FIXED DIVISORS OF PRODUCTS OF POLYNOMIALS

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Abstract

For a given polynomial $f \in \mathbb{Z}[x]$, $d(S, f)$ is defined as $\gcd\{f(a) : a \in S\}$ where $S \subseteq \mathbb{Z}$. In general, the equality $d(S, fg) = d(S, f)d(S, g)$ does not hold for given polynomials f and g. In this article, we get a necessary and sufficient condition for the aforementioned equality to hold. We also suggest a generalization to our results for an arbitrary domain.

1. Introduction

Let S be an arbitrary subset of $\mathbb{Z}$. For a given polynomial $f \in \mathbb{Z}[x]$, the *fixed divisor of f over S*, denoted by $d(S, f)$, is defined as the greatest common divisor of the values of f taken over S, that is,

$$d(S, f) = \gcd\{f(a) : a \in S\}.$$

For a solid summary of the literature on the topic, we refer to [11]. Some interesting results on the topic can be found in [6], [7], and [12]. This article is inspired by the following question asked by Prasad, Rajkumar and Reddy [11].

Question 1 (Prasad, Rajkumar and Reddy [11])**.** For a subset $S \subseteq \mathbb{Z}$, what are the polynomials f and g in $\mathbb{Z}[x]$ such that $d(S, fg) = d(S, f)d(S, g)$?

This question remains open to date. As per our knowledge, no necessary and sufficient condition is known as a positive answer to this question. We investigate a necessary and sufficient condition to Question 1 for any subset $S \subseteq \mathbb{Z}$. Our results remain true even if $\mathbb{Z}$ is replaced by any Dedekind domain.

For a given subset $S \subseteq \mathbb{Z}$, recall that the *ring of integer-valued polynomials* is defined as

$$\text{Int}(S, \mathbb{Z}) = \{f \in \mathbb{Q}[x] : f(S) \subseteq \mathbb{Z}\}.$$

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One of the most beautiful topics in ring theory is the study of irreducible polynomials, which has a venerable history. In the ring $\mathrm{Int}(S,\mathbb{Z})$, the study of irreducibility is also very interesting. We refer to Prasad, Rajkumar and Reddy [11] for some known results. Recently some new results are known (see Prasad [8], [9], and [10].). If for a pair of polynomials f and g, we have $d(S,fg)=d(S,f)d(S,g)$, then the polynomial $\frac{fg}{d(S,fg)}\in\mathrm{Int}(S,\mathbb{Z})$ is reducible. Conversely, if a polynomial $\frac{f}{d(S,f)}\in\mathrm{Int}(S,\mathbb{Z})$ is reducible in $\mathrm{Int}(S,\mathbb{Z})$, then we must have a factorization $f=f_1f_2$ such that $d(S,f)=d(S,f_1)d(S,f_2)$. Hence, apart from its mathematical beauty, Question 1 is also helpful in the study of the irreducibility of integer-valued polynomials.

Here is the organization of the paper. In Section 2 we recall some known concepts which are helpful in our study. In Section 3, we introduce the notion of fixed divisor matrices and use them to prove our main results. In Section 4, we show how the size of the fixed divisor matrices can be reduced in some cases. Finally, in Section 5, we suggest a further generalization of these results in more general domains.

2. Preliminaries

We recall the definition of fixed divisor sequences (FD sequences), which is one of the main tools of investigation throughout the study.

Let $\{a_i\}_{i\geq 0}$ be a sequence of distinct elements of S. Suppose that for every $k>0$, there exists $l_k\in\mathbb{Z}$, such that for every polynomial f of degree k,

$$d(S,f)=(f(a_0),f(a_1),\ldots,f(a_{l_k})),$$

and no proper subset of $\{a_0,a_1,\ldots,a_{l_k}\}$ determines the fixed divisor of all the degree k polynomials. Then we say that the sequence $\{a_i\}_{i\geq 0}$ is a *fixed divisor sequence* (see Prasad, Rajkumar and Reddy [11]).

For instance, the sequence $0,1,2,\ldots$ is a fixed divisor sequence in $\mathbb{Z}$ with $l_k=k$ for all $k>0$. A p-ordering of a given set $S\subseteq\mathbb{Z}$ is said to be a *simultaneous p-ordering* if it is a p-ordering (see the definition on the next page) for all primes of $\mathbb{Z}$ (see [1] and [3]). For a given set $S\subseteq\mathbb{Z}$, any simultaneous p-ordering of S is also an FD sequence.

Sometimes all elements of an FD sequence may not belong to the given subset S. In this case, we say that the sequence is an *External Fixed Divisor Sequence* (EFD sequence). Hence an EFD sequence is an FD sequence from the viewpoint of fixed divisors, but it has at least one element that does not belong to the given subset.

Let S be an arbitrary subset of a Dedekind domain R. Then we can always construct an EFD sequence with $l_k=k$ for any finite value of the integer $k>0$. This was shown in Rajkumar, Reddy and Semwal [12]. In the case when $S=R=\mathbb{Z}[i]$, Volkov and Petrov [13] proved that there does not exist an FD sequence such that as k tends to infinity, then $l_k=k$. They conjectured that l_k grows as $\frac{\pi}{2}k+o(k)$ and an asymptotically sharp example is realized on the set of integer points inside the circle of radius $\sqrt{n/2} + o(\sqrt{n})$. Byszewski, Fraczyk and Szumowicz [4] found the growth of l_k in the case of a Dedekind domain. They proved that we always have $l_k \leq k + 1$. Conclusively, in any Dedekind domain, we can always construct an FD sequence with $l_k = k + 1$ for sufficiently large values of k.

Now we recall Bhargava factorials. Readers are advised to go through [1], [2], and [3] for a deep study. A *p-ordering* $a_0, a_1, \ldots$ of $S \subset \mathbb{Z}$, is a sequence of elements of S in which $a_k \in S$ minimizes the highest power of p dividing $\prod_{i=0}^{k-1}(a_k - a_i)$ for every positive integer k. Denote the highest power of p dividing $a \in \mathbb{Z}$ by $w_p(a)$. Then the *Bhargava factorial of index* k is defined as

$$k!_S = \prod_p w_p \left(\prod_{i=0}^{k-1} (a_k - a_i) \right).$$

For every primitive polynomial $f \in \mathbb{Z}[x]$ of degree k, $d(S, f)$ always divides $k!_S$. Let $\mathrm{Int}_k(S, \mathbb{Z})$ denote the subset of polynomials of degree at most k in $\mathrm{Int}(S, \mathbb{Z})$. Then it turns out that

$$k!_S = \gcd\{a : a\mathrm{Int}_k(S, \mathbb{Z}) \subseteq \mathbb{Z}[x]\}.$$

We now mention the notation that we will use throughout the remainder of the paper. We let S an arbitrary subset of $\mathbb{Z}$ and the sequence $a_0, a_1, \ldots$ denotes an FD sequence of S. The number l_k denotes the smallest number of elements of the fixed divisor sequence which gives the fixed divisor of any degree k polynomial. We assume that the set of primes that divides $k!_S$ is $\{p_1, p_2, \ldots, p_{m_k}\}$ and is arranged in the increasing order. Hence the set $\{p_1, p_2, \ldots, p_{m_k}\}$ is, in fact, the set of the first m_k primes. For a given prime number p and $a \in \mathbb{Z}, \nu_p(a)$ denotes the p-adic valuation of a with the assumption $\nu_p(0) = \infty$. The term polynomial refers to a polynomial with integer coefficients.

3. Necessary and Sufficient Conditions

We start this section with the definition of fixed divisor matrices (FD matrices).

Definition 1. For a polynomial $f \in \mathbb{Z}[x]$ and a subset $S \subseteq \mathbb{Z}$ with an FD sequence $a_0, a_1, \ldots$, the *FD matrix of order* $r \times s$ is defined as follows:

$$M_f(S)_{r \times s} = [\nu_{p_i}(f(a_j))]_{1 \leq i \leq r, 1 \leq j \leq s}.$$

In the above definition, r and s are positive integers and r can exceed the degree of f. For instance, if we take $S = \mathbb{Z}$ and $0, 1, 2, 3, \ldots$ as an FD sequence, then for the polynomial $f = x^2 + x + 2$, we have

$$M_f(\mathbb{Z})_{2\times 4} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } M_f(\mathbb{Z})_{3\times 5} = \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Sometimes most of the columns in the FD matrix become redundant and the number of columns can be considerably small. We discuss this elaborately in Section 4.

For a given $m \times n$ matrix $M = [a_{ij}]_{m\times n}$ with all entries in $\mathbb{Z}$, let α_i denote the smallest entry in the ith row, that is, $\alpha_i = \min\{a_{ij} : 1 \le j \le n\}$. We define the matrix M^* as follows:

$$M^* = [a_{ij} - \alpha_i]_{m\times n}.$$

The following lemma follows by our way of defining the fixed divisor matrix.

Lemma 1. *Let $M_f(S)_{m\times n} = [a_{ij}]_{m\times n}$, where $m \ge m_k, n \ge l_k$, $f \in \mathbb{Z}[x]$ is a polynomial of degree k and $S \subseteq \mathbb{Z}$. Then we have*

$$\nu_{p_i}(d(S, f)) = \min\{a_{ij} : 1 \le j \le n\} \text{ for all } 1 \le i \le m_k.$$

Theorem 1. *Let S be an arbitrary subset of $\mathbb{Z}$, let f and g be given polynomials of degrees k_1 and k_2, respectively, and let $k = k_1 + k_2$. Then, $d(S, fg) = d(S, f)d(S, g)$ if and only if each row of the matrix $M_f(S)^*_{m_k\times l_k} + M_g(S)^*_{m_k\times l_k}$ contains at least one zero.*

Proof. Let us assume that $d(S, fg) = d(S, f)d(S, g)$. Assume $p_1 < p_2 < \ldots < p_{m_k}$ are the only prime numbers dividing $k!_S$, where k is the degree of fg. By Lemma 1, at least one entry of the rth row of $M_f(S)_{m_k\times l_k}$ must be $\nu_{p_r}(d(S, f))$ for all $1 \le r \le m_k$. Assume $(r, i_1), (r, i_2), \ldots, (r, i_k)$ are entries of the matrix $M_f(S)_{m_k\times l_k}$ which are equal to $\nu_{p_r}(d(S, f))$. Now we look at the $(r, i_1), (r, i_2), \ldots, (r, i_k)$ entries of the matrix $M_g(S)_{m_k\times l_k}$. At least one of these positions must have the entry $\nu_{p_r}(d(S, g))$, for otherwise $\nu_{p_r}(d(S, fg)) > \nu_{p_r}(d(S, f)) + \nu_{p_r}(d(S, g))$. Hence there exists a common position, say (r, i_j), such that $\nu_{p_r}(d(S, f))$ and $\nu_{p_r}(d(S, g))$ are the (r, i_j) entry of the matrices $M_f(S)_{m_k\times l_k}$ and $M_g(S)_{m_k\times l_k}$, respectively. As a consequence, the (r, i_j) entry of the matrices $M_f(S)^*_{m_k\times l_k}$ and $M_g(S)^*_{m_k\times l_k}$ is zero. In other words, the (r, i_j) entry of the matrix $M_f(S)^*_{m_k\times l_k} + M_g(S)^*_{m_k\times l_k}$ is zero. This completes the 'only if' part of the proof as p_r is arbitrary.

Conversely, assume that the (r, j) entry of the matrix $M_f(S)^*_{m_k\times l_k} + M_g(S)^*_{m_k\times l_k}$ is zero. Then the (r, j) entry of the matrices $M_f(S)^*$ and $M_g(S)^*_{m_k\times l_k}$ must be zero. Hence, $\nu_{p_r}(d(S, f))$ and $\nu_{p_r}(d(S, g))$ appear in the (r, j) position of the matrices $M_f(S)_{m_k\times l_k}$ and $M_g(S)_{m_k\times l_k}$, respectively. As a result, $\nu_{p_r}(fg(a_j)) = \nu_{p_r}(d(S, f)) + \nu_{p_r}(d(S, g))$, which is the smallest possible value of $\nu_{p_r}(fg(a_j))$, for $j = 0, 1, \ldots, l_k$. Since this is true for each r, it follows that $d(S, fg) = d(S, f)d(S, g)$. □

We explain our theorem with a few examples.

Example 1. Let us apply Theorem 1 to the polynomials $f = x^3 - 2x^2 + 2x + 3$ and $g = x^3 + 6x^2 + 2x + 9$. We have the following table:

x	$f(x)$	$g(x)$
0	$3 = 2^0 3^1 5^0$	$9 = 2^0 3^2 5^0$
1	$4 = 2^2 3^0 5^0$	$18 = 2^1 3^2 5^0$
2	$7 = 2^0 3^0 5^0 \times 7^1$	$45 = 2^0 3^2 5^1$
3	$18 = 2^1 3^2 5^0$	$96 = 2^5 3^1 5^0$
4	$43 = 2^0 3^0 5^0 \times 43$	$177 = 2^0 3^1 5^0 \times 59$
5	$88 = 2^3 3^0 5^0 \times 11$	$294 = 2^1 3^1 5^0 \times 7^2$
6	$159 = 2^0 3^1 5^0 \times 53$	$453 = 2^0 3^1 5^0 \times 151.$

Since we are interested only in the primes less than 6, the other primes are written after the " $\times$ " sign. Hence, we have the following matrices:

$$M_f(\mathbb{Z})_{3\times 7} = \begin{pmatrix} 0 & 2 & 0 & 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$M_g(\mathbb{Z})_{3\times 7} = \begin{pmatrix} 0 & 1 & 0 & 5 & 0 & 1 & 0 \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, $M_f(\mathbb{Z})_{3\times 7} = M_f(\mathbb{Z})^*_{3\times 7}$ since each row of the matrix $M_f(\mathbb{Z})$ contains at least one zero. Also, $M_g(\mathbb{Z})^*_{3\times 7} = \begin{pmatrix} 0 & 1 & 0 & 5 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$, hence, we have

$$M_f(\mathbb{Z})^*_{3\times 7} + M_g(\mathbb{Z})^*_{3\times 7} = \begin{pmatrix} 0 & 3 & 0 & 6 & 0 & 4 & 0 \\ 2 & 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since at least one zero appears in each row of the matrix $M_f(\mathbb{Z})^*_{3\times 7} + M_g(\mathbb{Z})^*_{3\times 7}$, we must have $d(\mathbb{Z}, fg) = d(\mathbb{Z}, f)d(\mathbb{Z}, g)$.

The computation of $M_f(\mathbb{Z})^* + M_g(\mathbb{Z})^*$ from $M_f(\mathbb{Z})$ and $M_g(\mathbb{Z})$ can be done in just one step. Hence, while working with examples we can directly write our matrices. We give one more example to illustrate our theorem.

Example 2. Let us apply Theorem 1 to the polynomials $f(x) = 3x^2 - 7x - 2$ and $g(x) = 7x^2 - 11x + 6$. Here the primes for which a row of the FD matrix is defined are only 2 and 3. We have

$$M_f(\mathbb{Z})_{2\times 5} = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix} \text{ and } M_g(\mathbb{Z})_{2\times 5} = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 0 \end{pmatrix}.$$

Observe that, $M_f(\mathbb{Z})^*_{2\times 5}+M_g(\mathbb{Z})^*_{2\times 5}=\begin{pmatrix}0&0&2&2&0\\1&1&1&2&2\end{pmatrix}$. Since the second row does not contain a zero, we cannot have the equality $d(\mathbb{Z},fg)=d(\mathbb{Z},f)d(\mathbb{Z},g)$.

Using similar arguments as in the proof of Theorem 1 we can prove the following Theorem.

Theorem 2. *Let $S\subset\mathbb{Z}$ be an arbitrary subset and f,g be given polynomials in $\mathbb{Z}[x]$. Then $d(S,fg)=d(S,f)d(S,g)$ if and only if $M_{fg}(S)^*_{m_k\times l_k}=M_f(S)^*_{m_k\times l_k}+M_g(S)^*_{m_k\times l_k}$, where k is the degree of the polynomial fg.*

In Theorem 1 and Theorem 2 we fixed the increasing order in the set of primes dividing $k!_S$. However, any fixed order in the set of primes dividing $k!_S$ gives the same result.

We have already explained how our theorems can be used as a criterion to test the irreducibility of integer-valued polynomials. More formally, we can rephrase Theorem 2 as follows.

Theorem 3. *A polynomial $\frac{f}{d(S,f)}$ of degree k is reducible in Int$(S,\mathbb{Z})$ if and only if there exists a factorization $f=f_1f_2$ such that $M_{f_1f_2}(S)^*_{m_k\times l_k}=M_{f_1}(S)^*_{m_k\times l_k}+M_{f_2}(S)^*_{m_k\times l_k}$.*

We can write a similar analog of Theorem 1. While working with FD matrices, it appears that when the degrees of the polynomials increase, then the number of columns also increases. However, this is not true always. We discuss this elaborately in the next section.

4. P_1 and P_2 Polynomials

By the definition of the fixed divisor sequences, it is obvious that the value of l_k cannot be smaller than k. However, the number of elements that determines $d(S,f)$ for a given polynomial f is a local property of the polynomial f. More explicitly, consider the polynomial $f=x^2+x+2$ with $d(\mathbb{Z},f)=2$. It is easy to observe that only one element determines $d(\mathbb{Z},f)$.

For a given polynomial $f\in\mathbb{Z}[x]$ of degree k, any $k+1$ consecutive integers completely determine $d(\mathbb{Z},f)$. This was proved for the first time by Hensel [5] in 1896. He also made an observation that sometimes a smaller number of elements may be sufficient to determine $d(\mathbb{Z},f)$. He explicitly gave examples of some polynomials for which $d(\mathbb{Z},f)$ is determined by a smaller number of elements. However, after his seminal work, no special attention was given to this observation. Recently, Rajkumar, Reddy and Semwal [12] proved that for a given polynomial $f\in\mathbb{Z}[x]$ and a given subset $S\subset\mathbb{Z}$, we can find a pair of integers a and b such that $d(\mathbb{Z}, f) = \gcd(f(a), f(b))$. They proved this result for multivariate polynomials which remains true in the case of one variable as well.

For a given subset $S \subseteq \mathbb{Z}$, we can make the following classification:

$$P_1(S) = \{f \in \mathbb{Z}[x] : d(\mathbb{Z}, f) = f(a) \text{ for some } a \in \mathbb{Z}\},$$

and

$$P_2(S) = \{f \in \mathbb{Z}[x] : d(\mathbb{Z}, f) = \gcd(f(a), f(b)) \text{ for } a, b \in \mathbb{Z}, a \neq b\}.$$

This classification is already done in [6]. We study both cases separately. If for given polynomials f and g there exists $a \in \mathbb{Z}$ such that $d(S, f) = f(a)$ and $d(S, g) = g(a)$, then we have $d(S, fg) = d(S, f)d(S, g)$. Hence we assume that $d(S, f) = f(a)$ and $d(S, g) = g(b)$, where a and b are distinct. Observe that we may not have $d(S, fg) = d(S, f)d(S, g)$. In this case, we slightly manipulate the definition of FD matrices. We write only two columns that correspond to those two elements which generate fixed divisors. More formally we study the following matrices.

If $d(S, f) = f(a)$ and $d(S, g) = g(b)$, then we consider only two columns in the FD matrices, that is,

$$M_f(\mathbb{Z})_{k\times 2} = [\nu_{p_i}(f(a)) \quad \nu_{p_i}(f(b))]_{1\leq i\leq m_k},$$

$$M_g(\mathbb{Z})_{k\times 2} = [\nu_{p_i}(g(a)) \quad \nu_{p_i}(g(b))]_{1\leq i\leq m_k}$$

and

$$M_{fg}(\mathbb{Z})_{k\times 2} = [\nu_{p_i}(fg(a)) \quad \nu_{p_i}(fg(b))]_{1\leq i\leq m_k}$$

This reduces our work. We give an example to illustrate this.

Example 3. Let us consider the polynomials $f(x) = 3x^2 - 7x - 2$ and $g(x) = 7x^2 - 11x + 6$ again. Clearly, $d(\mathbb{Z}, f) = f(0)$ and $d(\mathbb{Z}, g) = g(1)$. Hence we have the following matrices in which columns correspond to 0 and 1:

$$M_f(\mathbb{Z})_{2\times 2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } M_g(\mathbb{Z})_{2\times 2} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

This implies $M_f(\mathbb{Z})^*_{2\times 2} + M_g(\mathbb{Z})^*_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. As the second row does not contain a zero, we cannot have the equality $d(\mathbb{Z}, fg) = d(\mathbb{Z}, f)d(\mathbb{Z}, g)$.

Similarly, if both the polynomials f and g are P_2 polynomials, then we have at most four columns in the FD matrices. We end this section with the following proposition.

Proposition 1. *Let $S \subseteq \mathbb{Z}$ be an arbitrary subset. Then $d(S, fg) = d(S, f)d(S, g)$ for all polynomials $f, g \in \mathbb{Z}[x]$ if and only if S is a singleton.*

Proof. If S is a singleton, then one direction follows directly, and hence we assume that S contains more than one element. If S is a finite set, we write $S = A \cup B$ where A and B are non-empty proper subsets of S. Let f_A be a polynomial that vanishes in A and does not vanish in B, and let f_B be a polynomial that vanishes in B and does not vanish in A. Consider the polynomial $f = f_A f_B$ that vanishes in S and hence $d(S, f) = 0$. However, neither $d(S, f_A)$ is zero nor $d(S, f_B)$ is zero.

Now we handle the case when S is not a finite set. We again write $S = A \cup B$ where A is a non-empty proper finite subset of S. Let f_A be a polynomial that vanishes in A and there exists at least one $b \in B$ such that $f_A(b)$ is a unit. In this case, we assume that the values of the polynomial f_B are always multiples of a given prime p in B. As a result, the values of the polynomial $f(x) = f_A(x)f_B(x)$ are always multiples of p when $x \in S$ and hence $d(S, f)$ is a multiple of p. However, $d(S, f_A)d(S, f_B)$ is a divisor of p. Consequently, we cannot have the equality $d(S, f_A f_B) = d(S, f_A)d(S, f_B)$, completing the proof. □

5. Further Generalizations

Our main results can be obtained for more than two polynomials. Apart from this, our main results remain true in the case of UFDs and Dedekind domains. In the case of an arbitrary domain, the theorem may remain true depending on whether the subset S of the given domain has a fixed divisor sequence or not. It is clear by our method that our main results remain true for multivariate polynomials as well.

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