaping pattern of the natural leaping convergents from the set $\mathcal{C}(\sigma(\xi))$. Consider a family of continued fractions, $\xi(a)$, and an lft σ, where the above σ-relation holds so that

$$\sigma\left(\frac{p_{f(n)}}{q_{f(n)}}\right) = \frac{p'_{g(n)}}{q'_{g(n)}} \in \mathcal{C}\big(\sigma(\xi(a))\big),$$

for sufficiently large a. In general, the family $\sigma\big(\xi(a)\big) = [b_0, b_1, b_2, \ldots]$ has the leaping pattern $g(n)$, except possibly for the special cases when a is not sufficiently large, resulting in the occurrence of indices j such that $b_j \leq 0$. Letting $a = x$ be such a special case, we have two possibilities. There will be no natural leaping convergents whatsoever, or the leaping pattern will be transformed into one that is adverse or conformal. The behavior is described as follows.

When zeroes occur among the partial denominators of a continued fraction, we can treat strings of an odd number k, of consecutive zeroes using the simplification rule

$$[\ldots, a, b, \overbrace{0, \ldots, 0}^{k\text{-zeroes}}, c, d, \ldots]_{simplified} = [\ldots, a, b+c, d, \ldots].$$

We then obtain $\sigma\big(\xi(x)\big) = [b_0, b_1, b_2, \ldots]_{simplified} = [c_0, c_1, c_2, \ldots]$. Consequently, if we have $\sigma\big(\mathcal{C}_{f(n)}(\xi(x))\big) \cap \mathcal{C}_{g(n)}\big(\sigma\big(\xi(x)\big)\big) = \emptyset$, then for all $n \in \mathbb{N}$, it must be the case that $c_{h(n)} \neq b_{g(n)}$, where the indexing function $h(n)$ is determined from $g(n)$ after taking into consideration a collapsing of zeroes among the partial denominators in the continued fraction $\sigma(\xi(x)) = [b_0, b_1, b_2, \ldots]$. Thus we have

$$\sigma\left(\frac{p_{f(n)}}{q_{f(n)}}\right) = \frac{b_{g(n)}p'_{h(n)-1} + p'_{h(n)-2}}{b_{g(n)}q'_{h(n)-1} + q'_{h(n)-2}} \notin \mathcal{C}\big(\sigma(\xi(x))\big),$$

or equivalently,

$$\sigma\big(\mathcal{C}_{f(n)}(\xi(x))\big) = b_{g(n)}\mathcal{C}_{h(n)-1}\big(\sigma(\xi(x))\big) \oplus \mathcal{C}_{h(n)-2}\big(\sigma(\xi(x))\big) \notin \mathcal{C}\big(\sigma(\xi(x))\big),$$

for all $n \in \mathbb{N}$. This is demonstrated in the following example.

Example 6. Let $\xi(a) = [\,\overline{a}\,]$ and $\sigma = \left(\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right) \in \mathcal{L}$ so that for $a \equiv 0 \pmod 2$ we have

$$\sigma\big(\xi(a)\big) \;=\; \left[\,\overline{1, 1, \frac{a-2}{2}}\,\right] \;=\; \left[\,\overline{1, 1, k-1}\,\right] \;=\; \sigma\big(\xi(2k)\big)\,. \tag{57}$$

The statement in Equation (57) is easily seen by showing only that $\sigma\big(\xi(a)\big) = \left[\,\overline{1, 1, k-1}\,\right]$, as the remaining identities follow from $a = 2k$. We have

$$\sigma\big(\xi(a)\big) \;=\; \frac{2(k + \sqrt{k^2+1})}{k + \sqrt{k^2+1} + 1} \;=\; \frac{k - 1 + \sqrt{k^2+1}}{k}\,,$$

which is the positive root of the polynomial identity $kX^2 - 2(k-1)X - 2 = 0$, satisfying

$$X \;=\; 1 + \cfrac{1}{1 + \cfrac{1}{k - 1 + \cfrac{1}{X}}}\,.$$

It follows that $\sigma\big(\xi(a)\big) = \left[\,\overline{1, 1, k-1}\,\right]$, as desired. Now for sufficiently large $a \equiv 0$ (mod 2), we have

$$\sigma\left(\frac{p_n}{q_n}\right) \;=\; \frac{p'_{3n+4}}{q'_{3n+4}} \;=\; \frac{b_{3n+4}p'_{3n+3} + p'_{3n+2}}{b_{3n+4}q'_{3n+3} + q'_{3n+2}} \;\in\; \mathcal{C}\big(\sigma(\xi(a))\big)\,,$$

or equivalently,

$$\begin{aligned} \sigma\big(\mathcal{C}_n(\xi(a))\big) &= \mathcal{C}_{3n+4}\big(\sigma(\xi(a))\big) \\ &= b_{3n+4}\mathcal{C}_{3n+3}\big(\sigma(\xi(a))\big) \oplus \mathcal{C}_{3n+2}\big(\sigma(\xi(a))\big) \in \mathcal{C}\big(\sigma(\xi(a))\big)\,. \end{aligned}$$

Structurally the natural leaping pattern is $g(n) = 3n+4$, however by Equation (57) we see that when $a = 2$, an exception occurs, where we have

$$\sigma\left(\frac{p_n}{q_n}\right) \;=\; \frac{b_{3n+4}p'_{n+1} + p'_n}{b_{3n+4}q'_{n+1} + q'_n} \;\notin\; \mathcal{C}\big(\sigma(\xi(2))\big)\,,$$

or

$$\sigma\big(\mathcal{C}_n(\xi(2))\big) \;=\; b_{3n+4}\mathcal{C}_{n+1}\big(\sigma(\xi(2))\big) \oplus \mathcal{C}_n\big(\sigma(\xi(2))\big) \;\notin\; \mathcal{C}\big(\sigma(\xi(2))\big)\,.$$

From

$$\sigma\big(\xi(2)\big) \;=\; [\,\overline{1, 1, 0}\,] \;=\; [\,1, \overline{1, 0, 1}\,]_{simplified} \;=\; [1, \overline{2}\,]\,,$$

we see that $g(n)$ is essentially reduced by a factor of 3 through the collapsing of zeroes, and so in this case, after taking into consideration a preperiod of length 1, we have $h(n) = n + 2$. Because $b_{g(n)} = b_{3n+4} = 1 < 2 = c_{n+2} = c_{h(n)}$ for all $n \in \mathbb{N}$, we see that $\sigma\big(\mathcal{C}(\xi(2))\big)$ are minor convergents of $\sigma\big(\xi(2)\big)$.

Example 7. Consider σ and ξ from Example 6 and let $a \equiv 0 \pmod 2$ so that $a = 2k$. Then for all $n \geq 0$,

(i) we have the expressions

$$\begin{aligned}\frac{p'_{3n+4}}{q'_{3n+4}} &= \frac{4k(\varphi_1^{n+1} - \varphi_2^{n+1}) + 2(\varphi_1^n - \varphi_2^n)}{(2k+1)(\varphi_1^{n+1} - \varphi_2^{n+1}) + (\varphi_1^n - \varphi_2^n)} \\ &= \frac{2\varphi_1^{n+2} - 2\varphi_2^{n+2}}{\varphi_1^{n+1}(\varphi_1 + 1) - \varphi_2^{n+1}(\varphi_2 + 1)},\end{aligned}$$

(ii) for $k > 1$, the convergents p_n/q_n and p'_{3n+4}/q'_{3n+4} are σ-related,

(iii) for $k = 1$, we have the minor convergents, $\sigma\big(\mathcal{C}_n(\xi(2))\big)$, of the continued fraction $\sigma\big(\xi(2)\big)$ in closed form

$$\begin{aligned}&b_{3n+4}\mathcal{C}_{n+1}\big(\sigma(\xi(2))\big) \oplus \mathcal{C}_n\big(\sigma(\xi(2))\big) \\ &\quad= \frac{2(1+\sqrt{2})^{n+2} - 2(1-\sqrt{2})^{n+2}}{(2+\sqrt{2})(1+\sqrt{2})^{n+1} + (2-\sqrt{2})(1-\sqrt{2})^{n+1}}.\end{aligned}$$

Here we have $\varphi_1 = \psi_1 = k + \sqrt{k^2+1}$, and $\varphi_2 = \psi_2 = k - \sqrt{k^2+1}$.

Beginning with (i), we show that

$$\frac{p'_{3n+4}}{q'_{3n+4}} = \frac{4k(\varphi_1^{n+1} - \varphi_2^{n+1}) + 2(\varphi_1^n - \varphi_2^n)}{(2k+1)(\varphi_1^{n+1} - \varphi_2^{n+1}) + (\varphi_1^n - \varphi_2^n)}.$$

Using Proposition 1, we let $k \geq 2$ be an integer and

$$\sigma\big(\xi(2k)\big) = [\,\overline{1, 1, k-1}\,] = [1, \overline{1, k-1, 1}\,].$$

We then have the values

$$\rho = 0\,, \quad w = 3\,, \quad a_0 = 1 \quad T_1 = T_3 = 1\,, \quad T_2 = k-1\,.$$

The convergents of p'_n/q'_n for $n = 0, 1, 2, 3, 4$ and 7 are

$$\frac{p'_0}{q'_0} = \frac{1}{1}, \quad \frac{p'_1}{q'_1} = \frac{2}{1}, \quad \frac{p'_2}{q'_2} = \frac{2k-1}{k}, \quad \frac{p'_3}{q'_3} = \frac{2k+1}{k+1},$$

$$\frac{p'_4}{q'_4} = \frac{4k}{2k+1}, \qquad \text{and} \qquad \frac{p'_7}{q'_7} = \frac{8k^2+2}{4k^2+2k+1}.$$

Computing from Equations (36), (37) and (38) we have

$$\psi_1 \,=\, k + \sqrt{k^2+1} \qquad \text{and} \qquad \psi_2 \,=\, k - \sqrt{k^2+1}$$

where

$$U = \begin{vmatrix} -1 & -1 & 0 \\ 1 & (k-1) & -1 \\ 0 & 1 & -1 \end{vmatrix} + \begin{vmatrix} -(k-1) \end{vmatrix} = -2k\,.$$

Now from Equation (41) we have for all $m \geq 0$ and for $0 \leq i \leq 2$:

$$\frac{p'_{mw+i}}{q'_{mw+i}} = \frac{p'_{i+w}(\psi_1^m - \psi_2^m) + (-1)^{w-1} p'_i(\psi_1^{m-1} - \psi_2^{m-1})}{q'_{i+w}(\psi_1^m - \psi_2^m) + (-1)^{w-1} q'_i(\psi_1^{m-1} - \psi_2^{m-1})},$$

so that for $m = n+1$

$$\begin{aligned} \frac{p'_{3m+1}}{q'_{3m+1}} &= \frac{4k(\psi_1^m - \psi_2^m) + 2(\psi_1^{m-1} - \psi_2^{m-1})}{(2k+1)(\psi_1^m - \psi_2^m) + (\psi_1^{m-1} - \psi_2^{m-1})} \\ &= \frac{4k(\psi_1^{n+1} - \psi_2^{n+1}) + 2(\psi_1^n - \psi_2^n)}{(2k+1)(\psi_1^{n+1} - \psi_2^{n+1}) + (\psi_1^n - \psi_2^n)} = \frac{p'_{3n+4}}{q'_{3n+4}}. \end{aligned}$$

From the last line above and from $2k\varphi_i + 1 = \varphi_i^2$, we obtain

$$\begin{aligned} \frac{p'_{3n+4}}{q'_{3n+4}} &= \frac{2\varphi_1^n(2k\varphi_1 + 1) - 2\varphi_2^n(2k\varphi_2 + 1)}{\varphi_1^n\big((2k\varphi_1 + 1) + \varphi_1\big) - \varphi_2^n\big((2k\varphi_2 + 1) + \varphi_2\big)} \\ &= \frac{2\varphi_1^n\varphi_1^2 - 2\varphi_2^n\varphi_2^2}{\varphi_1^n(\varphi_1^2 + \varphi_1) - \varphi_2^n(\varphi_2^2 + \varphi_2)} \\ &= \frac{2(\varphi_1^{n+2} - \varphi_2^{n+2})}{\varphi_1^{n+1}(\varphi_1 + 1) - \varphi_2^{n+1}(\varphi_2 + 1)}, \end{aligned}$$

as claimed.

We are now ready to show that for all $n \geq 0$, the convergents p_n/q_n and p'_{3n+4}/q'_{3n+4} are σ-related. Let $S(n)$ be the statement that for all $n \geq 0$

$$\sigma\left(\frac{p_n}{q_n}\right) = \frac{p'_{3n+4}}{q'_{3n+4}}.$$

$S(0)$ is the statement that

$$\sigma\left(\frac{p_0}{q_0}\right) = \sigma\left(\frac{2k}{1}\right) = \frac{2(2k) + 0}{1(2k) + 1} = \frac{4k}{2k+1} = \frac{p'_4}{q'_4},$$

which is true. Similarly, $S(1)$ is the statement that

$$\sigma\left(\frac{p_1}{q_1}\right) = \sigma\left(\frac{4k^2+1}{2k}\right) = \frac{\dfrac{2(4k^2+1)}{2k} + 0}{\dfrac{1(4k^2+1)}{2k} + 1} = \frac{8k^2+2}{4k^2+2k+1} = \frac{p'_7}{q'_7},$$

which is also true.

Now suppose $S(n-1)$ holds, so that for all $n \geq 1$ we have

$$\sigma\left(\frac{p_{n-1}}{q_{n-1}}\right) = \frac{p'_{3n+1}}{q'_{3n+1}}.$$

For brevity, we will write $X_i = \varphi_1^{n+i} - \varphi_2^{n+i}$ so that $p_n/q_n = X_2/X_1$, and similarly $p_{n-1}/q_{n-1} = X_1/X_0$. Adding

$$\frac{2X_0X_2 - 2X_1^2}{(X_2+X_1)(X_1+X_0)}$$

to both sides, we obtain

$$\sigma\left(\frac{p_{n-1}}{q_{n-1}}\right) + \frac{2X_0X_2 - 2X_1^2}{(X_2+X_1)(X_1+X_0)} = \frac{p'_{3n+1}}{q'_{3n+1}} + \frac{2X_0X_2 - 2X_1^2}{(X_2+X_1)(X_1+X_0)}.$$

We first show that the left-hand side is equal to $\sigma(p_n/q_n)$ and following, that the right-hand side is equal to p'_{3n+4}/q'_{3n+4}. For $\sigma(p_n/q_n)$, we obtain

$$\sigma\left(\frac{p_n}{q_n}\right) = \frac{2X_2}{X_2+X_1} = \frac{2X_1}{X_1+X_0} + \frac{2X_2}{X_2+X_1} - \frac{2X_1}{X_1+X_0} \tag{58}$$

$$= \sigma\left(\frac{X_1}{X_0}\right) + \frac{2X_2}{X_2+X_1} - \frac{2X_1}{X_1+X_0}$$

$$= \sigma\left(\frac{p_{n-1}}{q_{n-1}}\right) + \frac{2X_0X_2 - 2X_1^2}{(X_2+X_1)(X_1+X_0)}. \tag{59}$$

Next, we have from (i) that

$$\begin{aligned}
\frac{p'_{3n+4}}{q'_{3n+4}} &= \frac{2(\varphi_1^{n+2} - \varphi_2^{n+2})}{\varphi_1^{n+1}(\varphi_1+1) - \varphi_2^{n+1}(\varphi_2+1)} \\
&= \frac{2(\varphi_1^{n+2} - \varphi_2^{n+2})}{\varphi_1^{n+2} - \varphi_2^{n+2} + \varphi_1^{n+1} - \varphi_2^{n+1}} \\
&= \frac{2X_2}{X_2+X_1} \\
&\overset{Eq.(58),(59)}{=} \frac{p'_{3n+1}}{q'_{3n+1}} + \frac{2X_0X_2 - 2X_1^2}{(X_2+X_1)(X_1+X_0)}.
\end{aligned}$$

This completes the inductive step, thus showing that for all $n \geq 0$, $S(n-1)$ implies $S(n)$. Therefore, $S(n)$ holds for all $n \geq 0$.

We claim in Example 6 that

$$\sigma\big(\mathcal{C}_n(\xi(2))\big) = b_{3n+4}\mathcal{C}_{n+1}\big(\sigma(\xi(2))\big) \oplus \mathcal{C}_n\big(\sigma(\xi(2))\big) \notin \mathcal{C}\big(\sigma(\xi(2))\big),$$

and indeed since $\xi(2k)$ is σ-related for $k \geq 1$ we have

$$\sigma\left(\frac{p_n}{q_n}\right) = \frac{p'_{3n+4}}{q'_{3n+4}} = \frac{b_{3n+4}p'_{3n+3} + p'_{3n+2}}{b_{3n+4}q'_{3n+3} + q'_{3n+2}}.$$

However, when $k = 1$ the indices for the convergents according to the natural leaping pattern $g(n)$ are transformed to $h(n) = n + 2$. Thus from Example 5 we have for $\sigma\big(\xi(2)\big)$ the closed form formulae:

$$\frac{p'_n}{q'_n} = \frac{(\psi_1+1)\psi_1^n - (\psi_2+1)\psi_2^n}{\psi_1^{n+1} - \psi_2^{n+1}},$$

and similarly

$$\frac{p'_{n+1}}{q'_{n+1}} = \frac{(\psi_1+1)\psi_1^{n+1} - (\psi_2+1)\psi_2^{n+1}}{\psi_1^{n+2} - \psi_2^{n+2}}. \tag{60}$$

We multiply the numerator and denominator of p'_{n+1}/q'_{n+1} by $b_{g(n)}$, where $g(n) = 3n + 4$ is the natural leaping pattern of the σ-related convergents of $\sigma\big(\xi(x)\big)$, for $x > 2$. Note that $b_{g(n)} = 1$. We then take the mediant to obtain

$$\begin{aligned}
\frac{b_{3n+4}p'_{n+1} + p'_n}{b_{3n+4}q'_{n+1} + q'_n} &= \frac{(\psi_1+1)\psi_1^{n+1} - (\psi_2+1)\psi_2^{n+1} + (\psi_1+1)\psi_1^n - (\psi_2+1)\psi_2^n}{\psi_1^{n+2} - \psi_2^{n+2} + \psi_1^{n+1} - \psi_2^{n+1}} \\
&= \frac{(\psi_1+1)(\psi_1+1)\psi_1^n - (\psi_2+1)(\psi_2+1)\psi_2^n}{\psi_1^{n+2} - \psi_2^{n+2} + \psi_1^{n+1} - \psi_2^{n+1}} \\
&= \frac{2\psi_1^{n+2} - 2\psi_2^{n+2}}{\psi_1^{n+2} - \psi_2^{n+2} + \psi_1^{n+1} - \psi_2^{n+1}} \\
&= \sigma\left(\frac{p_n}{q_n}\right).
\end{aligned}$$

Observing that $b_{g(n)} = 1 < 2 = c_{h(n)}$, the above shows that

$$\sigma\big(\mathcal{C}_n(\xi(2))\big) = b_{3n+4}\mathcal{C}_{n+1}\big(\sigma(\xi(2))\big) \oplus \mathcal{C}_n\big(\sigma(\xi(2))\big) \notin \mathcal{C}\big(\sigma(\xi(2))\big).$$

Finally, by Equation (60), we have

$$\begin{aligned}
&b_{3n+4}\mathcal{C}_{n+1}\big(\sigma(\xi(2))\big) \oplus \mathcal{C}_n\big(\sigma(\xi(2))\big) \\
&\quad = \frac{2\psi_1^{n+2} - 2\psi_2^{n+2}}{(\psi_1+1)\psi_1^{n+1} - (\psi_2+1)\psi_2^{n+1}} \\
&\quad = \frac{2(1+\sqrt{2})^{n+2} - 2(1-\sqrt{2})^{n+2}}{(2+\sqrt{2})(1+\sqrt{2})^{n+1} + (2-\sqrt{2})(1-\sqrt{2})^{n+1}}.
\end{aligned}$$

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