LUCAS TYPE SEQUENCES AND SUMS OF BINOMIAL COEFFICIENTS

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Abstract

For each integer $q = 2p + 1$, where p is a positive integer, we consider the sequence $\{s_n\}$ defined by $s_n = \sum_{k=1}^{p} \bigl(2\cos(2k\pi/q)\bigr)^n$. It is known that this sequence is integer-valued and satisfies a simple recursion relation arising from Chebyshev polynomials. We find explicit values for the first $2q$ terms of each sequence $\{s_n\}$ and show that each term of the sequence corresponds to a sum of binomial coefficients of the form $\sum_{k=0}^{n} \binom{qn+t}{qk+r}$, where t and r are nonnegative integers less than q.

1. Introduction

Let $q = 2p + 1$, where p is a positive integer. For each nonnegative integer n, let

$$s_n = \sum_{k=1}^{p} \bigl(2\cos(2k\pi/q)\bigr)^n.$$

In this paper, we explore the properties of the sequence $\{s_n\}$. To get a glimpse of the relevance of this sequence to the title of the paper, let $\alpha = \phi$ and $\beta = -1/\phi$, where ϕ is the golden ratio. Letting $q = 5$ and recalling the

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well-known facts that $2\cos(\pi/5) = \alpha$ and $2\cos(2\pi/5) = -\beta$, we find that

$$s_n = \sum_{k=1}^{2} \big(2\cos(2k\pi/5)\big)^n = (-\beta)^n + (-\alpha)^n = (-1)^n \ell_n,$$

where ℓ_n represents the nth Lucas number. In addition, the identity

$$5\sum_{k=0}^{n} \binom{5n}{5k} = 2^{5n} + 2s_{5n} = 2^{5n} + 2(-1)^n \ell_{5n}$$

is valid for all nonnegative integers n. For other values of q, we obtain sequences that share some properties with the Lucas sequence. For instance, each sequence $\{s_n\}$ is integer-valued and satisfies a simple recursion relation. These facts are verified by Wang in [6], but we include the details since our approach is a bit different. Furthermore, in Section 4, we show that each term of the sequence corresponds to a particular sum of binomial coefficients.

2. Initial Terms of the Sequence $\{s_n\}$

To get a sense for the various sequences $\{s_n\}$, we begin by determining the first few terms of each of the sequences. Given the definition of the sequence, it is somewhat surprising that these values are all integers. To find the terms of $\{s_n\}$, it is helpful to have other ways to represent the sequence. For one such representation, we observe that

$$\begin{aligned}
&\sum_{k=1}^{p} \big((-1)^k 2\cos(k\pi/q)\big)^n \\
&\qquad = \sum_{\substack{1\le k\le p \\ k \text{ odd}}} \big(-2\cos(k\pi/q)\big)^n + \sum_{\substack{1\le k\le p \\ k \text{ even}}} \big(2\cos(k\pi/q)\big)^n \\
&\qquad = \sum_{\substack{1\le k\le p \\ k \text{ odd}}} \big(2\cos((q-k)\pi/q)\big)^n + \sum_{\substack{1\le k\le p \\ k \text{ even}}} \big(2\cos(k\pi/q)\big)^n \\
&\qquad = \sum_{\substack{p< k< q \\ k \text{ even}}} \big(2\cos(k\pi/q)\big)^n + \sum_{\substack{1\le k\le p \\ k \text{ even}}} \big(2\cos(k\pi/q)\big)^n \\
&\qquad = \sum_{k=1}^{p} \big(2\cos(2k\pi/q)\big)^n.
\end{aligned}$$

We also have (letting $j = p+1-k$ in the second step)

$$\begin{aligned}\sum_{k=1}^{p}\big(2\cos(2k\pi/q)\big)^n &= \sum_{k=1}^{p}\Big(-2\cos\big((q-2k)\pi/q\big)\Big)^n\\ &= (-1)^n\sum_{j=1}^{p}\Big(2\cos\big((2j-1)\pi/q\big)\Big)^n.\end{aligned}$$

We have thus shown that

$$\begin{aligned}s_n &= \sum_{k=1}^{p}\big(2\cos(2k\pi/q)\big)^n && \text{form (1)}\\ &= \sum_{k=1}^{p}\big((-1)^k 2\cos(k\pi/q)\big)^n && \text{form (2)}\\ &= (-1)^n\sum_{k=1}^{p}\big(2\cos\big((2k-1)\pi/q\big)\big)^n && \text{form (3)}\end{aligned}$$

for all $n \geq 0$. As we shall see, sometimes one of these representations is better suited to verify a given property of the sequence. We note that our sequence $\{s_n\}$ corresponds to the sequence $\{b_n\}$ in [6] and that the sequence $\{(-1)^n s_n\}$ corresponds to the sequence $\{a_n\}$ defined in both [2] and [6] (using form (3)).

The following theorem allows us to find specific values of the sequence $\{s_n\}$ for small values of n.

Theorem 1. Let $q = 2p+1$, where p is a positive integer. For each positive integer n, we have

$$\sum_{k=1}^{p}(-1)^k 2\cos\big((2n-1)k\pi/q\big) = \begin{cases} 2p, & \text{if } 2n-1 \text{ is an odd multiple of } q;\\ -1, & \text{otherwise;}\end{cases}$$

$$\sum_{k=1}^{p} 2\cos(2nk\pi/q) = \begin{cases} 2p, & \text{if } n \text{ is a multiple of } q;\\ -1, & \text{otherwise.}\end{cases}$$

Proof. We consider the two special cases first. Let $2n-1 = mq$, where $m > 0$ is an odd integer. We then have

$$\sum_{k=1}^{p}(-1)^k 2\cos\big((2n-1)k\pi/q\big) = \sum_{k=1}^{p}(-1)^k 2(-1)^{mk} = \sum_{k=1}^{p} 2 = 2p.$$

Now let $n = mq$ for some positive integer m and compute

$$\sum_{k=1}^{p} 2\cos(2nk\pi/q) = \sum_{k=1}^{p} 2 = 2p.$$

Suppose $2n-1$ is not an odd multiple of q and let $r = -\exp\bigl((2n-1)\pi i/q\bigr)$, where $i = \sqrt{-1}$. Noting that $r \neq 1$ and $r^q = (-1)^q(-1) = 1$, we have

$$\begin{aligned}\sum_{k=1}^{p}(-1)^k 2\cos\bigl((2n-1)k\pi/q\bigr) &= \sum_{k=1}^{p}\bigl(r^k + r^{-k}\bigr)\\ &= \frac{r - r^{p+1}}{1-r} + \frac{r^{-1} - r^{-p-1}}{1-r^{-1}}\\ &= \frac{r - r^{p+1} - 1 + r^{-p}}{1-r}\\ &= -1 - \frac{r^{2p+1}-1}{r^p(1-r)} = -1.\end{aligned}$$

Finally, suppose that n is not a multiple of q and let $r = \exp(2n\pi i/q)$. Noting that $r \neq 1$ and $r^q = 1$, we find that (omitting some steps similar to those above)

$$\sum_{k=1}^{p} 2\cos(2nk\pi/q) = \sum_{k=1}^{p}\bigl(r^k + r^{-k}\bigr) = -1.$$

This completes the proof. □

We can use Theorem 1 to find the values of s_n for $0 \leq n < 2q$. These values of the sequence are recorded in the following theorem.

Theorem 2. Let $q = 2p+1$, where p is a positive integer. The first $2q$ values of the sequence $\{s_n\}_{n=0}^{\infty}$ are given by

$$\begin{aligned} s_{2n} &= \frac{q}{2}\binom{2n}{n} - 2^{2n-1} \quad \text{for } 0 \leq n \leq 2p;\\ s_{2n+1} &= -2^{2n} \quad \text{for } 0 \leq n < p;\\ s_{2n+1} &= q\binom{2n+1}{n-p} - 2^{2n} \quad \text{for } p \leq n \leq 2p.\end{aligned}$$

Proof. It is obvious from the definition that $s_0 = p$, which is consistent with the fact that

$$\frac{q}{2}\binom{0}{0} - 2^{-1} = \frac{2p+1-1}{2} = p,$$

and we see that $s_1 = -1 = -2^0$ by Theorem 1. For larger values of n, we use the basic trigonometric identities

$$2^{2n}\cos^{2n} x = \sum_{j=0}^{n-1}\binom{2n}{j} 2\cos\big((2n-2j)x\big) + \binom{2n}{n};$$
$$2^{2n+1}\cos^{2n+1} x = \sum_{j=0}^{n}\binom{2n+1}{j} 2\cos\big((2n+1-2j)x\big).$$

Suppose that $1 \le n \le 2p$. Noting that $n-j$ is not a multiple of q for $0 \le j \le n-1$, we can use Theorem 1 to compute (using form (2) for s_n)

$$\begin{aligned}
s_{2n} &= \sum_{k=1}^{p}\big((-1)^k 2\cos(k\pi/q)\big)^{2n} = \sum_{k=1}^{p} 2^{2n}\cos^{2n}(k\pi/q)\\
&= \sum_{k=1}^{p}\left(\sum_{j=0}^{n-1}\binom{2n}{j} 2\cos\big((2n-2j)k\pi/q\big) + \binom{2n}{n}\right)\\
&= \sum_{j=0}^{n-1}\binom{2n}{j}\sum_{k=1}^{p} 2\cos\big(2(n-j)k\pi/q\big) + p\binom{2n}{n}\\
&= -\sum_{j=0}^{n-1}\binom{2n}{j} + \frac{q-1}{2}\binom{2n}{n}\\
&= -\frac{1}{2}\left(\sum_{j=0}^{n-1}\binom{2n}{j} + \sum_{j=n+1}^{2n}\binom{2n}{j}\right) + \frac{q}{2}\binom{2n}{n} - \frac{1}{2}\binom{2n}{n}\\
&= \frac{q}{2}\binom{2n}{n} - \frac{1}{2}\sum_{j=0}^{2n}\binom{2n}{j}\\
&= \frac{q}{2}\binom{2n}{n} - 2^{2n-1}.
\end{aligned}$$

Now suppose that $1 \le n < p$ and note that $2n+1-2j$ is not an odd multiple of q for $0 \le j \le n$. By Theorem 1, we have (using form (2) again)

$$\begin{aligned} s_{2n+1} &= \sum_{k=1}^{p} \big((-1)^k 2\cos(k\pi/q)\big)^{2n+1} \\ &= \sum_{k=1}^{p} (-1)^k 2^{2n+1} \cos^{2n+1}(k\pi/q) \\ &= \sum_{k=1}^{p} \sum_{j=0}^{n} \binom{2n+1}{j} (-1)^k 2\cos\big((2n+1-2j)k\pi/q\big) \\ &= \sum_{j=0}^{n} \binom{2n+1}{j} \sum_{k=1}^{p} (-1)^k 2\cos\big((2n+1-2j)k\pi/q\big) \\ &= -\sum_{j=0}^{n} \binom{2n+1}{j} \\ &= -2^{2n}. \end{aligned}$$

Finally, suppose that $p \le n \le 2p$ and note that $2n+1-2j$ is an odd multiple of q only when $j = n-p$ (for $0 \le j \le n$). Using Theorem 1, we obtain (omitting some of the similar steps from the previous equation)

$$\begin{aligned} s_{2n+1} &= \sum_{j=0}^{n} \binom{2n+1}{j} \sum_{k=1}^{p} (-1)^k 2\cos\big((2n+1-2j)k\pi/q\big) \\ &= \left(-\sum_{j=0}^{n} \binom{2n+1}{j} + \binom{2n+1}{n-p} \right) + 2p\binom{2n+1}{n-p} \\ &= q\binom{2n+1}{n-p} - 2^{2n}. \end{aligned}$$

This completes the proof. □

Assuming that $p \ge 5$, the first eleven terms of the sequence $\{s_n\}_{n=0}^{\infty}$ are thus

$$p,\ -1,\ q-2,\ -4,\ 3q-8,\ -16,\ 10q-32,\ -64,\ 35q-128,\ -256,\ 126q-512.$$

It is interesting to note that the first few odd terms of the sequence $\{s_n\}$ are independent of the value of q.

Using form (1) for s_n, it is obvious that $s_n > 0$ for all even values of n. We claim that $s_n < 0$ for all odd values of n. If n is odd and $q = 4p+1$, then (using form (2) for s_n)

$$\begin{aligned} s_n &= \sum_{k=1}^{2p} \big((-1)^k 2\cos(k\pi/q)\big)^n \\ &= \sum_{k=1}^{p} \Big(\big(2\cos((2k)\pi/q)\big)^n - \big(2\cos((2k-1)\pi/q)\big)^n\Big). \end{aligned}$$

Since the cosine function is decreasing and positive on the interval $(0, \frac{1}{2}\pi)$ and $2p/q < \frac{1}{2}$, we know that

$$\cos((2k-1)\pi/q) > \cos((2k)\pi/q) > 0$$

for all values of k that satisfy $1 \le k \le p$. This shows that each term of the above sum is negative. If n is odd and $q = 4p+3$, then

$$\begin{aligned} s_n &= \sum_{k=1}^{2p+1} \big((-1)^k 2\cos(k\pi/q)\big)^n \\ &= \sum_{k=1}^{2p} \big((-1)^k 2\cos(k\pi/q)\big)^n - \big(2\cos((2p+1)\pi/q)\big)^n. \end{aligned}$$

The sum is negative as above and the single extra term that is subtracted is positive since the argument of cosine is less than $\pi/2$.

3. A Recurrence Relation for the Sequence $\{s_n\}$

We next determine a recurrence relation satisfied by the sequence $\{s_n\}$. The following results are given in Theorem 2.2 of [6], but we include a few details since our approach is a little different. As usual, let $q = 2p+1$, where p is a positive integer.

To simplify the notation, let $c_k = 2\cos(2k\pi/q)$ for $1 \le k \le p$. Consider the polynomial

$$\begin{aligned} R_p(x) &= (x-c_1)(x-c_2)\cdots(x-c_p) \\ &\equiv x^p + a_1x^{p-1} + a_2x^{p-2} + a_3x^{p-3} + \cdots + a_{p-1}x + a_p. \end{aligned}$$

The terms $s_n = c_1^n + c_2^n + c_3^n + \cdots + c_p^n$ represent the corresponding sequence $\{s_n\}$ and this sequence satisfies the recurrence relation

$$s_n = -\sum_{k=1}^{p} a_k s_{n-k}$$

for all $n \geq p$, along with the previously calculated values of s_k for $0 \leq k < p$. To verify the recurrence relation, we note that

$$\begin{aligned} -\sum_{k=1}^{p} a_k s_{n-k} &= -\sum_{k=1}^{p} a_k \sum_{j=1}^{p} c_j^{n-k} = -\sum_{j=1}^{p} c_j^{n-p} \sum_{k=1}^{p} a_k c_j^{p-k} \\ &= -\sum_{j=1}^{p} c_j^{n-p}(-c_j^p) = \sum_{j=1}^{p} c_j^n = s_n. \end{aligned}$$

To show that each term of the sequence $\{s_n\}$ is an integer, it is sufficient to prove that each of the coefficients a_k for $1 \leq k \leq p$ is an integer. This is our next goal.

The Chebyshev polynomials U_n of the second kind are defined through the identity $U_n(\cos\theta)\sin\theta = \sin((n+1)\theta)$. It is easy to see that $U_0(x) = 1$ and $U_1(x) = 2x$. These polynomials satisfy the recurrence relation

$$U_{n+1}(x/2) = xU_n(x/2) - U_{n-1}(x/2)$$

(replacing x with $x/2$) for all $n \geq 1$. From this recursion, we easily see that each $U_n(x/2)$ is a monic polynomial of degree n with integer coefficients. We claim that $R_p(x) = U_p(x/2) + U_{p-1}(x/2)$ for all integers $p \geq 1$. To verify this fact, it is sufficient to show that the two monic polynomials $R_p(x)$ and $U_p(x/2) + U_{p-1}(x/2)$ have the same roots. We know that the roots of $R_p(x)$ are the numbers c_k for $1 \leq k \leq p$. For each of these k values, the number $2k\pi/q$ lies in the interval $(0, \pi)$ so the sine of this value is nonzero. We thus have (recall that $q = 2p+1$)

$$\begin{aligned} \sin(2k\pi/q)\Big(U_p\big(\cos(2k\pi/q)\big) &+ U_{p-1}\big(\cos(2k\pi/q)\big)\Big) \\ &= \sin\big(2(p+1)k\pi/q\big) + \sin\big(2pk\pi/q\big) \\ &= \sin\big(k\pi + k\pi/q\big) + \sin\big(k\pi - k\pi/q\big) \\ &= 0, \end{aligned}$$

which indicates that c_k is a root of $U_p(x/2)+U_{p-1}(x/2)$ for all k that satisfy $1 \leq k \leq p$. This establishes the claim. With this result, we obtain the recurrence relation

$$\begin{aligned} R_{p+1}(x) &= U_{p+1}(x/2) + U_p(x/2) \\ &= \Big(xU_p(x/2) - U_{p-1}(x/2)\Big) + \Big(xU_{p-1}(x/2) - U_{p-2}(x/2)\Big) \\ &= x\Big(U_p(x/2) + U_{p-1}(x/2)\Big) - \Big(U_{p-1}(x/2) + U_{p-2}(x/2)\Big) \\ &= xR_p(x) - R_{p-1}(x) \end{aligned}$$

for all $p \geq 1$. It follows that the polynomial R_p has integer coefficients for all $p \geq 1$. The first few of these polynomials are listed below:

$$\begin{aligned} &R_1(x) = x + 1; \\ &R_2(x) = x^2 + x - 1; \\ &R_3(x) = x^3 + x^2 - 2x - 1; \\ &R_4(x) = x^4 + x^3 - 3x^2 - 2x + 1; \\ &R_5(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1; \\ &R_6(x) = x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1; \\ &R_7(x) = x^7 + x^6 - 6x^5 - 5x^4 + 10x^3 + 6x^2 - 4x - 1. \end{aligned}$$

Note that $a_1 = 1$, $a_2 = -(p-1)$, and $a_3 = -(p-2)$ for $p \geq 2$. The next theorem gives a specific expression for the polynomial R_p.

Theorem 3. For each positive integer p, the polynomial R_p can be expressed as

$$R_p(x) = \sum_{k=0}^{\lfloor p/2 \rfloor} (-1)^k \left[\binom{p-k}{k} x^{p-2k} + \binom{p-k-1}{k} x^{p-2k-1} \right].$$

Proof. It is known that (see page 454 of [7])

$$U_n(x/2) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k}$$

for each positive integer n. The explicit expression for $R_p(x)$ then follows from the fact that $R_p(x) = U_p(x/2) + U_{p-1}(x/2)$ for all $p \geq 1$. (Recall that $\binom{n}{k} = 0$ when $k > n$.) □

To illustrate our results with a specific example, we consider the case $q = 11$. For this value of q, the relevant polynomial and recurrence relation are given by

$$R_5(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1;$$
$$s_{n+1} = -s_n + 4s_{n-1} + 3s_{n-2} - 3s_{n-3} - s_{n-4}.$$

From the values given in Theorem 2, we know that

$$s_0 = 5, \quad s_1 = -1, \quad s_2 = 9, \quad s_3 = -4, \quad s_4 = 25.$$

We then have (for example)

$$s_5 = -s_4 + 4s_3 + 3s_2 - 3s_1 - s_0 = -25 - 16 + 27 + 3 - 5 = -16;$$
$$s_6 = -s_5 + 4s_4 + 3s_3 - 3s_2 - s_1 = 16 + 100 - 12 - 27 + 1 = 78;$$
$$s_7 = -s_6 + 4s_5 + 3s_4 - 3s_3 - s_2 = -78 - 64 + 75 + 12 - 9 = -64.$$

Using Theorem 2, we find that $s_6 = 10q - 32 = 78$, showing that our results are consistent.

4. The Sequence $\{s_n\}$ and Sums of Binomial Coefficients

In this section, we demonstrate how the terms of the sequence $\{s_n\}$ are related to sums of binomial coefficients. Using properties of complex numbers, it can be shown that

$$q\sum_{k=0}^{n}\binom{qn+t}{qk+r} = 2^{qn+t} + \sum_{k=1}^{q-1}(-1)^{nk}\cos\big((t-2r)k\pi/q\big)\big(2\cos(k\pi/q)\big)^{qn+t},$$

where t and r are nonnegative integers less than q. A version of this identity is stated without proof in [3] and also as equation 6.21 in Volume 6 of [4]. The Wikipedia pages for binomial coefficients and series multisection also list this result. The appearance of the term 2^{qn+t} should come as no surprise based upon the expansion of $(1+1)^{qn+t}$ using the binomial theorem.

These binomial identities vary a bit depending on whether q is even or odd. In this paper, we focus on the case in which q is odd. However, a few comments concerning the even case are given in Section 6. Suppose that $q = 2p + 1$, where p is a positive integer. We then have

$$q\sum_{k=0}^{n}\binom{qn+t}{qk+r}$$
$$= 2^{qn+t} + \sum_{k=1}^{q-1}(-1)^{nk}\cos\big((t-2r)k\pi/q\big)\big(2\cos(k\pi/q)\big)^{qn+t}$$
$$= 2^{qn+t} + \sum_{k=1}^{p}(-1)^{nk}\cos\big((t-2r)k\pi/q\big)\big(2\cos(k\pi/q)\big)^{qn+t}$$
$$+ \sum_{k=p+1}^{2p}(-1)^{nk}\cos\big((t-2r)k\pi/q\big)\big(2\cos(k\pi/q)\big)^{qn+t}$$

and (making the substitution $j = q - k$)

$$\sum_{k=p+1}^{2p}(-1)^{nk}\cos\big((t-2r)k\pi/q\big)\big(2\cos(k\pi/q)\big)^{qn+t}$$
$$= \sum_{j=p}^{1}(-1)^{n(q-j)}\cos\big((t-2r)(q-j)\pi/q\big)\big(2\cos((q-j)\pi/q)\big)^{qn+t}$$
$$= \sum_{j=1}^{p}(-1)^{n(q-j)}(-1)^{t-2r}\cos\big((t-2r)j\pi/q\big)(-1)^{qn+t}\big(2\cos(j\pi/q)\big)^{qn+t}$$
$$= \sum_{j=1}^{p}(-1)^{nj}\cos\big((t-2r)j\pi/q\big)\big(2\cos(j\pi/q)\big)^{qn+t}.$$

It follows that (we refer to this equation as the *basic binomial identity*)

$$q\sum_{k=0}^{n}\binom{qn+t}{qk+r} = 2^{qn+t} + \sum_{k=1}^{p}(-1)^{nk}2\cos\big((t-2r)k\pi/q\big)\big(2\cos(k\pi/q)\big)^{qn+t}.$$

We make note of the following three particularly simple forms:

$$q\sum_{k=0}^{n}\binom{qn+2r}{qk+r} = 2^{qn+2r} + 2\sum_{k=1}^{p}(-1)^{nk}\big(2\cos(k\pi/q)\big)^{qn+2r};$$
$$q\sum_{k=0}^{n}\binom{qn+2r+1}{qk+r} = 2^{qn+2r+1} + \sum_{k=1}^{p}(-1)^{nk}\big(2\cos(k\pi/q)\big)^{qn+2r+2};$$
$$q\sum_{k=0}^{n}\binom{qn+2r-1}{qk+r} = 2^{qn+2r-1} + \sum_{k=1}^{p}(-1)^{nk}\big(2\cos(k\pi/q)\big)^{qn+2r};$$

these forms occur when the value of $|t-2r|$ is either 0 or 1. From these equations, it becomes clear how the sequence $\{s_n\}$ comes into play. We identify a binomial sum for each term of the sequence in the following theorem.

Theorem 4. Let $q = 2p+1$, where p is a positive integer. For $0 \le t \le p$, we have

$$\sum_{k=1}^{p}\big(2\cos(2k\pi/q)\big)^{qn+2t} = s_{qn+2t} = q\sum_{k=0}^{n}\binom{qn+2t-1}{qk+t} - 2^{qn+2t-1}$$
$$= \frac{q}{2}\sum_{k=0}^{n}\binom{qn+2t}{qk+t} - 2^{qn+2t-1}$$

for all $n \ge 0$, and for $0 \le t \le p-1$, we have

$$\sum_{k=1}^{p}\big(2\cos(2k\pi/q)\big)^{qn+2t+1} = s_{qn+2t+1} = q\sum_{k=0}^{n}\binom{qn+2t}{qk+t+p+1} - 2^{qn+2t}$$
$$= \frac{q}{2}\sum_{k=0}^{n}\binom{qn+2t+1}{qk+t+p+1} - 2^{qn+2t}$$

for all $n \ge 0$.

Proof. We use form (2) for the terms s_n throughout the proof. For $0 \le t \le p$ and $n \ge 0$, the basic binomial identity yields

$$q\sum_{k=0}^{n}\binom{qn+2t}{qk+t} = 2^{qn+2t} + 2\sum_{k=1}^{p}(-1)^{nk}\big(2\cos(k\pi/q)\big)^{qn+2t}$$
$$= 2^{qn+2t} + 2\sum_{k=1}^{p}\big((-1)^{k}2\cos(k\pi/q)\big)^{qn+2t}$$
$$= 2^{qn+2t} + 2s_{qn+2t}$$

and

$$q\sum_{k=0}^{n}\binom{qn+2t-1}{qk+t}$$
$$= 2^{qn+2t-1} + \sum_{k=1}^{p}(-1)^{nk}2\cos(k\pi/q)\big(2\cos(k\pi/q)\big)^{qn+2t-1}$$
$$= 2^{qn+2t-1} + \sum_{k=1}^{p}\big((-1)^{k}2\cos(k\pi/q)\big)^{qn+2t}$$
$$= 2^{qn+2t-1} + s_{qn+2t}.$$

Since the $t = 0$ value in the second sum violates the condition that t is nonnegative in the expression $qn + t$, we show that the sum is still correct in this case (we assume $n \geq 1$ since $n = 0$ is trivial):

$$\begin{aligned}
q\sum_{k=0}^{n}\binom{qn-1}{qk} &= q\sum_{k=0}^{n-1}\binom{q(n-1)+2p}{qk} \\
&= 2^{qn-1} + \sum_{k=1}^{p}(-1)^{(n-1)k}2\cos\big((q-1)k\pi/q\big)\big(2\cos(k\pi/q)\big)^{qn-1} \\
&= 2^{qn-1} + \sum_{k=1}^{p}(-1)^{(n-1)k}(-1)^{k}\big(2\cos(k\pi/q)\big)^{qn} \\
&= 2^{qn-1} + \sum_{k=1}^{p}\big((-1)^{k}2\cos(k\pi/q)\big)^{qn} \\
&= 2^{qn-1} + s_{qn}.
\end{aligned}$$

Using computations similar to those above, for $0 \leq t \leq p-1$ and $n \geq 0$, we find that

$$\begin{aligned}
&q\sum_{k=0}^{n}\binom{qn+2t+1}{qk+t+p+1} \\
&\quad = 2^{qn+2t+1} + \sum_{k=1}^{p}(-1)^{nk}2\cos(qk\pi/q)\big(2\cos(k\pi/q)\big)^{qn+2t+1} \\
&\quad = 2^{qn+2t+1} + 2\sum_{k=1}^{p}(-1)^{qnk}(-1)^{k}\big(2\cos(k\pi/q)\big)^{qn+2t+1} \\
&\quad = 2^{qn+2t+1} + 2\sum_{k=1}^{p}\big((-1)^{k}2\cos(k\pi/q)\big)^{qn+2t+1} \\
&\quad = 2^{qn+2t+1} + 2s_{qn+2t+1};
\end{aligned}$$

and

$$\begin{aligned}
q\sum_{k=0}^{n}&\binom{qn+2t}{qk+t+p+1}\\
&= 2^{qn+2t} + \sum_{k=1}^{p}(-1)^{nk}2\cos\big(q+1)k\pi/q\big)\big(2\cos(k\pi/q)\big)^{qn+2t}\\
&= 2^{qn+2t} + \sum_{k=1}^{p}(-1)^{qnk}(-1)^{k}2\cos(k\pi/q)\big(2\cos(k\pi/q)\big)^{qn+2t}\\
&= 2^{qn+2t} + \sum_{k=1}^{p}\big((-1)^{k}2\cos(k\pi/q)\big)^{qn+2t+1}\\
&= 2^{qn+2t} + s_{qn+2t+1}.
\end{aligned}$$

This completes the proof. □

Theorem 4 gives us two ways to represent each term of the sequence $\{s_m\}$ for $m \geq 1$. Given a positive integer m, we write $m = qn+r$, where $n \geq 0$ and $0 \leq r < q$, and use the appropriate binomial sum for the given remainder r to find the value of s_m.

For the simplest set of results, we use these binomial sums to represent the values of s_n for $0 \leq n < 2q$ and compare the results with those obtained in Theorem 2. For $0 \leq m \leq p$, we have

$$s_{2m} = s_{q\cdot 0+2m} = \frac{q}{2}\sum_{k=0}^{0}\binom{2m}{qk+m} - 2^{2m-1} = \frac{q}{2}\binom{2m}{m} - 2^{2m-1}.$$

For $p+1 \leq m \leq 2p$, we have (using $t = m-p-1$ and noting that the inequality $q+m > 2m-1$ is valid since $m < q$)

$$\begin{aligned}
s_{2m} = s_{q+2m-q} &= q\sum_{k=0}^{1}\binom{q+2m-q-1}{qk+m} - 2^{q+2m-q-1}\\
&= q\sum_{k=0}^{1}\binom{2m-1}{qk+m} - 2^{2m-1}\\
&= q\binom{2m-1}{m} - 2^{2m-1}\\
&= \frac{q}{2}\binom{2m}{m} - 2^{2m-1}.
\end{aligned}$$

We have thus shown that

$$s_{2m} = \frac{q}{2}\binom{2m}{m} - 2^{2m-1}$$

for all values of m that satisfy $0 \le m \le 2p$, the same values listed in Theorem 2. This gives the even terms $s_0, s_2, s_4, \ldots, s_{2q-2}$.

Moving now to the odd terms, for $0 \le m \le p-1$, we have

$$s_{2m+1} = q\sum_{k=0}^{0}\binom{2m}{qk+m+p+1} - 2^{2m} = -2^{2m}$$

while for $p < m \le q-1$, we have

$$\begin{aligned}
s_{2m+1} = s_{q+2(m-p)} &= q\sum_{k=0}^{1}\binom{q+2(m-p)-1}{qk+m-p} - 2^{q+2(m-p)-1} \\
&= q\sum_{k=0}^{1}\binom{2m}{qk+m-p} - 2^{2m} \\
&= q\binom{2m}{m-p} + q\binom{2m}{q+m-p} - 2^{2m} \\
&= q\binom{2m}{m-p} + q\binom{2m}{m-p-1} - 2^{2m} \\
&= q\binom{2m+1}{m-p} - 2^{2m}.
\end{aligned}$$

For our final value when $m = p$, we see that Theorem 4 gives

$$s_{2p+1} = s_q = \frac{q}{2}\sum_{k=0}^{1}\binom{q}{qk} - 2^{q-1} = q - 2^{q-1}.$$

Note that letting $m = p$ in the previous equation generates this same value. These last computations give the odd terms $s_1, s_3, s_5, \ldots, s_{2q-1}$. Once again, these values are consistent with those listed in Theorem 2.

5. A Specific Example

Suppose that $q = 2p + 1$, where p is a positive integer, and consider the binomial sums

$$q \sum_{k=0}^{n} \binom{qn+t}{qk+r},$$

where t and r are nonnegative integers that are less than q. There are q^2 sums in this collection and each one of them is related to terms of the corresponding sequence $\{s_n\}$. To determine all of these sums, it is sufficient to know just p of them. We illustrate this fact with the case $q = 11$. The basic binomial identity for this value of q is

$$11 \sum_{k=0}^{n} \binom{11n+t}{11k+r} = 2^{11n+t} + \sum_{k=1}^{5} (-1)^{nk} 2\cos\Big(\frac{(t-2r)k\pi}{11}\Big)\Big(2\cos\Big(\frac{k\pi}{11}\Big)\Big)^{11n+t}$$

and the terms of the sequence $\{s_n\}$ are given by $s_n = u^n + v^n + x^n + y^n + z^n$, where (using form (2) for s_n)

$$\begin{aligned} u &= -2\cos(\pi/11);\\ v &= 2\cos(2\pi/11);\\ x &= -2\cos(3\pi/11);\\ y &= 2\cos(4\pi/11);\\ z &= -2\cos(5\pi/11). \end{aligned}$$

For the simplest binomial sum, we find that (use form (2) for s_n)

$$\begin{aligned} 11 \sum_{k=0}^{n} \binom{11n}{11k} &= 2^{11n} + \sum_{k=1}^{5} (-1)^{nk} 2\big(2\cos(k\pi/11)\big)^{11n}\\ &= 2^{11n} + 2\sum_{k=1}^{5} \big((-1)^k 2\cos(k\pi/11)\big)^{11n}\\ &= 2^{11n} + 2s_{11n}. \end{aligned}$$

This result also follows directly from Theorem 4. For the next sum, we first note that

$$\begin{aligned} v &= 2\cos(2\pi/11) = 2\big(2\cos^2(\pi/11) - 1\big) = u^2 - 2;\\ y &= 2\cos(4\pi/11) = 2\big(2\cos^2(2\pi/11) - 1\big) = v^2 - 2;\\ z &= 2\cos(6\pi/11) = 2\big(2\cos^2(3\pi/11) - 1\big) = x^2 - 2;\\ x &= 2\cos(8\pi/11) = 2\big(2\cos^2(4\pi/11) - 1\big) = y^2 - 2;\\ u &= 2\cos(10\pi/11) = 2\big(2\cos^2(5\pi/11) - 1\big) = z^2 - 2. \end{aligned}$$

It then follows that

$$\begin{aligned}
11\sum_{k=0}^{n}\binom{11n}{11k+10} &= 11\sum_{k=0}^{n}\binom{11n}{11k+1}\\
&= 2^{11n} + \sum_{k=1}^{5} 2\cos(2k\pi/11)\big((-1)^k 2\cos(k\pi/11)\big)^{11n}\\
&= 2^{11n} + vu^{11n} + yv^{11n} + zx^{11n} + xy^{11n} + uz^{11n}\\
&= 2^{11n} + (u^2-2)u^{11n} + (v^2-2)v^{11n} + (x^2-2)x^{11n}\\
&\qquad + (y^2-2)y^{11n} + (z^2-2)z^{11n}\\
&= 2^{11n} + s_{11n+2} - 2s_{11n}.
\end{aligned}$$

For the next sum, we use the trigonometric identity

$$2\cos(4\theta) = 4\cos^2(2\theta) - 2 = (4\cos^2\theta - 2)^2 - 2 = (2\cos\theta)^4 - 4(2\cos\theta)^2 + 2$$

to find that

$$\begin{aligned}
y &= 2\cos(4\pi/11) = u^4 - 4u^2 + 2;\\
x &= 2\cos(8\pi/11) = v^4 - 4v^2 + 2;\\
u &= 2\cos(12\pi/11) = x^4 - 4x^2 + 2;\\
z &= 2\cos(16\pi/11) = y^4 - 4y^2 + 2;\\
v &= 2\cos(20\pi/11) = z^4 - 4z^2 + 2.
\end{aligned}$$

It then follows that

$$\begin{aligned}
11\sum_{k=0}^{n}\binom{11n}{11k+9} &= 11\sum_{k=0}^{n}\binom{11n}{11k+2}\\
&= 2^{11n} + \sum_{k=1}^{5} 2\cos(4k\pi/11)\big((-1)^k 2\cos(k\pi/11)\big)^{11n}\\
&= 2^{11n} + yu^{11n} + xv^{11n} + ux^{11n} + zy^{11n} + vz^{11n}\\
&= 2^{11n} + (u^4-4u^2+2)u^{11n} + (v^4-4v^2+2)v^{11n}\\
&\qquad + (x^4-4x^2+2)x^{11n} + (y^4-4y^2+2)y^{11n}\\
&\qquad + (z^4-4z^2+2)z^{11n}\\
&= 2^{11n} + s_{11n+4} - 4s_{11n+2} + 2s_{11n}.
\end{aligned}$$

Skipping the $r = 3$ sum for the moment, we use the trigonometric identity

$$2\cos(3\theta) = 8\cos^3(2\theta) - 6\cos\theta = (2\cos\theta)^3 - 3(2\cos\theta)$$

to obtain

$$\begin{aligned}
x &= -2\cos(3\pi/11) = u^3 - 3u;\\
z &= 2\cos(6\pi/11) = v^3 - 3v;\\
v &= -2\cos(9\pi/11) = x^3 - 3x;\\
u &= 2\cos(12\pi/11) = y^3 - 3y;\\
y &= -2\cos(15\pi/11) = z^3 - 3z;
\end{aligned}$$

and thus

$$\begin{aligned}
11\sum_{k=0}^{n}\binom{11n}{11k+7} &= 11\sum_{k=0}^{n}\binom{11n}{11k+4}\\
&= 2^{11n} + \sum_{k=1}^{5} 2\cos(8k\pi/11)\big((-1)^k 2\cos(k\pi/11)\big)^{11n}\\
&= 2^{11n} + xu^{11n} + zv^{11n} + vx^{11n} + uy^{11n} + yz^{11n}\\
&= 2^{11n} + (u^3-3u)u^{11n} + (v^3-3v)v^{11n} + (x^3-3x)x^{11n}\\
&\qquad + (y^3-3y)y^{11n} + (z^3-3z)z^{11n}\\
&= 2^{11n} + s_{11n+3} - 3s_{11n+1}.
\end{aligned}$$

The next identity is much easier:

$$\begin{aligned}
11\sum_{k=0}^{n}\binom{11n}{11k+6} &= 11\sum_{k=0}^{n}\binom{11n}{11k+5}\\
&= 2^{11n} + \sum_{k=1}^{5} 2\cos(10k\pi/11)\big((-1)^k 2\cos(k\pi/11)\big)^{11n}\\
&= 2^{11n} + uu^{11n} + vv^{11n} + xx^{11n} + yy^{11n} + zz^{11n}\\
&= 2^{11n} + s_{11n+1}.
\end{aligned}$$

We can use another trigonometric identity (involving $\cos(5\theta)$) to find that

$$11\sum_{k=0}^{n}\binom{11n}{11k+8} = 11\sum_{k=0}^{n}\binom{11n}{11k+3} = 2^{11n} + s_{11n+5} - 5s_{11n+3} + 5s_{11n+1}.$$

However, we illustrate another approach using the fact that the sum of all eleven of these binomial sums is $11\cdot 2^{11n}$. Calling the above sum S (the

$r = 3$ case) and using all of the other sums, we find that

$$\begin{aligned} 11 \cdot 2^{11n} &= \left(2^{11n} + 2s_{11n}\right) + 2\left(2^{11n} + s_{11n+2} - 2s_{11n}\right) \\ &\quad + 2\left(2^{11n} + s_{11n+4} - 4s_{11n+2} + 2s_{11n}\right) + 2S \\ &\quad + 2\left(2^{11n} + s_{11n+3} - 3s_{11n+1}\right) + 2\left(2^{11n} + s_{11n+1}\right); \\ 2 \cdot 2^{11n} &= 2s_{11n+4} + 2s_{11n+3} - 6s_{11n+2} - 4s_{11n+1} + 2s_{11n} + 2S; \\ S &= 2^{11n} - s_{11n+4} - s_{11n+3} + 3s_{11n+2} + 2s_{11n+1} - s_{11n}. \end{aligned}$$

We have thus obtained two different expressions for the sum S. It must be the case that

$$s_{11n+5} - 5s_{11n+3} + 5s_{11n+1} = -s_{11n+4} - s_{11n+3} + 3s_{11n+2} + 2s_{11n+1} - s_{11n}$$

or

$$s_{11n+5} = -s_{11n+4} + 4s_{11n+3} + 3s_{11n+2} - 3s_{11n+1} - s_{11n}.$$

This equation is exactly the recurrence relation identified earlier in the paper for the $q = 11$ case. Hence, the two expressions are equal.

We have thus determined the following eleven binomial sums:

$$\sum_{k=0}^{n} \binom{11n}{11k} = \frac{1}{11}\left(2^{11n} + 2s_{11n}\right);$$

$$\sum_{k=0}^{n} \binom{11n}{11k+10} = \sum_{k=0}^{n} \binom{11n}{11k+1} = \frac{1}{11}\left(2^{11n} + s_{11n+2} - 2s_{11n}\right);$$

$$\sum_{k=0}^{n} \binom{11n}{11k+9} = \sum_{k=0}^{n} \binom{11n}{11k+2} = \frac{1}{11}\left(2^{11n} + s_{11n+4} - 4s_{11n+2} + 2s_{11n}\right);$$

$$\sum_{k=0}^{n} \binom{11n}{11k+8} = \sum_{k=0}^{n} \binom{11n}{11k+3} = \frac{1}{11}\left(2^{11n} + s_{11n+5} - 5s_{11n+3} + 5s_{11n+1}\right);$$

$$\sum_{k=0}^{n} \binom{11n}{11k+7} = \sum_{k=0}^{n} \binom{11n}{11k+4} = \frac{1}{11}\left(2^{11n} + s_{11n+3} - 3s_{11n+1}\right);$$

$$\sum_{k=0}^{n} \binom{11n}{11k+6} = \sum_{k=0}^{n} \binom{11n}{11k+5} = \frac{1}{11}\left(2^{11n} + s_{11n+1}\right).$$

Note that we needed to find just five of the sums in order to determine all eleven of them. For the remaining 110 sums, we have two options. We can either use the general formula and trigonometric identities or we can use properties of binomial coefficients. We have illustrated how to use the first approach. For two simple examples of the second approach, we note that

$$\begin{aligned}
11\sum_{k=0}^{n}\binom{11n+1}{11k+10} &= 11\sum_{k=0}^{n}\binom{11n+1}{11k+2}\\
&= 11\sum_{k=0}^{n}\binom{11n}{11k+2} + 11\sum_{k=0}^{n}\binom{11n}{11k+1}\\
&= \left(2^{11n} + s_{11n+4} - 4s_{11n+2} + 2s_{11n}\right)\\
&\quad + \left(2^{11n} + s_{11n+2} - 2s_{11n}\right)\\
&= 2^{11n+1} + s_{11n+4} - 3s_{11n+2};\\
11\sum_{k=0}^{n}\binom{11n+1}{11k+6} &= 11\sum_{k=0}^{n}\binom{11n}{11k+6} + 11\sum_{k=0}^{n}\binom{11n}{11k+5}\\
&= 2^{11n+1} + 2s_{11n+1}.
\end{aligned}$$

Hence, we are able to obtain all 121 of these binomial identities for $q = 11$ in various ways. Each of these binomial sums corresponds to a combination of terms from the sequence $\{s_n\}$.

6. Some Final Thoughts

We have specifically illustrated the sequence $\{s_n\}$ and its relationship to sums of binomial coefficients when $q = 11$. In the introduction, we made note of one such sum that appears when $q = 5$ and directly involves Lucas numbers. The reader may wish to fill in the details for this case and determine all 25 of these sums. As q increases, the recurrence relation becomes more involved and the values of some of the binomial sums require more terms of the corresponding sequence $\{s_n\}$. Although tedious at times, it is interesting to write out some of the recurrence relations and their relationships to sums of binomial coefficients for various small values of q. The paper [1] works with the sequence corresponding to $q = 7$, focusing on its connections with permutation properties.

As mentioned earlier, the papers [2] and [6] consider the sequences $\{s_n\}$ and/or $\{(-1)^n s_n\}$. They then apply these sequences to a certain class of permutation polynomials in a finite field. The paper [6] uses Dickson polynomials to obtain recurrence relations for sequences that generalize our sequences

$\{s_n\}$. Finally, some of the sequences $\{s_n\}$ appear in the On-Line Encyclopedia of Integer Sequences (see [5]). For instance, the sequence A094649 corresponds to $q = 9$, the sequence A094650 corresponds to $q = 11$, and the sequence A216605 corresponds to $q = 13$. For larger values of q, most of the sequences $\{s_n\}$ are not listed in this encyclopedia.

In this paper, we have focused on sums of binomial coefficients for the case in which q is odd. The situation is more complicated when q is even. We make a few observations about this case, leaving further exploration of this topic to the interested reader. Suppose that $q = 2p + 2$, where p is a positive integer. Modifying the derivation of our basic binomial identity, it can be shown that

$$q\sum_{k=0}^{n}\binom{qn+t}{qk+r} = 2^{qn+t} + \sum_{k=1}^{p}(-1)^{nk}2\cos\big((t-2r)k\pi/q\big)\big(2\cos(k\pi/q)\big)^{qn+t}.$$

Two special cases are

$$q\sum_{k=0}^{n}\binom{qn+2r}{qk+r} = 2^{qn+2r} + 2\sum_{k=1}^{p}(-1)^{nk}\big(2\cos(k\pi/q)\big)^{qn+2r};$$
$$q\sum_{k=0}^{n}\binom{qn+2r+1}{qk+r} = 2^{qn+2r+1} + \sum_{k=1}^{p}(-1)^{nk}\big(2\cos(k\pi/q)\big)^{qn+2r+2}.$$

Notice that the (-1) term cannot be combined with the cosine terms since the exponents $qn + 2r$ and $qn + 2r + 2$ are even. For this reason, and others as well, it is not as straightforward to define a corresponding sequence $\{s_n\}$ when q is even.

We make one final observation, using our $q = 11$ case for the sake of illustration. Let $\{s_n\}$ be the sequence that corresponds to this value of q. Starting with the sum (see the end of Section 5)

$$11\sum_{k=0}^{n}\binom{11n+1}{11k+6} = 2^{11n+1} + 2s_{11n+1},$$

we find that

$$\begin{aligned}
2^{22n+1} + 2s_{22n+1} &= 11 \sum_{k=0}^{2n} \binom{22n+1}{11k+6} \\
&= 11 \sum_{k=0}^{n} \binom{22n+1}{22k+6} + 11 \sum_{k=0}^{n} \binom{22n+1}{22k+17} \\
&= 22 \sum_{k=0}^{n} \binom{22n+1}{22k+6}.
\end{aligned}$$

Relationships like this occur whenever the value of q is twice an odd number. For example, using several of the $q = 5$ results, it can be shown that

$$\begin{aligned}
10 \sum_{k=0}^{n} \binom{10n+1}{10k+8} &= 10 \sum_{k=0}^{n} \binom{10n+1}{10k+3} = 2^{10n+1} - 2\ell_{10n+1}; \\
10 \sum_{k=0}^{n} \binom{10n+3}{10k+9} &= 10 \sum_{k=0}^{n} \binom{10n+3}{10k+4} = 2^{10n+3} - 2\ell_{10n+3}; \\
10 \sum_{k=0}^{n} \binom{10n+5}{10k+5} &= 10 \sum_{k=0}^{n} \binom{10n+5}{10k} = 2^{10n+5} - 2\ell_{10n+5}; \\
10 \sum_{k=0}^{n} \binom{10n+7}{10k+6} &= 10 \sum_{k=0}^{n} \binom{10n+7}{10k+1} = 2^{10n+7} - 2\ell_{10n+7}; \\
10 \sum_{k=0}^{n} \binom{10n+9}{10k+7} &= 10 \sum_{k=0}^{n} \binom{10n+9}{10k+2} = 2^{10n+9} - 2\ell_{10n+9}.
\end{aligned}$$

We list these results for $q = 10$ because they are directly related to Lucas numbers and seem particularly interesting. However, as mentioned above, we will not pursue the case in which q is even any further.

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