A NEW CLASS OF ORDINARY GENERATING FUNCTIONS FOR BINARY PRODUCTS OF MERSENNE NUMBERS AND GAUSSIAN NUMBERS WITH PARAMETERS p AND q

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Abstract

In this paper, we derive some new generating functions for the products of several special numbers including (p, q)-Fibonacci numbers, (p, q)-modified Pell numbers, and (p, q)-Jacobsthal Lucas numbers. We also give some new generating functions for the products of Mersenne and Gaussian numbers with parameters p and q.

1. Introduction

Asci and Gurel [1] introduced the concept of the complex Jacobsthal and Jacobsthal Lucas numbers as the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers, respectively. These Gaussian numbers are constructed with the same second-order linear recurrence relation but different initial values. Namely, the Gaussian Jacobsthal numbers $\{GJ_n\}_{n\in\mathbb{N}}$ satisfy the recurrence relation:

$$GJ_0 = \frac{i}{2}, GJ_1 = 1 \text{ and } GJ_n = pGJ_{n-1} + 2qGJ_{n-2}, \text{ for } n \geq 2,$$

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whereas Gaussian Jacobsthal Lucas numbers $\{Gj_n\}_{n\in\mathbb{N}}$ satisfy the recurrence relation

$$Gj_0 = 2 - \frac{i}{2},\ Gj_1 = 1 + 2i \text{ and } Gj_n = pGj_{n-1} + 2qGj_{n-2},\ \text{for } n \geq 2.$$

In the literature, there has been much research involving the study of generalizations of Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers. For example, the authors in [2], study the Gaussian (p, q)-Jacobsthal numbers $\{GJ_{p,q,n}\}_{n\in\mathbb{N}}$ and Gaussian (p, q)-Jacobsthal Lucas numbers $\{Gj_{p,q,n}\}_{n\in\mathbb{N}}$, respectively. Let p, q be any real number with $q \neq 0$ and $p > 0$. Let p and q be any real number with $q \neq 0$ and $p > 0$. The recurrence relations of $\{GJ_{p,q,n}\}_{n\in\mathbb{N}}$ and $\{Gj_{p,q,n}\}_{n\in\mathbb{N}}$ are respectively given by

$$GJ_{p,q,n} := \begin{cases} \frac{i}{2},\ \text{if } n = 0 \\ 1,\ \text{if } n = 1 \\ pGJ_{p,q,n-1} + 2qGJ_{p,q,n-2},\ \text{if } n \geq 2, \end{cases}$$

$$Gj_{p,q,n} := \begin{cases} 2 - \frac{ip}{2},\ \ \text{if } n = 0 \\ p + 2iq,\ \text{if } n = 1 \\ pGj_{p,q,n-1} + 2qGj_{p,q,n-2},\ \text{if } n \geq 2. \end{cases}$$

In [3], Karaaslan and Yagmur introduced the Gaussian (p, q)-Pell and Gaussian (p, q)-Pell Lucas numbers by means of the following recurrence relations, respectively. Let n be given as a positive number greater or equal to 2. Then

$$\begin{gathered} GP_{p,q,n} = 2pGP_{p,q,n-1} + qGP_{p,q,n-2} \\ (GP_{p,q,0} = i, GP_{p,q,1} = 1) \end{gathered} \tag{1}$$

and

$$\begin{gathered} GQ_{p,q,n} = 2pGQ_{p,q,n-1} + qGQ_{p,q,n-2} \\ (GQ_{p,q,0} = 2 - 2ip, GQ_{p,q,1} = 2p + 2iq)\,. \end{gathered} \tag{2}$$

It is worthwhile to note that

$$GP_{p,q,n} = P_{p,q,n} + iqP_{p,q,n-1} \text{ and } GQ_{p,q,n} = Q_{p,q,n} + iqQ_{p,q,n-1},$$

where $P_{p,q,n}$ and $Q_{p,q,n}$ are the (p, q)-Pell and (p, q)-Pell Lucas numbers defined in [4].

Remark 1. If $p = q = 1$ in Equations (1) and (2), then we get the recurrence relations of Gaussian Pell numbers and Gaussian Pell Lucas numbers, respectively (see [5]).

The (p, q)-modified Pell numbers are defined by the recurrence relation

$$MP_{p,q,n} = 2pMP_{p,q,n-1} + qMP_{p,q,n-2}, \ \ (n \geq 2)\,,$$

with the initial values $MP_{p,q,0} = 1$ and $MP_{p,q,1} = p$ (see [6, 7]).

The Binet formulas for Gaussian (p, q)-Pell numbers, Gaussian (p, q)-Pell-Lucas numbers and (p, q)-modified Pell numbers are given by (see [3, 6])

$$GP_{p,q,n} = \frac{x_1^n - x_2^n}{x_1 - x_2} + i\frac{x_1x_2^n - x_2x_1^n}{x_1 - x_2},$$

$$GQ_{p,q,n} = \left(x_{_1}^n + x_{_2}^n\right) - i\left(x_1x_2^n + x_2x_1^n\right),$$

and

$$MP_{p,q,n} = p\left(\frac{x_1^n + x_2^n}{x_1 + x_2}\right),$$

respectively, where $x_1 = p + \sqrt{p^2 + q}$ and $x_2 = p - \sqrt{p^2 + q}$ are roots of the characteristic equation $x^2 - 2px - q = 0$.

The Gaussian (p, q)-Fibonacci numbers and Gaussian(p, q)-Lucas numbers, denoted respectively by $\{GF_{p,q,n}\}_{n\in\mathbb{N}}$ and $\{GL_{p,q,n}\}_{n\in\mathbb{N}}$, are defined by

$$\begin{cases} GF_{p,q,0} = i, \ GF_{p,q,1} = 1 \\ GF_{p,q,n} = pGF_{p,q,n-1} + qGF_{p,q,n-2} \ \ (n \geq 2)\,, \end{cases} \tag{3}$$

and

$$\begin{cases} GL_{p,q,0} = 2 - ip, \ GL_{p,q,1} = p + 2iq \\ GL_{p,q,n} = pGL_{p,q,n-1} + qGL_{p,q,n-2} \ \ (n \geq 2)\,, \end{cases} \tag{4}$$

respectively, (see [8]).

Remark 2. If $p = q = 1$ in Equations (3) and (4), then we get the recurrence relations of Gaussian Fibonacci numbers and Gaussian Lucas numbers, respectively, (see [9]).

The Mersenne numbers, denoted by M_n, are numbers of the form $M_n = 2^n - 1$, where n is nonnegative number. The Mersenne sequence $\{M_n\}_{n\in\mathbb{N}}$ can be defined recursively as follows:

$$\begin{cases} M_0 = 0, \ M_1 = 1 \\ M_n = 3M_{n-1} - 2M_{n-2} \ \ (n \geq 2)\,. \end{cases}$$

The explicit formula of Mersenne numbers is given by (see [10])

$$M_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n-k-1}{k} 3^{n-2k-1}2^k,$$

and the negative index of them is given by

$$M_{-n} = \frac{-M_n}{2^n}. \tag{5}$$

For more details about this sequence, one can look at [11, 12].

We now give definitions for some special numbers with parameters p and q.

Definition 1 ([13]). *The (p,q)-Fibonacci numbers, denoted by $\{F_{p,q,n}\}_{n\in\mathbb{N}}$, are defined for $n \in \mathbb{N}$ by*

$$F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2} \qquad (n \geq 2)\,,$$

with initial conditions $F_{p,q,0} = 0$ and $F_{p,q,1} = 1$.

Definition 2 ([14]). *The (p,q)-Lucas numbers, denoted by $\{L_{p,q,n}\}_{n\in\mathbb{N}}$, are defined for $n \in \mathbb{N}$ by*

$$L_{p,q,n} = pL_{p,q,n-1} + qL_{p,q,n-2} \qquad (n \geq 2)\,,$$

with the initial conditions $L_{p,q,0} = 2$ and $L_{p,q,1} = p$.

Definition 3 ([15]). *The (p,q)-Jacobsthal numbers, denoted by $\{J_{p,q,n}\}_{n\in\mathbb{N}}$, are defined for $n \in \mathbb{N}$ by*

$$J_{p,q,n} = pJ_{p,q,n-1} + 2qJ_{p,q,n-2} \qquad (n \geq 2)\,,$$

with the initial conditions $J_{p,q,0} = 0$ and $J_{p,q,1} = 1$.

Definition 4 ([15]). *The (p,q)-Jacobsthal Lucas numbers, denoted by $\{j_{p,q,n}\}_{n\in\mathbb{N}}$, are defined for $n \in \mathbb{N}$ by*

$$j_{p,q,n} = pj_{p,q,n-1} + 2qj_{p,q,n-2} \qquad (n \geq 2)\,,$$

with the initial conditions $j_{p,q,0} = 2$ and $j_{p,q,1} = p$.

Definition 5 ([4]). *The (p,q)-Pell numbers, denoted by $\{P_{p,q,n}\}_{n\in\mathbb{N}}$, are defined for $n \in \mathbb{N}$ by*

$$P_{p,q,n} = 2pP_{p,q,n-1} + qP_{p,q,n-2} \qquad (n \geq 2)\,,$$

with the initial conditions $P_{p,q,0} = 0$ and $P_{p,q,1} = 1$.

Definition 6 ([4]). *The (p,q)-Pell Lucas numbers, denoted by $\{Q_{p,q,n}\}_{n\in\mathbb{N}}$, are defined for $n \in \mathbb{N}$ by*

$$Q_{p,q,n} = 2pQ_{p,q,n-1} + qQ_{p,q,n-2} \qquad (n \geq 2)\,,$$

with the initial conditions $Q_{p,q,0} = 2$ and $Q_{p,q,1} = 2p$.

We are now in a position to state the following definitions which will be useful for the main results of this paper.

Definition 7 ([16])**.** *Let A and B be any two alphabets. Then $S_n(A-B)$ is defined by the form*

$$\frac{\prod_{b\in B}(1-bz)}{\prod_{a\in A}(1-az)} = \sum_{n=0}^{\infty} S_n(A-B)z^n, \tag{6}$$

with the condition $S_n(A-B)=0$ for $n<0$.

Remark 3. Taking $A=\{0\}$ in (6) gives

$$\prod_{b\in B}(1-bz) = \sum_{n=0}^{\infty} S_n(-B)z^n.$$

Definition 8 ([17])**.** *Let n be a positive integer and $A=\{a_1,a_2\}$ an alphabet. Then, the n^{th} symmetric function $S_n(a_1+a_2)$ is defined by*

$$S_n(A) = S_n(a_1+a_2) = \frac{a_1^{n+1}-a_2^{n+1}}{a_1-a_2}.$$

It immediately follows from Definition 8 that

$$\begin{aligned}
S_0(A) &= S_0(a_1+a_2) = 1,\\
S_1(A) &= S_1(a_1+a_2) = a_1+a_2,\\
S_2(A) &= S_2(a_1+a_2) = a_1^2+a_1a_2+a_2^2.
\end{aligned}$$

Lemma 1 ([18, 19])**.** *We have, for $n\in\mathbb{N}$:*

$$\begin{aligned}
F_{p,q,n} &= S_{n-1}\left(a_1+[-a_2]\right) \ \textit{and}\ L_{p,q,n} = 2S_n\left(a_1+[-a_2]\right) - pS_{n-1}\left(a_1+[-a_2]\right),\\
\textit{with}\ \ a_{1,2} &= \frac{p\pm\sqrt{p^2+4q}}{2},\\
P_{p,q,n} &= S_{n-1}\left(a_1+[-a_2]\right) \ \textit{and}\ Q_{p,q,n} = 2S_n\left(a_1+[-a_2]\right) - 2pS_{n-1}\left(a_1+[-a_2]\right),\\
\textit{with}\ \ a_{1,2} &= p\pm\sqrt{p^2+q},\\
MP_{p,q,n} &= S_n(a_1+[-a_2]) - pS_{n-1}(a_1+[-a_2]),\\
\textit{with}\ \ a_{1,2} &= p\pm\sqrt{p^2+q},\\
J_{p,q,n} &= S_{n-1}\left(a_1+[-a_2]\right) \ \textit{and}\ j_{p,q,n} = 2S_n\left(a_1+[-a_2]\right) - pS_{n-1}\left(a_1+[-a_2]\right),\\
\textit{with}\ a_{1,2} &= \frac{p\pm\sqrt{p^2+8q}}{2},\\
M_n &= S_{n-1}\left(a_1+[-a_2]\right),\\
\textit{with}\ \ a_1 &= 2\ \textit{and}\ a_2 = 1.
\end{aligned}$$

Lemma 2 ([2, 8]). *We have, for $n \in \mathbb{N}$:*

$$\begin{aligned}
GF_{p,q,n} &= iS_n(a_1+[-a_2]) + (1-ip)\,S_{n-1}(a_1+[-a_2]),\\
\textit{with } a_{1,2} &= \frac{p \pm \sqrt{p^2+4q}}{2},\\
GL_{p,q,n} &= (2-ip)\,S_n(a_1+[-a_2]) + \left(i\left(p^2+2q\right)-p\right)S_{n-1}(a_1+[-a_2]),\\
\textit{with } a_{1,2} &= \frac{p \pm \sqrt{p^2+4q}}{2},\\
GP_{p,q,n} &= iS_n(a_1+[-a_2]) + (1-2ip)\,S_{n-1}(a_1+[-a_2]),\\
\textit{with } a_{1,2} &= p \pm \sqrt{p^2+q},\\
GQ_{p,q,n} &= (2-2ip)\,S_n(a_1+[-a_2]) + \left(i\left(4p^2+2q\right)-2p\right)S_{n-1}(a_1+[-a_2]),\\
\textit{with } a_{1,2} &= p \pm \sqrt{p^2+q},\\
GJ_{p,q,n} &= \frac{i}{2}S_n(a_1+[-a_2]) + \left(1-\frac{ip}{2}\right)S_{n-1}(a_1+[-a_2]),\\
\textit{with } a_{1,2} &= \frac{p \pm \sqrt{p^2+8q}}{2},\\
Gj_{p,q,n} &= \left(2-\frac{ip}{2}\right)S_n(a_1+[-a_2]) + \left(i\left(\frac{p^2}{2}+2q\right)-p\right)S_{n-1}(a_1+[-a_2]),\\
\textit{with } a_{1,2} &= \frac{p \pm \sqrt{p^2+8q}}{2}.
\end{aligned}$$

2. Generating Functions for the Squares of (p,q)-Numbers and Gaussian (p,q)-Numbers

The following two theorems are well-known from [21]. So we give them without proof.

Theorem 1. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, we have*

$$\sum_{n=0}^{\infty} S_n(A)S_{n-1}(E)z^n = \frac{(a_1+a_2)z - a_1a_2(e_1+e_2)z^2}{\left(\sum_{n=0}^{\infty} S_n(-A)e_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)e_2^n z^n\right)}. \tag{7}$$

Theorem 2. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, we have*

$$\sum_{n=0}^{\infty} S_n(A)S_n(E)z^n = \frac{1 - a_1a_2e_1e_2z^2}{\left(\sum_{n=0}^{\infty} S_n(-A)e_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)e_2^n z^n\right)}. \tag{8}$$

The following Proposition is a special case of Theorem 2.

Proposition 1. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, we have*

$$\sum_{n=0}^{\infty} S_{n-1}(A) S_{n-1}(E) z^n = \frac{z - a_1 a_2 e_1 e_2 z^3}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)}. \quad (9)$$

By making use of Theorems 1 and 2 and Proposition 1, we give some special cases which will be used to obtain the generating functions for the products of several special numbers. We consider the following sets,

$$A = \{a_1, -a_2\} \text{ and } E = \{e_1, -e_2\}.$$

By changing a_2 to $(-a_2)$ and e_2 to $(-e_2)$ in Equations (7)-(9), we find that

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n = \frac{(a_1 - a_2)z + a_1 a_2 (e_1 - e_2) z^2}{(1 - a_1 e_1 z)(1 + a_2 e_1 z)(1 + a_1 e_2 z)(1 - a_2 e_2 z)}, \quad (10)$$

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n = \frac{1 - a_1 a_2 e_1 e_2 z^2}{(1 - a_1 e_1 z)(1 + a_2 e_1 z)(1 + a_1 e_2 z)(1 - a_2 e_2 z)}, \quad (11)$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n = \frac{z - a_1 a_2 e_1 e_2 z^3}{(1 - a_1 e_1 z)(1 + a_2 e_1 z)(1 + a_1 e_2 z)(1 - a_2 e_2 z)}. \quad (12)$$

Firstly, let us now consider the following conditions for Equations (10)-(12):

$$\begin{cases} a_1 - a_2 = p \\ a_1 a_2 = q, \end{cases} \text{ and } \begin{cases} e_1 - e_2 = p \\ e_1 e_2 = q. \end{cases}$$

The conditions above would result in the following equations:

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n = \frac{pz + pqz^2}{1 - p^2 z - 2q(p^2 + q) z^2 - p^2 q^2 z^3 + q^4 z^4}, \quad (13)$$

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n = \frac{1 - q^2 z^2}{1 - p^2 z - 2q(p^2 + q) z^2 - p^2 q^2 z^3 + q^4 z^4}, \quad (14)$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n = \frac{z - q^2 z^3}{1 - p^2 z - 2q(p^2 + q) z^2 - p^2 q^2 z^3 + q^4 z^4}. \quad (15)$$

respectively.
Therefore we state the following theorems.

Theorem 3. *The following are generating functions for the squares of (p, q)-Lucas and (p, q)-Fibonacci numbers, respectively:*

$$\sum_{n=0}^{\infty} L_{p,q,n}^2 z^n = \frac{4 - 3p^2 z - 4q\left(p^2 + q\right) z^2 - p^2 q^2 z^3}{1 - p^2 z - 2q(p^2 + q) z^2 - p^2 q^2 z^3 + q^4 z^4}, \quad (16)$$

$$\sum_{n=0}^{\infty} F_{p,q,n}^2 z^n = \frac{z - q^2 z^3}{1 - p^2 z - 2q(p^2 + q) z^2 - p^2 q^2 z^3 + q^4 z^4}, \quad (17)$$

with $n \in \mathbb{N}$.

Proof. It is well-known that

$$L_{p,q,n} = 2S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2]).$$

Then, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{p,q,n}^2 z^n &= \sum_{n=0}^{\infty} \begin{pmatrix} (2S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2])) \\ \times(2S_n(e_1 + [-e_2]) - pS_{n-1}(e_1 + [-e_2])) \end{pmatrix} z^n \\
&= 4\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n \\
&\quad -2p\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\
&\quad -2p\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n \\
&\quad +p^2\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n.
\end{aligned}$$

By Equations (13), (14), and (15), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{p,q,n}^2 z^n &= \frac{4\left(1 - q^2z^2\right)}{1 - p^2z - 2q(p^2 + q)z^2 - p^2q^2z^3 + q^4z^4} \\
&\quad - \frac{4p\left(pz + pqz^2\right)}{1 - p^2z - 2q(p^2 + q)z^2 - p^2q^2z^3 + q^4z^4} \\
&\quad + \frac{p^2\left(z - q^2z^3\right)}{1 - p^2z - 2q(p^2 + q)z^2 - p^2q^2z^3 + q^4z^4} \\
&= \frac{4 - 3p^2z - 4q\left(p^2 + q\right)z^2 - p^2q^2z^3}{1 - p^2z - 2q(p^2 + q)z^2 - p^2q^2z^3 + q^4z^4},
\end{aligned}$$

which is the first equation. The second equation can be proved similarly. □

Theorem 4. *The following are generating functions for the squares of Gaussian (p, q)-Lucas and Gaussian (p, q)-Fibonacci numbers, respectively:*

$$\begin{aligned}
\sum_{n=0}^{\infty} GL_{p,q,n}^2 z^n = &\frac{4 - p^2 - 4ip + \left(p^4 - 3p^2 - 4q^2 + 4ip\left(p^2 + q\right)\right)z}{1 - p^2z - 2q(p^2 + q)z^2 - p^2q^2z^3 + q^4z^4} \\
&+ \frac{q\left(2p^4 + 5p^2q - 4p^2 - 4q + 6ip\left(p^2 + 2q\right)\right)z^2}{1 - p^2z - 2q(p^2 + q)z^2 - p^2q^2z^3 + q^4z^4} \qquad (18) \\
&+ \frac{q^2\left(p^4 + 4p^2q - p^2 + 4q^2 + 2ip\left(p^2 + 2q\right)\right)z^3}{1 - p^2z - 2q(p^2 + q)z^2 - p^2q^2z^3 + q^4z^4},
\end{aligned}$$

$$\sum_{n=0}^{\infty} GF_{p,q,n}^2 z^n = \frac{-1+\left(p^2+1\right)z+q\left(2p^2+q+2ip\right)z^2+q^2\left(p^2-1+2ip\right)z^3}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}, \tag{19}$$

with $n \in \mathbb{N}$,

Proof. Recall that

$$GL_{p,q,n} = (2-ip)\,S_n(a_1+[-a_2]) + \left(i\left(p^2+2q\right)-p\right)S_{n-1}(a_1+[-a_2]).$$

We see that

$$\begin{aligned}
\sum_{n=0}^{\infty} GL_{p,q,n}^2 z^n &= \sum_{n=0}^{\infty}\left(\begin{array}{c}\left((2-ip)\,S_n(a_1+[-a_2])+\left(i\left(2q+p^2\right)-p\right)\right.\\ \left.S_{n-1}(a_1+[-a_2])\right)\left((2-ip)\,S_n(e_1+[-e_2])\right.\\ \left.+\left(i\left(2q+p^2\right)-p\right)S_{n-1}(e_1+[-e_2])\right)\end{array}\right)z^n \\
&= (2-ip)^2\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&\quad + (2-ip)\left(i\left(2q+p^2\right)-p\right)\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&\quad + (2-ip)\left(i\left(2q+p^2\right)-p\right)\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&\quad + \left(i\left(2q+p^2\right)-p\right)^2\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n.
\end{aligned}$$

Using the relationships $(13)\,,(14)\,,$ and (15), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} GL_{p,q,n}^2 z^n \quad &= \quad \frac{(2-ip)^2\left(1-q^2z^2\right)}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4} \\
&\quad + \frac{2\,(2-ip)\left(i\left(2q+p^2\right)-p\right)\left(pz+pqz^2\right)}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4} \\
&\quad + \frac{\left(i\left(2q+p^2\right)-p\right)^2\left(z-q^2z^3\right)}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4} \\
&= \quad \frac{4-p^2-4ip+\left(p^4-3p^2-4q^2+4ip\left(p^2+q\right)\right)z}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4} \\
&\quad + \frac{q\left(2p^4+5p^2q-4p^2-4q+6ip\left(p^2+2q\right)\right)z^2}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4} \\
&\quad + \frac{q^2\left(p^4+4p^2q-p^2+4q^2+2ip\left(p^2+2q\right)\right)z^3}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4},
\end{aligned}$$

which gives Equation (18). Using the same procedure, we can obtain Equation (19). □

Remark 4. Setting $p=q=1$ in the Equations (16)-(19) yields the generating functions for the squares of Lucas, Fibonacci, Gaussian Lucas, and Gaussian Fibonacci numbers, respectively.

Secondly, let us now consider the following conditions for Equations (10)-(12):

$$\begin{cases} a_1 - a_2 = p \\ a_1 a_2 = 2q, \end{cases} \text{ and } \begin{cases} e_1 - e_2 = p \\ e_1 e_2 = 2q. \end{cases}$$

The condition above would result in the following equations:

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{pz+2pqz^2}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}, \quad (20)$$

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{1-4q^2z^2}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}, \quad (21)$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{z-4q^2z^3}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}, \quad (22)$$

respectively.
Therefore, we state the following results.

Proposition 2. *The following are generating functions for the squares of (p,q)-Jacobsthal Lucas and (p,q)-Jacobsthal numbers, respectively:*

$$\sum_{n=0}^{\infty} j_{p,q,n}^2 z^n = \frac{4-3p^2z-8q\left(p^2+2q\right)z^2-4p^2q^2z^3}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}, \quad (23)$$

$$\sum_{n=0}^{\infty} J_{p,q,n}^2 z^n = \frac{z-4q^2z^3}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}, \quad (24)$$

with $n \in \mathbb{N}$.

Proof. We have

$$j_{p,q,n} = 2S_n(a_1+[-a_2]) - pS_{n-1}(a_1+[-a_2]).$$

Then, we can see that

$$\begin{aligned} \sum_{n=0}^{\infty} j_{p,q,n}^2 z^n &= \sum_{n=0}^{\infty} \begin{pmatrix} (2S_n(a_1+[-a_2]) - pS_{n-1}(a_1+[-a_2])) \\ \times(2S_n(e_1+[-e_2]) - pS_{n-1}(e_1+[-e_2]) \end{pmatrix} z^n \\ &= 4\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\ &\quad -2p\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \end{aligned}$$

$$-2p\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n$$
$$+p^2\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n.$$

According to the relationships (20), (21), and (22), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}j_{p,q,n}^2z^n &= \frac{4\left(1-4q^2z^2\right)}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}\\ &\quad -\frac{4p\left(pz+2pqz^2\right)}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}\\ &\quad +\frac{p^2\left(z-4q^2z^3\right)}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}\\ &= \frac{4-3p^2z-8q\left(p^2+2q\right)z^2-4p^2q^2z^3}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4},\end{aligned}$$

which is the first equation. The second equation can be proved similarly. □

Remark 5. Putting $p=q=1$ in the Equations (23) and (24) yields the generating functions for the squares of Jacobsthal Lucas and Jacobsthal numbers, respectively.

Thirdly, let us consider the following conditions for Equations (10)-(12):

$$\begin{cases} a_1-a_2=2p \\ a_1a_2=q, \end{cases} \quad \text{and} \quad \begin{cases} e_1-e_2=2p \\ e_1e_2=q. \end{cases}$$

The conditions above would result in the following equations:

$$\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{2pz+2pqz^2}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4}, \quad (25)$$

$$\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{1-q^2z^2}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4}, \quad (26)$$

$$\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{z-q^2z^3}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4}. \quad (27)$$

respectively.
Therefore, we state the following theorems.

Theorem 5. *The following are generating functions for the squares of (p,q)-Pell Lucas, (p,q)-Pell and (p,q)-modified Pell numbers, respectively:*

$$\sum_{n=0}^{\infty}Q_{p,q,n}^2z^n = \frac{4-12p^2z-4q\left(4p^2+q\right)z^2-4p^2q^2z^3}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4}, \quad (28)$$

$$\sum_{n=0}^{\infty} P_{p,q,n}^2 z^n = \frac{z - q^2 z^3}{1 - 4p^2 z - 2q(4p^2+q)z^2 - 4p^2q^2z^3 + q^4z^4}, \tag{29}$$

$$\sum_{n=0}^{\infty} MP_{p,q,n}^2 z^n = \frac{1 - 3p^2 z - q\left(4p^2+q\right)z^2 - p^2q^2z^3}{1 - 4p^2 z - 2q(4p^2+q)z^2 - 4p^2q^2z^3 + q^4z^4}, \tag{30}$$

with $n \in \mathbb{N}$.

Proof. We have

$$Q_{p,q,n} = 2S_n(a_1 + [-a_2]) - 2pS_{n-1}(a_1 + [-a_2]).$$

We see that

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{p,q,n}^2 z^n &= \sum_{n=0}^{\infty} \begin{pmatrix} (2S_n(a_1+[-a_2]) - 2pS_{n-1}(a_1+[-a_2])) \\ \times(2S_n(e_1+[-e_2]) - 2pS_{n-1}(e_1+[-e_2])) \end{pmatrix} z^n \\
&= 4\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&\quad -4p\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&\quad -4p\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&\quad +4p^2\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&= \frac{4\left(1-q^2z^2\right)}{1 - 4p^2 z - 2q(4p^2+q)z^2 - 4p^2q^2z^3 + q^4z^4} \\
&\quad -\frac{8p\left(2pz + 2pqz^2\right)}{1 - 4p^2 z - 2q(4p^2+q)z^2 - 4p^2q^2z^3 + q^4z^4} \\
&\quad +\frac{4p^2\left(z - q^2z^3\right)}{1 - 4p^2 z - 2q(4p^2+q)z^2 - 4p^2q^2z^3 + q^4z^4},
\end{aligned}$$

and after necessary calculations, we get

$$\sum_{n=0}^{\infty} Q_{p,q,n}^2 z^n = \frac{4 - 12p^2 z - 4q\left(4p^2+q\right)z^2 - 4p^2q^2z^3}{1 - 4p^2 z - 2q(4p^2+q)z^2 - 4p^2q^2z^3 + q^4z^4},$$

which is the first equation. The other equations can be proved similarly. □

Theorem 6. *The following are generating functions for the squares of Gaussian (p,q)-Pell Lucas and Gaussian (p,q)-Pell numbers, respectively:*

$$\sum_{n=0}^{\infty} GQ_{p,q,n}^2 z^n = \frac{4-4p^2-8ip+4\left(4p^4-3p^2-q^2+2ip\left(4p^2+q\right)\right)z}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4} + \frac{4q\left(8p^4+5p^2q-4p^2-q+6ip\left(2p^2+q\right)\right)z^2}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4} + \frac{4q^2(4p^4+4p^2q-p^2+q^2+2ip\left(2p^2+q\right))z^3}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4}, \tag{31}$$

$$\sum_{n=0}^{\infty} GP_{p,q,n}^2 z^n = \frac{-1+\left(4p^2+1\right)z+q\left(8p^2+q+4ip\right)z^2+q^2\left(4p^2-1+4ip\right)z^3}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4}, \tag{32}$$

with $n \in \mathbb{N}$.

Proof. We have

$$GQ_{p,q,n} = (2-2ip)\, S_n(a_1+[-a_2]) + \left(i\left(4p^2+2q\right)-2p\right) S_{n-1}(a_1+[-a_2]).$$

Hence, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} GQ_{p,q,n}^2 z^n &= \sum_{n=0}^{\infty} \begin{pmatrix} \left((2-2ip)\, S_n(a_1+[-a_2]) + \left(i\left(4p^2+2q\right)-2p\right)\right. \\ \times S_{n-1}(a_1+[-a_2])\right)\left((2-2ip)\, S_n(e_1+[-e_2])\right. \\ \left. + \left(i\left(4p^2+2q\right)-2p\right) S_{n-1}(e_1+[-e_2])\right) \end{pmatrix} z^n \\
&= (2-2ip)^2 \sum_{n=0}^{\infty} S_n(a_1+[-a_2]) S_n(e_1+[-e_2]) z^n \\
&\quad + (2-2ip)(i(4p^2+2q)-2p) \sum_{n=0}^{\infty} S_n(a_1+[-a_2]) S_{n-1}(e_1+[-e_2]) z^n \\
&\quad + (2-2ip)(i(4p^2+2q)-2p) \sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2]) S_n(e_1+[-e_2]) z^n \\
&\quad + \left(i(4p^2+2q)-2p\right)^2 \sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2]) S_{n-1}(e_1+[-e_2]) z^n \\
&= \frac{(2-2ip)^2\left(1-q^2z^2\right)}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4} \\
&\quad + \frac{2\,(2-2ip)\left(i\left(4p^2+2q\right)-2p\right)\left(2pz+2pqz^2\right)}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4} \\
&\quad + \frac{\left(i\left(4p^2+2q\right)-2p\right)^2\left(z-q^2z^3\right)}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4},
\end{aligned}$$

and after a simple calculation, we get

$$\sum_{n=0}^{\infty} GQ_{p,q,n}^{2} z^n = \frac{4-4p^2-8ip+4\left(4p^4-3p^2-q^2+2ip\left(4p^2+q\right)\right)z}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4} + \frac{4q\left(8p^4+5p^2q-4p^2-q+6ip\left(2p^2+q\right)\right)z^2}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4} + \frac{4q^2(4p^4+4p^2q-p^2+q^2+2ip\left(2p^2+q\right))z^3}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4},$$

which gives Equation (31). Using the same procedure, we can obtain Equation (32). □

Remark 6. Putting $p = q = 1$ in the Equations (28)-(32) yields the generating functions for the squares of Pell Lucas, Pell, modified Pell, Gaussian Pell Lucas and Gaussian Pell numbers, respectively.

3. Generating Functions for the Products of Mersenne Numbers with (p, q)-Numbers and Gaussian (p, q)-Numbers

In this section, we now derive the new generating functions for the products of Mersenne numbers with (p, q)-Fibonacci numbers, (p, q)-Lucas numbers, (p, q)-Pell numbers, (p, q)-Pell Lucas numbers, (p, q)-Jacobsthal numbers, (p, q)-Jacobsthal Lucas numbers, (p, q)-modified Pell numbers, Gaussian (p, q)-Pell numbers, Gaussian (p, q)-Pell Lucas numbers, Gaussian (p, q)-Jacobsthal numbers, Gaussian (p, q)-Jacobsthal Lucas numbers, Gaussian (p, q)-Fibonacci numbers and Gaussian (p, q)-Lucas numbers.

Firstly, let us now consider the following conditions for Equations (10) and (12):

$$\begin{cases} a_1 - a_2 = p \\ a_1 a_2 = q, \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 3 \\ e_1 e_2 = -2 \end{cases}$$

The conditions above would result in the following equations:

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right) S_{n-1}\left(e_1+[-e_2]\right) z^n = \frac{pz+3qz^2}{1-3pz-(5q-2p^2)z^2+6pqz^3+4q^2z^4}, \quad (33)$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right) S_{n-1}\left(e_1+[-e_2]\right) z^n = \frac{z+2qz^3}{1-3pz-(5q-2p^2)z^2+6pqz^3+4q^2z^4}. \quad (34)$$

respectively.
Therefore, we state the following theorems.

Theorem 7. *The following are generating functions for the products of (p,q)-Lucas and (p,q)-Fibonacci numbers with Mersenne numbers, respectively:*

$$\sum_{n=0}^{\infty} L_{p,q,n} M_n z^n = \frac{pz + 6qz^2 - 2pqz^3}{1 - 3pz - (5q - 2p^2)z^2 + 6pqz^3 + 4q^2z^4}, \tag{35}$$

$$\sum_{n=0}^{\infty} F_{p,q,n} M_n z^n = \frac{z + 2qz^3}{1 - 3pz - (5q - 2p^2)z^2 + 6pqz^3 + 4q^2z^4}, \tag{36}$$

with $n \in \mathbb{N}$.

Proof. We have

$$L_{p,q,n} = 2S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2]).$$

Then, from this equation, we get

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n} M_n z^n &= \sum_{n=0}^{\infty} \left(2S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2])\right) S_{n-1}(e_1 + [-e_2]) z^n \\ &= 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n \\ &\quad -p\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n\end{aligned}$$

By using the relationships (33) and (34), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n} M_n z^n &= \frac{2\left(pz + 3qz^2\right)}{1 - 3pz - (5q - 2p^2)z^2 + 6pqz^3 + 4q^2z^4} \\ &\quad - \frac{p\left(z + 2qz^3\right)}{1 - 3pz - (5q - 2p^2)z^2 + 6pqz^3 + 4q^2z^4} \\ &= \frac{pz + 6qz^2 - 2pqz^3}{1 - 3pz - (5q - 2p^2)z^2 + 6pqz^3 + 4q^2z^4},\end{aligned}$$

which gives Equation (35). Using the same procedure, we can obtain Equation (36). □

Theorem 8. *The following are generating functions for the products of Gaussian (p,q)-Lucas and Gaussian (p,q)-Fibonacci numbers with Mersenne numbers, respectively:*

$$\sum_{n=0}^{\infty} GL_{p,q,n} M_n z^n = \frac{(p + 2iq)\,z + 3q\,(2 - ip)\,z^2 + 2q\left(i\left(p^2 + 2q\right) - p\right)z^3}{1 - 3pz - (5q - 2p^2)z^2 + 6pqz^3 + 4q^2z^4}, \tag{37}$$

$$\sum_{n=0}^{\infty} GF_{p,q,n} M_n z^n = \frac{z + 3iqz^2 + 2q\,(1 - ip)\,z^3}{1 - 3pz - (5q - 2p^2)z^2 + 6pqz^3 + 4q^2z^4}, \tag{38}$$

with $n \in \mathbb{N}$.

Proof. It is well-known that

$$GL_{p,q,n} = (2 - ip)\, S_n(a_1 + [-a_2]) + \left(i\left(p^2 + 2q\right) - p\right) S_{n-1}(a_1 + [-a_2]).$$

Then,

$$\begin{aligned}\sum_{n=0}^{\infty} GL_{p,q,n} M_n z^n &= \sum_{n=0}^{\infty} \left(\begin{array}{l} ((2 - ip)\, S_n(a_1 + [-a_2]) + \left(i\left(p^2 + 2q\right) - p\right) \\ \times S_{n-1}(a_1 + [-a_2]))\, S_{n-1}(e_1 + [-e_2]) \end{array} \right) z^n \\ &= (2 - ip) \sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n \\ &\quad + \left(i\left(p^2 + 2q\right) - p\right) \sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n.\end{aligned}$$

By using the relationships (33) and (34), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} GL_{p,q,n} M_n z^n &= \frac{(2 - ip)\left(pz + 3qz^2\right)}{1 - 3pz - (5q - 2p^2)z^2 + 6pqz^3 + 4q^2z^4} \\ &\quad + \frac{\left(i\left(p^2 + 2q\right) - p\right)\left(z + 2qz^3\right)}{1 - 3pz - (5q - 2p^2)z^2 + 6pqz^3 + 4q^2z^4} \\ &= \frac{(p + 2iq)\, z + 3q\,(2 - ip)\, z^2 + 2q\left(i\left(p^2 + 2q\right) - p\right) z^3}{1 - 3pz - (5q - 2p^2)z^2 + 6pqz^3 + 4q^2z^4},\end{aligned}$$

which is the first equation. The second equation can be proved similarly. □

Theorem 9. *The following are generating functions for the products of (p, q)-Lucas, (p, q)-Fibonacci, Gaussian (p, q)-Lucas and Gaussian (p, q)-Fibonacci numbers with Mersenne numbers at negative indices, respectively:*

$$\sum_{n=0}^{\infty} L_{p,q,n} M_{-n} z^n = \frac{-2pz - 6qz^2 + pqz^3}{4 - 6pz - (5q - 2p^2)z^2 + 3pqz^3 + q^2z^4}, \tag{39}$$

$$\sum_{n=0}^{\infty} F_{p,q,n} M_{-n} z^n = \frac{-2z - qz^3}{4 - 6pz - (5q - 2p^2)z^2 + 3pqz^3 + q^2z^4}, \tag{40}$$

$$\sum_{n=0}^{\infty} GL_{p,q,n} M_{-n} z^n = \frac{-2\,(p + 2iq)\, z - 3q\,(2 - ip)\, z^2 - q\left(i\left(p^2 + 2q\right) - p\right) z^3}{4 - 6pz - (5q - 2p^2)z^2 + 3pqz^3 + q^2z^4}, \tag{41}$$

$$\sum_{n=0}^{\infty} GF_{p,q,n} M_{-n} z^n = \frac{-2z - 3iqz^2 - q\,(1 - ip)\, z^3}{4 - 6pz - (5q - 2p^2)z^2 + 3pqz^3 + q^2z^4}, \tag{42}$$

with $n \in \mathbb{N}$.

Proof. We use the change of variable $z = \frac{z}{2}$ in Equation (35) and according to relationship (5), we get

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n} M_{-n} z^n &= -\sum_{n=0}^{\infty} L_{p,q,n} M_n \left(\frac{z}{2}\right)^n \\ &= \frac{-\left(p\left(\frac{z}{2}\right) + 6q\left(\frac{z}{2}\right)^2 - 2pq\left(\frac{z}{2}\right)^3\right)}{1 - 3\left(\frac{z}{2}\right) - (5q - 2p^2)\left(\frac{z}{2}\right)^2 + 6pq\left(\frac{z}{2}\right)^3 + 4q^2\left(\frac{z}{2}\right)^4} \\ &= \frac{-2pz - 6qz^2 + pqz^3}{4 - 6pz - (5q - 2p^2)z^2 + 3pqz^3 + q^2z^4},\end{aligned}$$

which gives Equation (39). The other equations can be proved similarly. □

Put $p = q = 1$ in relationships (35)-(42), we obtain Table 1.

Coefficient of z^n	Generating function
L_nM_n	$\frac{z+6z^2-2z^3}{1-3z-3z^2+6z^3+4z^4}$
F_nM_n	$\frac{z+2z^3}{1-3z-3z^2+6z^3+4z^4}$
GL_nM_n	$\frac{(1+2i)z+3(2-i)z^2+2(3i-1)z^3}{1-3z-3z^2+6z^3+4z^4}$
GF_nM_n	$\frac{z+3iz^2+2(1-i)z^3}{1-3z-3z^2+6z^3+4z^4}$
L_nM_{-n}	$\frac{-2z-6z^2+z^3}{4-6z-3z^2+3z^3+z^4}$
F_nM_{-n}	$\frac{-2z-z^3}{4-6z-3z^2+3z^3+z^4}$
GL_nM_{-n}	$\frac{-2(1+2i)z-3(2-i)z^2-(3i-1)z^3}{4-6z-3z^2+3z^3+z^4}$
GF_nM_{-n}	$\frac{-2z-3iz^2-(1-i)z^3}{4-6z-3z^2+3z^3+z^4}$

Table 1: New generating functions for the products of Mersenne numbers with Fibonacci and Gaussian Fibonacci numbers.

Secondly, let us now consider the following conditions for Equations (10) and (12):

$$\begin{cases} a_1 - a_2 = p \\ a_1a_2 = 2q, \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = 3 \\ e_1e_2 = -2. \end{cases}$$

The conditions above would yield the following equations:

$$\sum_{n=0}^{\infty} S_n\left(a_1 + [-a_2]\right) S_{n-1}\left(e_1 + [-e_2]\right) z^n = \frac{pz+6qz^2}{1-3pz-2(5q-p^2)z^2+12pqz^3+16q^2z^4}, \quad (43)$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1 + [-a_2]\right) S_{n-1}\left(e_1 + [-e_2]\right) z^n = \frac{z+4qz^3}{1-3pz-2(5q-p^2)z^2+12pqz^3+16q^2z^4}. \quad (44)$$

Therefore, we state the following results.

Theorem 10. *The following are generating functions for the products of (p, q)-Jacobsthal Lucas and (p, q)-Jacobsthal numbers with Mersenne numbers, respectively:*

$$\sum_{n=0}^{\infty} j_{p,q,n} M_n z^n = \frac{pz + 12qz^2 - 4pqz^3}{1 - 3pz - 2(5q - p^2)z^2 + 12pqz^3 + 16q^2z^4}, \tag{45}$$

$$\sum_{n=0}^{\infty} J_{p,q,n} M_n z^n = \frac{z + 4qz^3}{1 - 3pz - 2(5q - p^2)z^2 + 12pqz^3 + 16q^2z^4}, \tag{46}$$

with $n \in \mathbb{N}$.

Proof. We have

$$j_{p,q,n} = 2S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2]).$$

Then, we can see that

$$\begin{aligned} \sum_{n=0}^{\infty} j_{p,q,n} M_n z^n &= \sum_{n=0}^{\infty} \left(2S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2])\right) S_{n-1}(e_1 + [-e_2]) z^n \\ &= 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n \\ &\quad - p\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n. \end{aligned}$$

By using the relationships (43) and (44), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} j_{p,q,n} M_n z^n &= \frac{2\left(pz + 6qz^2\right)}{1 - 3pz - 2(5q - p^2)z^2 + 12pqz^3 + 16q^2z^4} \\ &\quad - \frac{p\left(z + 4qz^3\right)}{1 - 3pz - 2(5q - p^2)z^2 + 12pqz^3 + 16q^2z^4} \\ &= \frac{pz + 12qz^2 - 4pqz^3}{1 - 3pz - 2(5q - p^2)z^2 + 12pqz^3 + 16q^2z^4}, \end{aligned}$$

which gives Equation (45). Using the same procedure, we can obtain Equation (46). □

Theorem 11. *The following are generating functions for the products of Gaussian (p, q)-Jacobsthal Lucas and Gaussian (p, q)-Jacobsthal numbers with Mersenne numbers, respectively:*

$$\sum_{n=0}^{\infty} Gj_{p,q,n} M_n z^n = \frac{(p + 2iq)\, z + 3q\,(4 - ip)\, z^2 + 2q\left(i\left(p^2 + 4q\right) - 2p\right) z^3}{1 - 3pz - 2(5q - p^2)z^2 + 12pqz^3 + 16q^2z^4}, \tag{47}$$

$$\sum_{n=0}^{\infty} GJ_{p,q,n} M_n z^n = \frac{z + 3iqz^2 + 2q\,(2 - ip)\, z^3}{1 - 3pz - 2(5q - p^2)z^2 + 12pqz^3 + 16q^2z^4}, \tag{48}$$

with $n \in \mathbb{N}$.

Proof. We have

$$Gj_{p,q,n} = \left(2 - \frac{ip}{2}\right) S_n(a_1 + [-a_2]) + \left(i\left(\frac{p^2}{2} + 2q\right) - p\right) S_{n-1}(a_1 + [-a_2]).$$

Then, from this equation, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} Gj_{p,q,n} M_n z^n &= \sum_{n=0}^{\infty} \left(\begin{array}{l} \left(\left(2 - \frac{ip}{2}\right) S_n(a_1 + [-a_2]) + \left(i\left(\frac{p^2}{2} + 2q\right) - p\right) \right. \\ \left. S_{n-1}(a_1 + [-a_2])\right) S_{n-1}(e_1 + [-e_2]) \end{array} \right) z^n \\ &= \left(2 - \frac{ip}{2}\right) \sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n \\ &\quad + \left(i\left(\frac{p^2}{2} + 2q\right) - p\right) \sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n.\end{aligned}$$

By using the relationships (43) and (44), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} Gj_{p,q,n} M_n z^n &= \frac{(4 - ip)\left(pz + 6qz^2\right)}{2\left(1 - 3pz - 2(5q - p^2)z^2 + 12pqz^3 + 16q^2z^4\right)} \\ &\quad + \frac{\left(i\left(p^2 + 4q\right) - 2p\right)\left(z + 4qz^3\right)}{2\left(1 - 3pz - 2(5q - p^2)z^2 + 12pqz^3 + 16q^2z^4\right)} \\ &= \frac{(p + 2iq)\, z + 3q\,(4 - ip)\, z^2 + 2q\left(i\left(p^2 + 4q\right) - 2p\right) z^3}{1 - 3pz - 2(5q - p^2)z^2 + 12pqz^3 + 16q^2z^4},\end{aligned}$$

which gives Equation (47). Using the same procedure, we can obtain Equation (48). □

Theorem 12. *The following are generating functions for the products of* (p,q)*-Jacobsthal Lucas,* (p,q)*-Jacobsthal, Gaussian* (p,q)*-Jacobsthal Lucas and Gaussian* (p,q)*-Jacobsthal numbers with Mersenne numbers at negative indices, respectively:*

$$\sum_{n=0}^{\infty} j_{p,q,n} M_{-n} z^n = \frac{-2pz - 12qz^2 + 2pqz^3}{4 - 6pz - 2(5q - p^2)z^2 + 6pqz^3 + 4q^2z^4}, \tag{49}$$

$$\sum_{n=0}^{\infty} J_{p,q,n} M_{-n} z^n = \frac{-2z - 2qz^3}{4 - 6pz - 2(5q - p^2)z^2 + 6pqz^3 + 4q^2z^4}, \tag{50}$$

$$\sum_{n=0}^{\infty} Gj_{p,q,n} M_{-n} z^n = \frac{-2\,(p + 2iq)\, z - 3q\,(4 - ip)\, z^2 - q\left(i\left(p^2 + 4q\right) - 2p\right) z^3}{4 - 6pz - 2(5q - p^2)z^2 + 6pqz^3 + 4q^2z^4}, \tag{51}$$

$$\sum_{n=0}^{\infty} GJ_{p,q,n} M_{-n} z^n = \frac{-2z - 3iqz^2 - q\,(2 - ip)\, z^3}{4 - 6pz - 2(5q - p^2)z^2 + 6pqz^3 + 4q^2z^4}, \tag{52}$$

with $n \in \mathbb{N}$.

Proof. We use the change of variable $z = \frac{z}{2}$ in Equation (45) and according to relationship (5), we get

$$\begin{aligned}\sum_{n=0}^{\infty} j_{p,q,n} M_{-n} z^n &= -\sum_{n=0}^{\infty} j_{p,q,n} M_n \left(\frac{z}{2}\right)^n \\ &= \frac{-\left(p\left(\frac{z}{2}\right) + 12q\left(\frac{z}{2}\right)^2 - 4pq\left(\frac{z}{2}\right)^3\right)}{1 - 3p\left(\frac{z}{2}\right) - 2(5q-p^2)\left(\frac{z}{2}\right)^2 + 12pq\left(\frac{z}{2}\right)^3 + 16q^2\left(\frac{z}{2}\right)^4} \\ &= \frac{-2pz - 12qz^2 + 2pqz^3}{4 - 6pz - 2(5q-p^2)z^2 + 6pqz^3 + 4q^2z^4},\end{aligned}$$

which gives Equation (49). Using the same procedure, we can obtain the other equations. □

Put $p = q = 1$ in relationships (45)-(52), we obtain Table 2.

Coefficient of z^n	Generating function
$j_n M_n$	$\frac{z+12z^2-4z^3}{1-3z-8z^2+12z^3+16z^4}$
$J_n M_n$	$\frac{z+4z^3}{1-3z-8z^2+12z^3+16z^4}$
$Gj_n M_n$	$\frac{(1+2i)z+3(4-i)z^2+2(5i-2)z^3}{1-3z-8z^2+12z^3+16z^4}$
$GJ_n M_n$	$\frac{z+3iz^2+2(2-i)z^3}{1-3z-8z^2+12z^3+16z^4}$
$j_n M_{-n}$	$\frac{-2z-12z^2+2z^3}{4-6z-8z^2+6z^3+4z^4}$
$J_n M_{-n}$	$\frac{-2z-2z^3}{4-6z-8z^2+6z^3+4z^4}$
$Gj_n M_{-n}$	$\frac{-2(1+2i)z-3(4-i)z^2-(5i-2)z^3}{4-6z-8z^2+6z^3+4z^4}$
$GJ_n M_{-n}$	$\frac{-2z-3iz^2-(2-i)z^3}{4-6z-8z^2+6z^3+4z^4}$

Table 2: New generating functions for the products of Mersenne numbers with Jacobsthal and Gaussian Jacobsthal numbers.

Thirdly, let us now consider the following conditions for Equations (10) and (12):

$$\begin{cases} a_1 - a_2 = 2p \\ a_1 a_2 = q, \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = 3 \\ e_1 e_2 = -2. \end{cases}$$

The conditions above would yield the following equations:

$$\sum_{n=0}^{\infty} S_n\left(a_1 + [-a_2]\right) S_{n-1}\left(e_1 + [-e_2]\right) z^n = \frac{2pz+3qz^2}{1-6pz-(5q-8p^2)z^2+12pqz^3+4q^2z^4}, \tag{53}$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1 + [-a_2]\right) S_{n-1}\left(e_1 + [-e_2]\right) z^n = \frac{z+2qz^3}{1-6pz-(5q-8p^2)z^2+12pqz^3+4q^2z^4}. \tag{54}$$

Thus we derive the following theorems.

Theorem 13. *The following are generating functions for the products of (p,q)-Pell Lucas, (p,q)-Pell and (p,q)-modified Pell numbers with Mersenne numbers, respectively:*

$$\sum_{n=0}^{\infty} Q_{p,q,n} M_n z^n = \frac{2pz + 6qz^2 - 4pqz^3}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4}, \tag{55}$$

$$\sum_{n=0}^{\infty} P_{p,q,n} M_n z^n = \frac{z + 2qz^3}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4}, \tag{56}$$

$$\sum_{n=0}^{\infty} MP_{p,q,n} M_n z^n = \frac{pz + 3qz^2 - 2pqz^3}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4}, \tag{57}$$

with $n \in \mathbb{N}$.

Proof. We know that

$$Q_{p,q,n} = 2S_n(a_1 + [-a_2]) - 2pS_{n-1}(a_1 + [-a_2]).$$

Then, from this equation, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{p,q,n} M_n z^n &= \sum_{n=0}^{\infty} (2S_n(a_1 + [-a_2]) - 2pS_{n-1}(a_1 + [-a_2]))S_{n-1}(e_1 + [-e_2])z^n \\
&= 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\
&\quad - 2p\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\
&= \frac{2\left(2pz + 3qz^2\right)}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4} \\
&\quad - \frac{2p\left(z + 2qz^3\right)}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4} \\
&= \frac{2pz + 6qz^2 - 4pqz^3}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4},
\end{aligned}$$

which is the first equation. The other equation can be proved similarly. □

Theorem 14. *The following are generating functions for the products of Gaussian (p,q)-Pell Lucas and Gaussian (p,q)-Pell numbers with Mersenne numbers, respectively:*

$$\sum_{n=0}^{\infty} GQ_{p,q,n} M_n z^n = \frac{2\left(p + iq\right)z + 6q\left(1 - ip\right)z^2 + 4q\left(i\left(2p^2 + q\right) - p\right)z^3}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4}, \tag{58}$$

$$\sum_{n=0}^{\infty} GP_{p,q,n} M_n z^n = \frac{z + 3iqz^2 + 2q(1-2ip)z^3}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4}, \tag{59}$$

with $n \in \mathbb{N}$.

Proof. We have

$$GQ_{p,q,n} = (2 - 2ip) S_n(a_1 + [-a_2]) + \left(i\left(4p^2 + 2q\right) - 2p\right) S_{n-1}(a_1 + [-a_2]).$$

Then, we see that

$$\begin{aligned}\sum_{n=0}^{\infty} GQ_{p,q,n} M_n z^n &= \sum_{n=0}^{\infty} \begin{pmatrix} ((2-2ip) S_n(a_1 + [-a_2]) + \left(i\left(4p^2+2q\right) - 2p\right) \\ \times S_{n-1}(a_1 + [-a_2])) S_{n-1}(e_1 + [-e_2]) \end{pmatrix} z^n \\ &= (2-2ip) \sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n \\ &\quad + \left(i\left(4p^2+2q\right) - 2p\right) \sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n.\end{aligned}$$

Then, according to the relationships (53) and (54), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} GQ_{p,q,n} M_n z^n &= \frac{(2-2ip)\left(2pz + 3qz^2\right)}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4} \\ &\quad + \frac{\left(i\left(4p^2+2q\right) - 2p\right)\left(z + 2qz^3\right)}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4} \\ &= \frac{2(p+iq)z + 6q(1-ip)z^2 + 4q\left(i\left(2p^2+q\right) - p\right)z^3}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4},\end{aligned}$$

which gives Equation (58). Using the same procedure, we can obtain Equation (59). □

Theorem 15. *The following are generating functions for the products of* (p, q)*-Pell Lucas,* (p, q)*-Pell,* (p, q)*-modified Pell,Gaussian* (p, q)*-Pell Lucas and Gaussian* (p, q)*-Pell numbers with Mersenne numbers with negative indices, respectively:*

$$\sum_{n=0}^{\infty} Q_{p,q,n} M_{-n} z^n = \frac{-4pz - 6qz^2 + 2pqz^3}{4 - 12pz - (5q - 8p^2)z^2 + 6pqz^3 + q^2z^4}, \tag{60}$$

$$\sum_{n=0}^{\infty} P_{p,q,n} M_{-n} z^n = \frac{-2z - qz^3}{4 - 12pz - (5q - 8p^2)z^2 + 6pqz^3 + q^2z^4}, \tag{61}$$

$$\sum_{n=0}^{\infty} q_{p,q,n} M_{-n} z^n = \frac{-2pz - 3qz^2 + pqz^3}{4 - 12pz - (5q - 8p^2)z^2 + 6pqz^3 + q^2z^4}, \tag{62}$$

$$\sum_{n=0}^{\infty} GQ_{p,q,n}M_{-n}z^n = \frac{-4\,(p+iq)\,z - 6q\,(1-ip)\,z^2 - 2q\left(i\left(2p^2+q\right)-p\right)z^3}{4-12pz-(5q-8p^2)z^2+6pqz^3+q^2z^4}, \quad (63)$$

$$\sum_{n=0}^{\infty} GP_{p,q,n}M_{-n}z^n = \frac{-2z-3iqz^2-q\,(1-2ip)\,z^3}{4-12pz-(5q-8p^2)z^2+6pqz^3+q^2z^4}, \quad (64)$$

with $n \in \mathbb{N}$.

Proof. We use the change of variable $z = \frac{z}{2}$ in Equation (55) and according to relationship (5), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{p,q,n}M_{-n}z^n &= -\sum_{n=0}^{\infty} Q_{p,q,n}M_n\left(\frac{z}{2}\right)^n \\
&= \frac{-\left(2p\left(\frac{z}{2}\right)+6q\left(\frac{z}{2}\right)^2-4pq\left(\frac{z}{2}\right)^3\right)}{1-6p\left(\frac{z}{2}\right)-(5q-8p^2)\left(\frac{z}{2}\right)^2+12pq\left(\frac{z}{2}\right)^3+4q^2\left(\frac{z}{2}\right)^4} \\
&= \frac{-4pz-6qz^2+2pqz^3}{4-12pz-(5q-8p^2)z^2+6pqz^3+q^2z^4},
\end{aligned}$$

which gives Equation (60). The other equations can be proved similarly. □

By taking $p = q = 1$ in Equations (55)-(64), we obtain Table 3.

Coefficient of z^n	Generating function
Q_nM_n	$\frac{2z+6z^2-4z^3}{1-6z+3z^2+12z^3+4z^4}$
P_nM_n	$\frac{z+2z^3}{1-6z+3z^2+12z^3+4z^4}$
q_nM_n	$\frac{z+3z^2-2z^3}{1-6z+3z^2+12z^3+4z^4}$
GQ_nM_n	$\frac{2(1+i)z+6(1-i)z^2+4(3i-1)z^3}{1-6z+3z^2+12z^3+4z^4}$
GP_nM_n	$\frac{z+3iz^2+2(1-2i)z^3}{1-6z+3z^2+12z^3+4z^4}$
Q_nM_{-n}	$\frac{-4z-6z^2+2z^3}{4-12z+3z^2+6z^3+z^4}$
P_nM_{-n}	$\frac{-2z-z^3}{4-12z+3z^2+6z^3+z^4}$
q_nM_{-n}	$\frac{-2z-3z^2+z^3}{4-12z+3z^2+6z^3+z^4}$
GQ_nM_{-n}	$\frac{-4(1+i)z-6(1-i)z^2-2(3i-1)z^3}{4-12z+3z^2+6z^3+z^4}$
GP_nM_{-n}	$\frac{-2z-3iz^2-(1-2i)z^3}{4-12z+3z^2+6z^3+z^4}$

Table 3: New generating functions for the products of Mersenne numbers with Pell and Gaussian Pell numbers.

4. Conclusion

In this paper, by making use of Theorems 1 and 2, we have derived some new generating functions for the squares of (p, q)-numbers and Gaussian (p, q)-numbers and the products of Mersenne numbers with (p, q)-numbers and Gaussian (p, q)-numbers. The derived theorems are based on symmetric functions and products of these numbers.

In our forthcoming investigation, we plan to establish further results and properties associated with some generalized forms of the above-mentioned families of numbers.

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