COMPOSITIONS THAT ARE PALINDROMIC MODULO m

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Abstract

In recent work, G. E. Andrews and G. Simay prove a surprising relation involving parity palindromic compositions, and ask whether a combinatorial proof can be found. We extend their results to a more general class of compositions that are palindromic modulo m, that includes the parity palindromic case when $m = 2$. We provide combinatorial proofs for the cases $m = 2$ and $m = 3$.

1. Introduction

Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ be a sequence of positive integers such that $\sum \sigma_i = n$. The sequence σ is called a *composition* of n of length k. The numbers σ_i are called the *parts* of the composition. If $\sigma_i = \sigma_{k-i+1}$ for all i, then σ is called a *palindromic* composition. If instead σ satisfies the weaker condition that $\sigma_i \equiv \sigma_{k-i+1}$ modulo m for some $m \geq 1$ and all i, then σ is said to be *palindromic modulo* m. Let $pc(n, m)$ be the number of compositions of n that are palindromic modulo m.

Andrews and Simay [2] have shown that $pc(1, 2) = 1$, and

$$pc(2n, 2) = pc(2n + 1, 2) = 2 \cdot 3^{n-1}$$

for $n > 1$. Our main theorem generalizes their result by giving an ordinary generating function for $pc(n, m)$ for all m. Of particular interest is that $pc(1, 3) = 1$, and

$$pc(n, 3) = 2 \cdot f(n - 1)$$

for $n > 1$, where $f(n)$ is the nth Fibonacci number (here $f(1) = f(2) = 1$). We then give a combinatorial proof of the above identities for $pc(n, 2)$ and $pc(n, 3)$, and conclude with some general properties and asymptotic analysis of $pc(n, m)$ for larger m.

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Theorem 1. *We have*

$$F_m(q) := \sum_{n\geq 1} pc(n,m)q^n = \frac{q+2q^2-q^{m+1}}{1-2q^2-q^m}.$$

Remark. Though we mean this to be a formal definition, it can be verified that the series converges for all $|q| < 1/2$. Indeed,

$$|F_m(q)| \leq F_1(|q|) = \frac{|q|}{1-2|q|}.$$

Proof of Theorem 1. Notice that to form a composition that is palindromic modulo m, we need to form a sequence of pairs of positive integers whose difference is a multiple of m. Furthermore, there may or may not be a central part that will always be congruent to itself. If $G_m(q)$ is the ordinary generating function for pairs of positive integers whose difference is a multiple of m, then

$$F_m(q) = \frac{G_m(q)}{1-G_m(q)} \cdot \frac{1}{1-q} + \frac{q}{1-q}.$$

Now for the function $G_m(q)$, either the pair of positive integers is the same, or we must choose to add a multiple of m to either the first or second number. Therefore

$$G_m(q) = \frac{q^2}{1-q^2} + \frac{2q^2}{1-q^2} \cdot \frac{q^m}{1-q^m}.$$

Simplifying gives the result. $\square$

When $m = 1$, every composition will be palindromic modulo m. Therefore $pc(n,1) = 2^{n-1}$, as is evident from $F_1(q)$. We next consider the case when $m = 2$.

2. The Case $m = 2$

When $m = 2$, Theorem 1 gives

$$F_2(q) = \frac{2q^3(1+q)}{1-3q^2},$$

and after expanding we see that $p(1,2) = 1$, and $p(2n,2) = p(2n+1,2) = 2 \cdot 3^{n-1}$. Our goal in this section will be to give a combinatorial proof of this formula.

For $n \geq 1$, we start by considering the 3^{n-1} elements of the set $\{0,1,2\}^{n-1}$. Our goal will be to embed two disjoint copies of this set into the set of compositions of $2n$ that are palindromic modulo 2, and show that every composition of $2n$ that is palindromic modulo 2 has a preimage. We then give a bijection between compositions of $2n$ that are palindromic modulo 2 and compositions of $2n+1$ that are palindromic modulo 2.

As an intermediate step, we define a one-to-two mapping from the elements of the set $\{0, 1, 2\}^{n-1}$ to a special set of sequences of triples which we define below.

Definition 1. Let $n \geq 1$. Define an n-triple sequence to be a sequence of triples (b_i, c_i, d_i) such that the following hold.

1. b_i is a positive integer for all i.

2. c_i is a nonnegative integer for all i.

3. $\sum b_i + \sum c_i = n$.

4. $d_i \in \{0, 1\}$ for all i.

5. If k is the number of triples in the sequence, then for all $i < k$ with $c_i = 0$ we have $d_i = 0$.

Lemma 1. *Let $n \geq 1$. Then there is a one-to-two and onto mapping between the set $\{0, 1, 2\}^{n-1}$ and the set of n-triple sequences. In particular, we have that the number of n-triple sequences is $2 \cdot 3^{n-1}$.*

Proof. For $n \geq 1$, let

$$a = (a_1, a_2, \ldots, a_{n-1}) \in \{0, 1, 2\}^{n-1}.$$

As an example, we take $a = (0, 1, 2, 2, 1) \in \{0, 1, 2\}^5$.

`Step 1:` Initialize the process with the triple $(b_1, c_1, d_1) = (1, 0, 0)$. This triple will always serve as the starting point for the algorithm. If $n = 1$, then we are done. Now if $a_1 = 0$, we create a new triple $(b_2, c_2, d_2) = (1, 0, 0)$. If $a_1 = 1$, then we increase b_1 by one. If $a_1 = 2$, then we increase c_1 by one. In our example of $(0, 1, 2, 2, 1)$, we see that $a_1 = 0$, so after the first step we have the following list of triples:

$$(1, 0, 0), \quad (1, 0, 0).$$

`Step 2:` Now for $i > 1$, we define the instructions given by each a_i recursively. We assume that at this step we have a total of j triples, for some $j \geq 1$.

1. If $a_i = 0$, then we create the new triple $(b_{j+1}, c_{j+1}, d_{j+1}) = (1, 0, 0)$.

2. If $a_i = 2$, then increase c_j by one.

3. If $a_i = 1$ and $a_{i-1} \neq 2$, then increase b_j by one.

4. If $a_i = 1$ and $a_{i-1} = 2$, then set $d_j = 1$ and create the new triple $(b_{j+1}, c_{j+1}, d_{j+1}) = (1, 0, 0)$.

For our example $(0, 1, 2, 2, 1)$, we form the list of triples

$$(1, 0, 0), \quad (2, 2, 1), \quad (1, 0, 0).$$

Now $\sum b_j + \sum c_j = n$ since we started with $b_1 = 1$ and either a b_j or a c_j was increased by one for each element of the sequence a. Furthermore, Rules (2) and (4) in Step 2 ensure that $d_j = 0$ when $c_j = 0$. Therefore, our algorithm transforms the sequence a into an n-triple sequence.

After performing this algorithm on each of the 3^{n-1} sequences in $\{0, 1, 2\}^{n-1}$, we have a set of 3^{n-1} n-triple sequences. By construction, all of the n-triple sequences are unique. For each sequence $a \in \{0, 1, 2\}^{n-1}$, we denote by k_a the number of triples in the corresponding n-triple sequence.

`Step 3:` We create a second n-triple sequence by setting $d_{k_a} = 1$ for all a. For our example $(0, 1, 2, 2, 1)$, we get a second 4-sequence triple:

$$(1, 0, 0), \quad (2, 2, 1), \text{ and } (1, 0, 1).$$

We now have $2 \cdot 3^{n-1}$ n-triple sequences, all of which are distinct. It is straightforward to verify that this algorithm can be reversed by pairing up n-triple sequences that differ only by the final element of the final triple, and adapting the rules described in Step 2 (see Example 2). □

Combinatorial proof that $pc(2n, 2) = 2 \cdot 3^{n-1}$. By Lemma 1, we know that there are $2 \cdot 3^{n-1}$ n-triple sequences. Therefore, we need to define a bijection between the set of n-triple sequences and the compositions of $2n$ that are palindromic modulo 2.

`Step 4:` Let $(b_1, c_1, d_1), \ldots, (b_{k_a}, c_{k_a}, d_{k_a})$ be an n-triple sequence. We show how to form a unique composition σ of $2n$ that is palindromic modulo 2. Start with the triple $(b_{k_a}, c_{k_a}, d_{k_a})$.

1. If $d_{k_a} = 0$ and $c_{k_a} = 0$, then set $\sigma_{k_a} = 2b_{k_a}$.
2. If $d_{k_a} = 0$ and $c_{k_a} > 0$, then set $\sigma_{k_a} = b_{k_a} + 2c_{k_a}$ and $\sigma_{k_a+1} = b_{k_a}$.
3. If $d_{k_a} = 1$, then set $\sigma_{k_a} = b_{k_a}$ and $\sigma_{k_a+1} = b_{k_a} + 2c_{k_a}$.

These cases will determine whether the length of the composition is even or odd. If $d_{k_a} = 0$ and $c_{k_a} = 0$, then the length of σ will be $2k_a - 1$. Otherwise, the length of σ will be $2k_a$. For convenience we refer to the length of σ as k in either case. For our example $(0, 1, 2, 2, 1)$, we have initialized the two compositions

$$(0, 0, 2, 0, 0) \text{ and } (0, 0, 1, 1, 0, 0).$$

`Step 5:` Now if $k_a = 1$ we are done. Assuming $k_a > 1$, for each $1 \leq j < k_a$ we consider the triple (b_j, c_j, d_j). If $d_j = 0$, then set $\sigma_j = b_j + 2c_j$ and $\sigma_{k-j+1} = b_j$.

If $d_j = 1$, then set $\sigma_j = b_j$ and $\sigma_{k-j+1} = b_j + 2c_j$. Also notice if $d_j = 1$, then by construction $c_j > 0$, so that $b_j \neq b_j + 2c_j$. This ensures that each sequence of triples will be associated to a unique composition. Also notice that the sum of the parts in σ equals $2\sum b_j + 2\sum c_j = 2n$, and that $|\sigma_j - \sigma_{k-j+1}|$ is even for all j. Thus σ is a composition of $2n$ that is palindromic modulo 2. Showing that every composition of $2n$ that is palindromic modulo 2 can be formed from an n-triple sequence is done by reversing the described algorithm (see Example 2). Returning to our example sequence $(0, 1, 2, 2, 1)$, we form the two compositions

$$(1, 2, 1, 1, 6, 1) \text{ and } (1, 2, 2, 6, 1),$$

each of which is a composition of 12 that is palindromic modulo 2. To aid the reader, we have performed the algorithm on the sequences in $\{0, 1, 2\}^3$ $(n = 4)$ in Table 2. □

We conclude by walking through the steps of the algorithm on a specific sequence to illustrate when the composition has even or odd length, and, in the case of odd length, how the central part of the composition is formed. Then, we show an example of how the algorithm is reversed starting with a composition of $2n$ that is palindromic modulo 2.

Example 1. The sequence $(2, 1, 1)$. Here $n = 4$, and so our algorithm will construct two compositions of 8 that are palindromic modulo 2.

`Step 1:` Start with the triple $(1, 0, 0)$. Since $a_1 = 2$, we increase the middle entry of this triple by 1, resulting in the triple $(1, 1, 0)$.

`Step 2:` Let $i = 2$. Then $a_2 = 1$ and $a_1 = 2$, so we set the last entry of the triple to 1, resulting in the triple $(1, 1, 1)$ and create a second triple $(1, 0, 0)$. Now let $i = 3$. Then $a_3 = 1$ and $a_2 \neq 2$, so we increase the first entry of the second triple by 1. We now have two triples,

$$(1, 1, 1) \text{ and } (2, 0, 0).$$

`Step 3:` We also have the following two triples by setting the last entry of the last triple to be equal to 1:

$$(1, 1, 1) \text{ and } (2, 0, 1).$$

Both of the sequences of triples we created are 4-triple sequences.

`Step 4:` For the first 4-triple sequence (found in Step 2), we have that in the second (last) triple the middle and last entries are both equal to 0, so the length of the composition will be 3. The second (last) triple will determine the middle part of the composition, which in this case is equal to 4 (twice the first entry of the last triple). The middle part of the composition will always be even since the composition we are forming is of an even number and the other parts come in pairs of the same parity.

For the second 4-triple sequence (found in Step 3), we have that the last entry of the last triple is equal to 1, so the length of the composition will be 4 twice the number of triples). The last triple determines the middle two parts of the compositions (the second part is equal to 2, the first entry of the last triple; and the third part is also equal to 2, the first entry of the last triple plus twice the middle entry of the last triple).

`Step 5:` For each of the remaining triples, we create a pair of parts extending from the middle of the composition outward. For the triples found in Step 2 and Step 3 the first triple is the same, and the last entry is equal to 1. Thus the first and last part of the compositions will be equal to 1 (the first entry of the triple) and the last part will be equal to 3 (the first entry of the triple plus twice the second entry of the triple). Starting with the sequence $(2, 1, 1)$ we have now formed the two compositions

$$(1, 4, 3) \text{ and } (1, 2, 2, 3),$$

both of which are compositions of 8 that are palindromic modulo 2.

Example 2. The composition $(1, 1, 2, 1, 3)$. This is a composition of 8, so we have $n = 4$. We will construct a sequence in $\{0, 1, 2\}^3$.

The length of the composition is 5, so we will form a 4-triple sequence consisting of three triples; the first triple for the first and fifth part, the second triple for the second and fourth part, and the third triple for the middle part. We begin with a summary of how the triples determine the parts of the composition.

Other than the last triple, the first element of each triple gives the smaller of the pairs of parts, the second element of the triple gives half the difference between the two parts, and the last element of the triple determines whether the larger (last element 0) or smaller (last element 1) part comes first. The last triple works the same as the other triples, except when the second element is 0. In this case, the last element determines whether there is a central part (last element 0) or a pair of equal central parts (last element 1). Now we return to the composition in our example.

Since the fifth part is greater than the first part, we set $d_1 = 1$. Furthermore, the absolute value of the difference of the first and fifth part is 2, so set $c_1 = 1$ (half of the difference). Finally, we set $b_1 = 1$, since the first part is equal to 1 and is the smaller of the two parts.

Since the second and fourth parts are equal, we set $c_2 = d_2 = 0$. Since both parts are equal to 1, we set $b_2 = 1$.

Since we have a central part, we set $c_3 = 0$ and $d_3 = 0$. Since the central part is 2, we set $b_3 = 1$.

We have now formed the 4-triple sequence $(1, 1, 1), (1, 0, 0), (1, 0, 0)$. We now reverse the rules described in Step 2. Since the last triple is $(1, 0, 0)$, we must have $a_3 = 0$. Since the first triple ends in 1 and the second triple is $(1, 0, 0)$, we must have $a_2 = 2$ and $a_1 = 1$. Therefore, the corresponding sequence is $(2, 1, 0)$.

Combinatorial proof that $pc(2n, 2) = pc(2n + 1, 2)$. We split the proof into two cases. Starting with a composition σ of $2n$ that is palindromic modulo 2, suppose the length of σ is $2k + 1$. Then adding one to σ_{k+1} gives a composition of $2n + 1$ that is still palindromic modulo 2. Now if the length of σ is $2k$, form a composition σ' of $2n + 1$ of length $2k + 1$ by setting

$$\sigma'_j = \begin{cases} \sigma_j & 1 \leq j \leq k \\ 1 & j = k + 1 \\ \sigma_{j-1} & k + 2 \leq j \leq 2k + 1. \end{cases}$$

We note that σ' is still palindromic modulo 2. It is straightforward to verify that this map is a bijection. For our examples of $(1, 2, 2, 6, 1)$ and $(1, 2, 1, 1, 6, 1)$ that are compositions of 12, we form the two compositions of 13 given by $(1, 2, 1, 1, 1, 6, 1)$ and $(1, 2, 3, 6, 1)$, each of which is palindromic modulo 2. □

3. The Case $m = 3$

When $m = 3$, Theorem 1 gives

$$F_3(q) = \frac{q + 2q^2 - q^4}{1 - 2q^2 - q^3} = q + \frac{2q^2}{1 - q - q^2},$$

and expanding this function shows $pc(1, 3) = 1$ and $pc(n, 3) = 2 \cdot f(n-1)$ for $n > 1$. Our goal of this section is to give a combinatorial proof of this formula.

It is well-known [1] that $f(n+1)$ is equal to the number of compositions of n with parts that are equal to 1 or 2. For $n > 1$, our goal will be to embed two disjoint copies of the compositions of $n - 2$ with parts equal to 1 or 2 into the compositions of n that are palindromic modulo 3. Then we will show each composition of n that is palindromic modulo 3 has a preimage.

Combinatorial proof that $pc(n, 3) = 2 \cdot f(n - 1)$. For $n > 1$, let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ be a composition of $n - 2$, where each $\sigma_i \in \{1, 2\}$. Form two distinct compositions of n by setting

$$\sigma' = (1, 1, \sigma_1, \sigma_2, \ldots, \sigma_k) \text{ and } \sigma'' = (2, \sigma_1, \sigma_2, \ldots, \sigma_k).$$

Doing this for each composition gives us a set of $2 \cdot f(n - 1)$ compositions of n with parts equal to 1 or 2, the set of which we will denote by A_n. For each of these compositions, we will form a unique composition of n that is palindromic modulo 3.

Let $+$ denote sequence concatenation, as in $(1,2)+(3,4)=(1,2,3,4)$. Now for any composition $a \in A_n$, we can decompose a in the following way:

$$a = a_1 + a_2 + \ldots + a_s,$$

where the substrings $\{a_i\}_{i=1}^{s}$ are determined by the following rules. We will write k_i to be the length of each substring a_i.

1. If a contains no 2's, then $a_1 = a$.
2. Suppose a begins with a 2, and this is the only 2 in a. Then $a_1 = (2)$.
3. Suppose a begins with the substring $(2,2)$ or $(2,1,1)$. Then $a_1 = (2)$.
4. Suppose a begins with the substring $(1,1)$ or $(2,1,2)$. Then a_1 terminates with the first 2 such that
   (a) it is the final 2 in a;
   (b) it is immediately followed by the substring (2) or $(1,1)$.

For $i > 1$, the substring a_i is determined recursively using these 4 rules, after first deleting the substrings a_1, a_2, $\ldots$, a_{i-1} from a. The resulting string cannot begin with the substring $(1,2)$, so it still makes sense to apply these rules. To aid the reader, Table 3 shows the decomposition of the compositions in the set A_8. As an example, take the composition $a = (1,1,1,2,2,1,1,2,1,2,1,1)$ in A_{16}. By applying Rule (4) we have $a_1 = (1,1,1,2)$, and we now consider the string $(2,1,1,2,1,2,1,1)$. By applying Rule (3) we have $a_2 = (2)$, and we now consider the string $(1,1,2,1,2,1,1)$. By applying Rule (4) we have $a_3 = (1,1,2,1,2)$, and we now consider the string $(1,1)$. By applying Rule (1) we have $a_4 = (1,1)$, and are now done. Thus

$$(1,1,1,2,2,1,1,2,1,2,1,1) = (1,1,1,2) + (2) + (1,1,2,1,2) + (1,1).$$

Let B_n be the set of compositions of n that are palindromic modulo 3. For each composition $a = a_1 + \ldots + a_s \in A_n$, we will show how to construct a unique $b \in B_n$. If a_s contains a 2, the length of b will be $2s$; if a_s does not contain a 2, the length of b will be $2s-1$. Either way, we will denote k_b to be the length of b and write $b = (b_1, b_2, \ldots, b_{k_b})$ (recall we have also set k_i to be the length of a_i).

Now assume a_i has o_i 1's, and t_i 2's. If $t_i = 0$ (which can only be the case for a_s), then set $b_s = k_s$. For what follows we will assume $t_i > 0$. Note that $k_i = o_i + t_i$, and

$$n = \sum_{i=1}^{s} (o_i + 2 \cdot t_i).$$

We will now form the triple (c_i, d_i, e_i) using the following rules.

Let o_i' be the number of 1's in a_i preceding the first 2. Then if

1. o'_i is even (possibly zero), set $c_i = \frac{o'_i}{2} + 1$, $d_i = 3 \cdot (t_i - 1)$, and $e_i = 0$;

2. o'_i is odd, set $c_i = \frac{o'_i - 1}{2}$, $d_i = 0$, and $e_i = 3 \cdot t_i$.

By construction this sequence of triples is unique to the composition $a \in A_n$ we began with. We now show how our example composition with decomposition $a_1 = (1,1,1,2)$, $a_2 = (2)$, $a_3 = (1,1,2,1,2)$, and $a_4 = (1,1)$ maps to three triples (the final substring a_4 contains no 2's, and therefore we set $b_4 = k_4 = 2$). The following table gives the relevant values:

a_i	o_i	o'_i	t_i	c_i	d_i	e_i
$(1,1,1,2)$	3	3	1	1	0	3
(2)	0	0	1	1	0	0
$(1,1,2,1,2)$	3	2	2	2	3	0

Therefore, we have the triples $(1,0,3), (1,0,0)$, and $(2,3,0)$.

We now form b by setting $b_i = c_i + d_i$ and $b_{k_b - i+1} = c_i + e_i$, adjusting slightly for the case when a_s has no 2's (recall that we just set $b_s = k_s$ in this case). The composition is uniquely determined from the triples, and by construction d_i and e_i are both multiples of 3. Furthermore,

$$b_i + b_{k_b - i+1} = 2c_i + d_i + e_i = o'_i + 3t_i - 1 = o_i + 2t_i.$$

To see why the last equality holds, consider first the case when $t_i = 1$. Then $o'_i = o_i$, and $3t_i - 1 = 2t_i$. Now if $t_i > 1$, we can pair each 2 in a_i with a 1 immediately preceding it. However, we have used a 1 counted by o'_1 in this pairing, so we must subtract one. Therefore, we have embedded A_n into B_n. For our example

$$a = (1,1,1,2,2,1,1,2,1,2,1,1) = (1,1,1,2) + (2) + (1,1,2,1,2) + (1,1),$$

we have the triples $(1,0,3)$, $(1,0,0)$, and $(2,3,0)$, and we obtain the composition

$$b = (1,1,5,2,2,1,4),$$

which is a composition of 16 that is palindromic modulo 3 ($b \in B_{16}$). It is straightforward to construct a composition $a \in A_n$ from a composition $b \in B_n$ by reversing this construction, which proves the result. □

4. The Case $m > 3$

If $m > 3$, the formula for $pc(n, m)$ can also be deduced from Theorem 1. For $m = 4$, Theorem 1 gives

$$F_4(q) = \frac{q + 2q^2 - q^5}{1 - 2q^2 - q^4},$$

and after expanding we see $pc(1,4)=1$, and for $n \geq 1$

$$pc(2n,4) = pc(2n+1,4) = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{\sqrt{2}}.$$

The familiar reader will recognize these numbers as twice the Pell numbers [4]. While the correct context for a combinatorial proof is not immediately clear, we pose the following question.

Question 1. Is there a combinatorial proof of the formula for $pc(n,4)$?

We also include some properties of $pc(n,m)$ that we observed for general m.

Proposition 1. *If m is even, then for $n \geq 1$ we have $pc(2n,m) = pc(2n+1,m)$.*

Proof. The proof of this is identical in spirit to the combinatorial proof that $pc(2n,2) = pc(2n+1,2)$ in Section 2. □

Proposition 2. *If $n > 1$, then $pc(n,m)$ is even for all m.*

Proof. A quick, one line proof of this fact comes from the identity

$$F_m(q) - q = \frac{2q^2(1+q)}{1-2q^2-q^m}.$$

This can also be seen combinatorially by pairing up compositions of n that are palindromic modulo m. Let a be a composition of $n > 1$ of length s that is palindromic modulo m. If $a_i \neq a_{s-i+1}$ for some i, then we can pair this composition with the composition formed by switching a_i and a_{s-i+1}.

Now assume $a_i = a_{s-i+1}$ for all i (i.e., a is a palindrome). If s is odd, than $a_{(s+1)/2}$ is even, and we can pair this composition with the composition of n of length $s+1$ formed by removing $a_{(s+1)/2}$ and appending $a_{(s+1)/2}/2$ to the beginning and the end, respectively. □

Proposition 3. *If $2n \leq m$, then $p(2n,m) = p(2n+1,m) = 2^n$.*

Proof. The condition that $2n \leq m$ requires all compositions of $2n$ and $2n+1$ that are palindromic modulo m to be palindromes. Let $pc(n,\infty)$ denote the number of palindromic compositions of n. Then it is well known [3] that $pc(n,\infty) = 2^{\lfloor n/2 \rfloor}$. □

Proposition 4. *Let α_m be the unique positive root of $1-2q^2-q^m=0$, and set*

$$c_m = \lim_{q \to \alpha_m} (1-\alpha_m^{-1}q) \cdot F_m(q) \quad \text{and} \quad d_m = \lim_{q \to \alpha_m} (1+\alpha_m^{-1}q) \cdot F_m(q).$$

If m is even, then

$$\limsup_{n\to\infty} \alpha_m^n \cdot pc(n,m) = c_m + d_m \quad \text{and} \quad \liminf_{n\to\infty} \alpha_m^n \cdot pc(n,m) = c_m - d_m.$$

If m is odd, $d_m = 0$, and thus $pc(n,m) \sim c_m \alpha_m^{-n}$ as $n \to \infty$.

Proof. This is a routine analysis of the rational function $F_m(q)$. For convenience, we set $\phi_m(q) = 1 - 2q^2 - q^m$. The asymptotics of $pc(n, m)$ are determined by the poles of $F_m(q)$, which are precisely the zeros of $\phi_m(q)$. We will assume $m > 4$, as the result can be verified directly for $m \leq 4$.

First note that $\phi_m(q)$ has exactly one positive zero, α_m, since $\phi'_m(q) < 0$ for all $q > 0$, $\phi_m(0) > 0$, and $\phi_m(1) < 0$. When m is even, $\phi_m(q)$ is an even function and thus $-\alpha_m$ is also a zero. When m is odd, $\phi_m(q)$ has a unique zero in the interval $(-1, -\alpha_m)$, since $\phi'_m(q) > 0$ on $(-1, 0)$, $\phi_m(-1) < 0$, and $\phi_m(\alpha_m) > 0$. Our next goal is to show that these are the only two zeros inside the disc $|q| < 0.9$.

Note that $q^2(2 + q^{m-2})$ has one zero (with multiplicity equal to 2) inside the disc $|q| < 0.9$. Therefore, if we can show

$$|q^2(2 + q^{m-2})| > 1$$

for all q with $|q| = 0.9$, we can apply Rouche's theorem to conclude that $\phi_m(q)$ has exactly two zeros inside the disc $|q| < 0.9$. For $m > 4$ we have $|q^{m-2}| \leq |q^3|$, so that $|2 + q^{m-2}| \geq 2 - |q^3| = 1.279$ and

$$|q^2(2 + q^{m-2})| > 0.81 \cdot 1.729 = 1.03599 > 1.$$

We can now perform a partial fraction decomposition on $F_m(q)$,

$$F_m(q) = \frac{c_m}{1 - \alpha_m^{-1} q} + \frac{d_m}{1 + \alpha_m^{-1} q} + G(q),$$

where $G(q)$ is a rational function with no poles in the disc $|q| < 0.9$. Therefore,

$$pc(n, m) = (c_m + (-1)^n d_m)\alpha_m^{-n} + o(\alpha_m^{-n})$$

as $n \to \infty$, noting that $d_m = 0$ if m is odd. The result follows. $\square$

Table 1 shows the values of α_m, c_m, and d_m for various m.

m	α_m^{-1}	c_m	d_m
1	2	$\frac{1}{2} = 0.5$	0
2	$\sqrt{3} \approx 1.73$	$\frac{3+\sqrt{3}}{9} \approx 0.53$	$\frac{3-\sqrt{3}}{9} \approx 0.14$
3	$\frac{\sqrt{5}+1}{2} \approx 1.61$	$\frac{5-\sqrt{5}}{5} = 0.55$	0
4	≈ 1.55	≈ 0.58	≈ 0.13
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	$\sqrt{2} \approx 1.41$	$\frac{2+\sqrt{2}}{4} \approx 0.85$	$\frac{2-\sqrt{2}}{4} \approx 0.14$

Table 1: The values of α_m, c_m, and d_m for various m.

$a \in \{0,1,2\}^3$	4-triple sequences	Composition of 8	Composition of 9
(0,0,0)	(1,0,0),(1,0,0),(1,0,0),(1,0,0)	(1,1,1,2,1,1,1)	(1,1,1,3,1,1,1)
	(1,0,0),(1,0,0),(1,0,0),(1,0,1)	(1,1,1,1,1,1,1,1)	(1,1,1,1,1,1,1,1,1)
(0,0,1)	(1,0,0),(1,0,0),(2,0,0)	(1,1,4,1,1)	(1,1,5,1,1)
	(1,0,0),(1,0,0),(2,0,1)	(1,1,2,2,1,1)	(1,1,2,1,2,1,1)
(0,0,2)	(1,0,0),(1,0,0),(1,1,0)	(1,1,3,1,1,1)	(1,1,3,1,1,1,1)
	(1,0,0),(1,0,0),(1,1,1)	(1,1,1,3,1,1)	(1,1,1,1,3,1,1)
(0,1,0)	(1,0,0),(2,0,0),(1,0,0)	(1,2,2,2,1)	(1,2,3,2,1)
	(1,0,0),(2,0,0),(1,0,1)	(1,2,1,1,2,1)	(1,2,1,1,1,2,1)
(0,1,1)	(1,0,0),(3,0,0)	(1,6,1)	(1,7,1)
	(1,0,0),(3,0,1)	(1,3,3,1)	(1,3,1,3,1)
(0,1,2)	(1,0,0),(2,1,0)	(1,4,2,1)	(1,4,1,2,1)
	(1,0,0),(2,1,1)	(1,2,4,1)	(1,2,1,4,1)
(0,2,0)	(1,0,0),(1,1,0),(1,0,0)	(1,3,2,1,1)	(1,3,3,1,1)
	(1,0,0),(1,1,0),(1,0,1)	(1,3,1,1,1,1)	(1,3,1,1,1,1,1)
(0,2,1)	(1,0,0),(1,1,1),(1,0,0)	(1,1,2,3,1)	(1,1,3,3,1)
	(1,0,0),(1,1,1),(1,0,1)	(1,1,1,1,3,1)	(1,1,1,1,1,3,1)
(0,2,2)	(1,0,0),(1,2,0)	(1,5,1,1)	(1,5,1,1,1)
	(1,0,0),(1,2,1)	(1,1,5,1)	(1,1,1,5,1)
(1,0,0)	(2,0,0),(1,0,0),(1,0,0)	(2,1,2,1,2)	(2,1,3,1,2)
	(2,0,0),(1,0,0),(1,0,1)	(2,1,1,1,1,2)	(2,1,1,1,1,1,2)
(1,0,1)	(2,0,0),(2,0,0)	(2,4,2)	(2,5,2)
	(2,0,0),(2,0,1)	(2,2,2,2)	(2,2,1,2,2)
(1,0,2)	(2,0,0),(1,1,0)	(2,3,1,2)	(2,3,1,1,2)
	(2,0,0),(1,1,1,)	(2,1,3,2)	(2,1,1,3,2)
(1,1,0)	(3,0,0),(1,0,0)	(3,2,3)	(3,3,3)
	(3,0,0),(1,0,1)	(3,1,1,3)	(3,1,1,1,3)
(1,1,1)	(4,0,0)	(8)	(9)
	(4,0,1)	(4,4)	(4,1,4)
(1,1,2)	(3,1,0)	(5,3)	(5,1,3)
	(3,1,1)	(3,5)	(3,1,5)
(1,2,0)	(2,1,0),(1,0,0)	(4,2,2)	(4,3,2)
	(2,1,0),(1,0,1)	(4,1,1,2)	(4,1,1,1,2)
(1,2,1)	(2,1,1),(1,0,0)	(2,2,4)	(2,3,4)
	(2,1,1),(1,0,1)	(2,1,1,4)	(2,1,1,1,4)
(1,2,2)	(2,2,0)	(6,2)	(6,1,2)
	(2,2,1)	(2,6)	(2,1,6)
(2,0,0)	(1,1,0),(1,0,0),(1,0,0)	(3,1,2,1,1)	(3,1,3,1,1)
	(1,1,0),(1,0,0),(1,0,1)	(3,1,1,1,1,1)	(3,1,1,1,1,1,1)
(2,0,1)	(1,1,0),(2,0,0)	(3,4,1)	(3,5,1)
	(1,1,0),(2,0,1)	(3,2,2,1)	(3,2,1,2,1)
(2,0,2)	(1,1,0),(1,1,0)	(3,3,1,1)	(3,3,1,1,1)
	(1,1,0),(1,1,1)	(3,1,3,1)	(3,1,1,3,1)
(2,1,0)	(1,1,1),(1,0,0),(1,0,0)	(1,1,2,1,3)	(1,1,3,1,3)
	(1,1,1),(1,0,0),(1,0,1)	(1,1,1,1,1,3)	(1,1,1,1,1,1,3)
(2,1,1)	(1,1,1),(2,0,0)	(1,4,3)	(1,5,3)
	(1,1,1),(2,0,1)	(1,2,2,3)	(1,2,1,2,3)
(2,1,2)	(1,1,1),(1,1,0)	(1,3,1,3)	(1,3,1,1,3)
	(1,1,1),(1,1,1)	(1,1,3,3)	(1,1,1,3,3)
(2,2,0)	(1,2,0),(1,0,0)	(5,2,1)	(5,3,1)
	(1,2,0),(1,0,1)	(5,1,1,1)	(5,1,1,1,1)
(2,2,1)	(1,2,1),(1,0,0)	(1,2,5)	(1,3,5)
	(1,2,1),(1,0,1)	(1,1,1,5)	(1,1,1,1,5)
(2,2,2)	(1,3,0)	(7,1)	(7,1,1)
	(1,3,1)	(1,7)	(1,1,7)

Table 2: The sequences in $\{0,1,2\}^{n-1}$ ($n=4$), mapped to two 4-triple sequences, mapped to a composition of $2n=8$ and a composition of $2n+1=9$, each of which is palindromic modulo 2.

$a \in A_8$	$a = a_1 + a_2 + \ldots + a_s$	$b \in B_8$
$(1,1,1,1,1,1,1,1)$	$a_1 = (1,1,1,1,1,1,1,1)$	(8)
$(1,1,2,1,1,1,1)$	$a_1 = (1,1,2)$ $a_2 = (1,1,1,1)$	$(2,4,2)$
$(1,1,1,2,1,1,1)$	$a_1 = (1,1,1,2)$ $a_2 = (1,1,1)$	$(1,3,4)$
$(1,1,1,1,2,1,1)$	$a_1 = (1,1,1,1,2)$ $a_2 = (1,1)$	$(3,2,3)$
$(1,1,1,1,1,2,1)$	$a_1 = (1,1,1,1,1,2)$ $a_2 = (1)$	$(2,1,5)$
$(1,1,1,1,1,1,2)$	$a_1 = (1,1,1,1,1,1,2)$	$(4,4)$
$(1,1,2,2,1,1)$	$a_1 = (1,1,2)$; $a_2 = (2)$ $a_3 = (1,1)$	$(2,1,2,1,2)$
$(1,1,2,1,2,1)$	$a_1 = (1,1,2,1,2)$ $a_2 = (1)$	$(5,1,2)$
$(1,1,2,1,1,2)$	$a_1 = (1,1,2)$ $a_2 = (1,1,2)$	$(2,2,2,2)$
$(1,1,1,2,2,1)$	$a_1 = (1,1,1,2)$ $a_2 = (2)$; $a_3 = (1)$	$(1,1,1,4)$
$(1,1,1,2,1,2)$	$a_1 = (1,1,1,2,1,2)$	$(1,7)$
$(1,1,1,1,2,2)$	$a_1 = (1,1,1,1,2)$ $a_2 = (2)$	$(3,1,1,3)$
$(1,1,2,2,2)$	$a_1 = (1,1,2)$; $a_2 = (2)$ $a_3 = (2)$	$(2,1,1,1,1,2)$
$(2,1,1,1,1,1,1)$	$a_1 = (2)$ $a_2 = (1,1,1,1,1,1)$	$(1,6,1)$
$(2,2,1,1,1,1)$	$a_1 = (2)$; $a_2 = (2)$ $a_3 = (1,1,1,1)$	$(1,1,4,1,1)$
$(2,1,2,1,1,1)$	$a_1 = (2,1,2)$ $a_2 = (1,1,1)$	$(4,3,1)$
$(2,1,1,2,1,1)$	$a_1 = (2)$ $a_2 = (1,1,2)$ $a_3 = (1,1)$	$(1,2,2,2,1)$
$(2,1,1,1,2,1)$	$a_1 = (2)$; $a_2 = (1,1,1,2)$ $a_3 = (1)$	$(1,1,1,4,1)$
$(2,1,1,1,1,2)$	$a_1 = (2)$ $a_2 = (1,1,1,1,2)$	$(1,3,3,1)$
$(2,2,2,1,1)$	$a_1 = (2)$; $a_2 = (2)$ $a_3 = (2)$; $a_4 = (1,1)$	$(1,1,1,2,1,1,1)$
$(2,2,1,2,1)$	$a_1 = (2)$; $a_2 = (2,1,2)$ $a_3 = (1)$	$(1,4,1,1,1)$
$(2,2,1,1,2)$	$a_1 = (2)$; $a_2 = (2)$ $a_3 = (1,1,2)$	$(1,1,2,2,1,1)$
$(2,1,2,2,1)$	$a_1 = (2,1,2)$; $a_2 = (2)$ $a_3 = (1)$	$(4,1,1,1,1)$
$(2,1,2,1,2)$	$a_1 = (2,1,2,1,2)$	$(7,1)$
$(2,1,1,2,2)$	$a_1 = (2)$; $a_2 = (1,1,2)$ $a_3 = (2)$	$(1,2,1,1,2,1)$
$(2,2,2,2)$	$a_1 = (2)$; $a_2 = (2)$ $a_3 = (2)$; $a_4 = (2)$	$(1,1,1,1,1,1,1,1)$

Table 3: The decomposition of each composition $a \in A_8$, and the resulting composition $b \in B_8$.

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