HARMONIC NUMBERS: COMBINATORIAL IDENTITIES AND SERIES REPRESENTATIONS

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Abstract

We present several combinatorial identities and infinite series involving the harmonic numbers. Among others, we show that

$$\frac{2}{\binom{2n}{n}}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^8}\binom{2n}{n+k}=\sum_{k=1}^{n}\frac{H_n^{(2)}-H_{k-1}^{(2)}}{k^2}\sum_{\nu=1}^{k}\frac{H_\nu^{(2)}}{\nu^2}\quad (n\geq 1)$$

and

$$\frac{H_n}{n}=\sum_{\nu=0}^{\infty}\sum_{j=0}^{\nu}(-1)^j\mu^{j+1}\binom{\nu}{j}\frac{H_{n+j+1}}{n+j+1}\quad (n\geq 1,\, 0<\mu<2).$$

Here, $H_n=\sum_{k=1}^{n}1/k$ and $H_n^{(2)}=\sum_{k=1}^{n}1/k^2$ denote the harmonic numbers of order 1 and 2.

1. Introduction and Statement of the Results

I. The classical harmonic numbers are defined by

$$H_0=0,\quad H_n=\sum_{k=1}^{n}\frac{1}{k}\quad (n\in\mathbb{N}),$$

and the generalized harmonic numbers of order $r\in\mathbb{C}\setminus\{0\}$ are given by

$$H_0^{(r)}=0,\quad H_n^{(r)}=\sum_{k=1}^{n}\frac{1}{k^r}\quad (n\in\mathbb{N}).$$

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H_n can be expressed in terms of the digamma function, $\psi = \Gamma'/\Gamma$,

$$H_n = \psi(n+1) + \gamma,$$

where γ denotes Euler's constant. Moreover, we have the elegant integral representation

$$H_n = -n \int_0^1 t^{n-1} \log(1-t) dt, \tag{1.1}$$

see Furdui [7].

The harmonic numbers have interesting applications in various fields, like, for example, number theory, analysis and combinatorics. Lagarias [10] discovered a remarkable connection between H_n and the famous Riemann hypothesis. He proved that the Riemann hypothesis is equivalent to the elementary inequality

$$\sigma(n) \leq H_n + e^{H_n} \log(H_n) \quad (n \in \mathbb{N}),$$

where $\sigma(n)$ denotes the sum of the divisors of n.

The properties of the harmonic numbers have been studied by numerous authors. Hassani [8] discovered the following limit relation for Apéry's constant $\zeta(3)$,

$$\zeta(3) = \lim_{n\to\infty} \Big(H_n^3 - 3 \sum_{k=1}^{n} \frac{H_{k-1} H_k}{k} \Big). \tag{1.2}$$

The reciprocity formula

$$H_{n-1} + \log(n) + \frac{1}{n} \sum_{k=0}^{n-1} \psi\Big(\frac{m+k}{n}\Big) = H_{m-1} + \log(m) + \frac{1}{m} \sum_{k=0}^{m-1} \psi\Big(\frac{n+k}{m}\Big) \quad (m, n \in \mathbb{N})$$

was given by Ramanujan [5, p. 185]. In the literature, we can find many interesting identities for sums and infinite series involving harmonic numbers. Here are a few examples:

$$\sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \big(1 + 2kH_{n+k} + 2kH_{n-k} - 4kH_k\big) = 0,$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 H_k H_{n-k} = \binom{2n}{n} \Big((H_{2n} - 2H_n)^2 + H_n^{(2)} - H_{2n}^{(2)}\Big),$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{H_k}{\binom{k+5}{5}^2} = \frac{75}{4}\zeta(3) - \frac{1075}{8}\zeta(2) + \frac{57175}{288},$$

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} \big(H_k^3 - 3H_k H_k^{(2)} + 2H_k^{(3)}\big) = 6.$$

These identities are due to Ahlgren and Ono [1], Batir et al. [4], Sofo [11], Choi [6].

II. In 2015, Kórus [9] applied an identity given by Retkes to present the following combinatorial identities.

Proposition 1. *For all integers $n \geq 1$, we have*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} = H_n, \tag{1.3}$$

$$\frac{2}{\binom{2n}{n}} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^2} \binom{2n}{n+k} = H_n^{(2)}, \tag{1.4}$$

$$3(n!)^2 \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^3} \frac{\binom{n}{k}}{\prod_{j=1}^{n}(k^2+kj+j^2)} = H_n^{(3)}. \tag{1.5}$$

One aim of this paper is to generalize these results.

Theorem 1. *For all integers $m \geq 2$ and $n \geq 1$, we have*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^m} \binom{n}{k} = \sum_{k_{m-1}=1}^{n} \frac{1}{k_{m-1}} \sum_{k_{m-2}=1}^{k_{m-1}} \frac{1}{k_{m-2}} \cdots \frac{1}{k_2} \sum_{k_1=1}^{k_2} \frac{1}{k_1} \sum_{k_0=1}^{k_1} \frac{1}{k_0}. \tag{1.6}$$

Theorem 2. *For all integers $m \geq 2$ and $n \geq 1$, we have*

$$\frac{2}{\binom{2n}{n}} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^{2m}} \binom{2n}{n+k} = \sum_{k_{m-1}=1}^{n} \frac{1}{k_{m-1}^2} \sum_{k_{m-2}=1}^{k_{m-1}} \frac{1}{k_{m-2}^2} \cdots \frac{1}{k_2^2} \sum_{k_1=1}^{k_2} \frac{1}{k_1^2} \sum_{k_0=1}^{k_1} \frac{1}{k_0^2}. \tag{1.7}$$

Theorem 3. *For all integers $m \geq 2$ and $n \geq 1$, we have*

$$3(n!)^2 \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^{3m}} \frac{\binom{n}{k}}{\prod_{j=1}^{n}(k^2+kj+j^2)} = \sum_{k_{m-1}=1}^{n} \frac{1}{k_{m-1}^3} \sum_{k_{m-2}=1}^{k_{m-1}} \frac{1}{k_{m-2}^3} \cdots \frac{1}{k_2^3} \sum_{k_1=1}^{k_2} \frac{1}{k_1^3} \sum_{k_0=1}^{k_1} \frac{1}{k_0^3}. \tag{1.8}$$

From Theorems 1–3 with $m = 2, 3, 4$, we obtain the following identities.

Corollary 1. *For all integers $n \geq 1$, we have*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^2} \binom{n}{k} = \sum_{k=1}^{n} \frac{H_k}{k}, \tag{1.9}$$

$$\frac{2}{\binom{2n}{n}}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^4}\binom{2n}{n+k}=\sum_{k=1}^{n}\frac{H_k^{(2)}}{k^2}, \tag{1.10}$$

$$3(n!)^2\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^6}\frac{\binom{n}{k}}{\prod_{j=1}^{n}(k^2+kj+j^2)}=\sum_{k=1}^{n}\frac{H_k^{(3)}}{k^3}, \tag{1.11}$$

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^3}\binom{n}{k}=\sum_{k=1}^{n}\frac{H_k(H_n-H_{k-1})}{k}, \tag{1.12}$$

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^4}\binom{n}{k}=\sum_{k=1}^{n}\frac{H_n-H_{k-1}}{k}\sum_{\nu=1}^{k}\frac{H_\nu}{\nu}, \tag{1.13}$$

$$\frac{2}{\binom{2n}{n}}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^6}\binom{2n}{n+k}=\sum_{k=1}^{n}\frac{H_k^{(2)}\big(H_n^{(2)}-H_{k-1}^{(2)}\big)}{k^2}, \tag{1.14}$$

$$\frac{2}{\binom{2n}{n}}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^8}\binom{2n}{n+k}=\sum_{k=1}^{n}\frac{H_n^{(2)}-H_{k-1}^{(2)}}{k^2}\sum_{\nu=1}^{k}\frac{H_\nu^{(2)}}{\nu^2}. \tag{1.15}$$

$$3(n!)^2\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^9}\frac{\binom{n}{k}}{\prod_{j=1}^{n}(k^2+kj+j^2)}=\sum_{k=1}^{n}\frac{H_k^{(3)}\big(H_n^{(3)}-H_{k-1}^{(3)}\big)}{k^3}, \tag{1.16}$$

$$3(n!)^2\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^{12}}\frac{\binom{n}{k}}{\prod_{j=1}^{n}(k^2+kj+j^2)}=\sum_{k=1}^{n}\frac{H_n^{(3)}-H_{k-1}^{(3)}}{k^3}\sum_{\nu=1}^{k}\frac{H_\nu^{(3)}}{\nu^3}. \tag{1.17}$$

Remark 1. Using (1.3), (1.9) and (1.12) gives

$$H_n^3-3\sum_{k=1}^{n}\frac{H_{k-1}H_k}{k}=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}\Big(H_n^2-\frac{3H_n}{k}+\frac{3}{k^2}\Big),$$

so that (1.2) leads to

$$\zeta(3)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}\Big(H_n^2-\frac{3H_n}{k}+\frac{3}{k^2}\Big).$$

III. The second aim is to provide one-parameter series representations for H_n/n, the unweighted arithmetic mean of the first n unit fractions, and for the sum $H_1+H_2/2+\cdots+H_n/n$.

Theorem 4. *For all integers $n\geq 1$ and real numbers $\lambda<1/2$ and $\mu\in(0,2)$, we have*

$$\frac{H_n}{n}=\sum_{\nu=0}^{\infty}\frac{1}{(1-\lambda)^{\nu+1}}\sum_{j=0}^{\nu}(-\lambda)^{\nu-j}\binom{\nu}{j}S(n,j) \tag{1.18}$$

with

$$S(n,j)=\sum_{k=0}^{n}\frac{(-1)^k}{(k+j+1)^2}\binom{n}{k}$$

and

$$\frac{H_n}{n}=\sum_{\nu=0}^{\infty}\sum_{j=0}^{\nu}(-1)^j\mu^{j+1}\binom{\nu}{j}\frac{H_{n+j+1}}{n+j+1}. \tag{1.19}$$

Theorem 5. *For all integers $n\geq 1$ and real numbers $\lambda<1/2$ and $\mu\in(0,2)$, we have*

$$\sum_{k=1}^{n}\frac{H_k}{k}=\sum_{\nu=0}^{\infty}\frac{1}{(1-\lambda)^{\nu+1}}\sum_{j=0}^{\nu}(-\lambda)^{\nu-j}\binom{\nu}{j}\Big(\frac{H_{j+1}}{j+1}-\frac{H_{n+j+1}}{n+j+1}\Big) \tag{1.20}$$

and

$$\sum_{k=1}^{n}\frac{H_k}{k}=\sum_{\nu=0}^{\infty}\sum_{j=0}^{\nu}(-\mu)^{j+1}\binom{\nu}{j}T(n,j) \tag{1.21}$$

with

$$T(n,j)=\sum_{k=1}^{n}\frac{(-1)^k}{(k+j+1)^2}\binom{n}{k}.$$

We highlight two special cases. Setting $\mu=1$ in (1.19) and $\lambda=-1$ in (1.20), we obtain the following identities.

Corollary 2. *For all integers $n\geq 1$, we have*

$$\frac{H_n}{n}=\sum_{\nu=0}^{\infty}\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}\frac{H_{n+j+1}}{n+j+1}$$

and

$$\sum_{k=1}^{n}\frac{H_k}{k}=\sum_{\nu=0}^{\infty}\frac{1}{2^{\nu+1}}\sum_{j=0}^{\nu}\binom{\nu}{j}\Big(\frac{H_{j+1}}{j+1}-\frac{H_{n+j+1}}{n+j+1}\Big).$$

2. Proofs

We apply a clever variant of the classical proof by induction to establish Theorems 1–3. This method was introduced by Turán [12] to prove an inequality for a sine polynomial with binomial coefficients.

Method. We want to show that $D_{m,n}=0$ for all integers $m\geq 2$ and $n\geq 1$.

Step 1. We prove that $D_{m,1}=0$ for $m\geq 2$ and that $D_{1,n}=0$ for $n\geq 1$.

Step 2. Let $m\geq 2$, $n\geq 1$. We prove that if $D_{m,n}=0$ and $D_{m-1,n+1}=0$, then $D_{m,n+1}=0$.

From Step 1 and Step 2 we conclude that $D_{m,n}=0$ for $m\geq 2$ and $n\geq 1$.

Proof of Theorem 1. We call $L^{(1)}_{m,n}$ the left-hand side of (1.6) and $R^{(1)}_{m,n}$ the right-hand side of (1.6).

`Step 1.` We have $L^{(1)}_{m,1} = R^{(1)}_{m,1} = 1$ for $m \geq 2$, and from (1.3) we obtain $L^{(1)}_{1,n} = R^{(1)}_{1,n}$ for $n \geq 1$.

`Step 2.` Let $m \geq 2$, $n \geq 1$ and $L^{(1)}_{m,n} = R^{(1)}_{m,n}$, $L^{(1)}_{m-1,n+1} = R^{(1)}_{m-1,n+1}$. Then we obtain

$$\begin{aligned}
L^{(1)}_{m,n+1} &= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k^m} \binom{n+1}{k} \\
&= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k^m} \left[\binom{n}{k} + \binom{n}{k-1} \right] \\
&= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^m} \binom{n}{k} + \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k^m} \binom{n}{k-1} \\
&= L^{(1)}_{m,n} + \frac{1}{n+1} \left[\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^{m-1}} \binom{n+1}{k} + \frac{(-1)^n}{(n+1)^{m-1}} \right] \\
&= L^{(1)}_{m,n} + \frac{1}{n+1} \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k^{m-1}} \binom{n+1}{k} \\
&= L^{(1)}_{m,n} + \frac{1}{n+1} L^{(1)}_{m-1,n+1} \\
&= R^{(1)}_{m,n} + \frac{1}{n+1} R^{(1)}_{m-1,n+1} \\
&= R^{(1)}_{m,n+1}.
\end{aligned}$$

□

Proof of Theorem 2. Let $L^{(2)}_{m,n}$ and $R^{(2)}_{m,n}$ be the expressions on the left-hand side and on the right-hand side of (1.7), respectively.

`Step 1.` We have $L^{(2)}_{m,1} = R^{(2)}_{m,1} = 1$ for $m \geq 2$, and using (1.4) gives $L^{(2)}_{1,n} = R^{(2)}_{1,n}$ for $n \geq 1$.

`Step 2.` Let $m \geq 2$, $n \geq 1$. We assume that $L^{(2)}_{m,n} = R^{(2)}_{m,n}$, $L^{(2)}_{m-1,n+1} = R^{(2)}_{m-1,n+1}$. Applying the identity

$$\frac{\binom{2n+2}{n+1+k}}{\binom{2n+2}{n+1}} = \frac{\binom{2n}{n+k}}{\binom{2n}{n}} + \left(\frac{k}{n+1}\right)^2 \frac{\binom{2n+2}{n+1+k}}{\binom{2n+2}{n+1}} \quad (1 \leq k \leq n+1)$$

yields

$$\begin{aligned}
L^{(2)}_{m,n+1} &= \frac{2}{\binom{2n+2}{n+1}} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^{2m}} \binom{2n+2}{n+1+k} + \frac{2(-1)^n}{(n+1)^{2m}\binom{2n+2}{n+1}} \\
&= \frac{2}{\binom{2n}{n}} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^{2m}} \binom{2n}{n+k} + \frac{2}{(n+1)^2\binom{2n+2}{n+1}} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^{2m-2}} \binom{2n+2}{n+1+k} \\
&\quad + \frac{2(-1)^n}{(n+1)^{2m}\binom{2n+2}{n+1}} \\
&= L^{(2)}_{m,n} + \frac{1}{(n+1)^2} L^{(2)}_{m-1,n+1} \\
&= R^{(2)}_{m,n} + \frac{1}{(n+1)^2} R^{(2)}_{m-1,n+1} \\
&= R^{(2)}_{m,n+1}. \qquad \square
\end{aligned}$$

Proof of Theorem 3. We denote by $L^{(3)}_{m,n}$ and $R^{(3)}_{m,n}$ the left-hand side and the right-hand side of (1.8), respectively.

`Step 1.` We have $L^{(3)}_{m,1} = R^{(3)}_{m,1} = 1$ for $m \geq 1$. Using (1.5) gives $L^{(3)}_{1,n} = R^{(3)}_{1,n}$ for $n \geq 1$.

`Step 2.` Let $m \geq 2$, $n \geq 1$ and $L^{(3)}_{m,n} = R^{(3)}_{m,n}$, $L^{(3)}_{m-1,n+1} = R^{(3)}_{m-1,n+1}$. We apply the identity

$$\frac{(n+1)^2\binom{n+1}{k}}{\prod_{j=1}^{n+1}(k^2+kj+j^2)} = \frac{\binom{n}{k}}{\prod_{j=1}^{n}(k^2+kj+j^2)} + \frac{k^3\binom{n+1}{k}}{(n+1)\prod_{j=1}^{n+1}(k^2+kj+j^2)}$$

for $1 \leq k \leq n+1$, and obtain

$$\begin{aligned}
L^{(3)}_{m,n+1} &= 3((n+1)!)^2 \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k^{3m}} \frac{\binom{n+1}{k}}{\prod_{j=1}^{n+1}(k^2+kj+j^2)} \\
&= 3(n!)^2 \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^{3m}} \frac{\binom{n}{k}}{\prod_{j=1}^{n}(k^2+kj+j^2)} \\
&\quad + \frac{3(n!)^2}{n+1} \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k^{3m-3}} \frac{\binom{n+1}{k}}{\prod_{j=1}^{n+1}(k^2+kj+j^2)} \\
&= L^{(3)}_{m,n} + \frac{1}{(n+1)^3} L^{(3)}_{m-1,n+1} \\
&= R^{(3)}_{m,n} + \frac{1}{(n+1)^3} R^{(3)}_{m-1,n+1} \\
&= R^{(3)}_{m,n+1}. \qquad \square
\end{aligned}$$

Proof of Corollary 1. The validity of (1.9)–(1.11) is obvious. We only prove (1.12) and (1.13). The proofs of (1.14)–(1.17) are similar. Applying (1.6) with $m = 3$ and the summation formula

$$\sum_{k=1}^{n}\sum_{\nu=1}^{k} A(k,\nu) = \sum_{k=1}^{n}\sum_{\nu=k}^{n} A(\nu,k) \tag{2.1}$$

gives

$$\begin{aligned}\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^3}\binom{n}{k} &= \sum_{k=1}^{n} \frac{1}{k}\sum_{\nu=1}^{k} \frac{H_\nu}{\nu} = \sum_{k=1}^{n}\sum_{\nu=k}^{n} \frac{H_k}{\nu k} = \sum_{k=1}^{n} \frac{H_k}{k}\sum_{\nu=k}^{n} \frac{1}{\nu}\\ &= \sum_{k=1}^{n} \frac{H_k(H_n - H_{k-1})}{k}.\end{aligned}$$

Next, we use (1.6) with $m = 4$ and (2.1). This leads to

$$\begin{aligned}\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^4}\binom{n}{k} &= \sum_{k_3=1}^{n} \frac{1}{k_3}\sum_{k_2=1}^{k_3} \frac{1}{k_2}\sum_{k_1=1}^{k_2} \frac{H_{k_1}}{k_1}\\ &= \sum_{k=1}^{n}\sum_{\nu=1}^{k}\sum_{j=1}^{\nu} \frac{H_j}{k\nu j}\\ &= \sum_{k=1}^{n}\sum_{\nu=k}^{n}\sum_{j=1}^{k} \frac{H_j}{k\nu j}\\ &= \sum_{k=1}^{n} \frac{1}{k}\sum_{\nu=k}^{n} \frac{1}{\nu}\sum_{j=1}^{k} \frac{H_j}{j}\\ &= \sum_{k=1}^{n} \frac{H_n - H_{k-1}}{k}\sum_{j=1}^{k} \frac{H_j}{j}.\end{aligned}$$

□

To prove Theorems 4 and 5 we apply a method which has been used in other papers to deduce series representations for certain special functions and for mathematical constants, like, for example, γ, π and Catalan's constant G; see Alzer and Koumandos [2], Alzer and Richards [3].

Proof of Theorem 4. (i) Let $0 < x < 1$ and $\lambda < 1/2$. Then $-1 < (x-\lambda)/(1-\lambda) < 1$. It follows that

$$\frac{1}{1-x} = \frac{1}{1-\lambda} \cdot \frac{1}{1-(x-\lambda)/(1-\lambda)} = \frac{1}{1-\lambda}\sum_{\nu=0}^{\infty}\Big(\frac{x-\lambda}{1-\lambda}\Big)^{\nu}. \tag{2.2}$$

From (1.1) and (2.2) we obtain

$$
\begin{aligned}
\frac{H_n}{n} &= -\int_0^1 \frac{(1-x)^n \log(x)}{1-x} dx \\
&= -\int_0^1 \frac{1}{1-\lambda} \sum_{\nu=0}^{\infty} \Big(\frac{x-\lambda}{1-\lambda}\Big)^{\nu} (1-x)^n \log(x) dx \\
&= \sum_{\nu=0}^{\infty} \frac{1}{(1-\lambda)^{\nu+1}} U(\nu, n) \qquad (2.3)
\end{aligned}
$$

with

$$
\begin{aligned}
U(\nu, n) &= -\int_0^1 (x-\lambda)^{\nu} (1-x)^n \log(x) dx \\
&= -\int_0^1 \sum_{j=0}^{\nu} \binom{\nu}{j} (-\lambda)^{\nu-j} x^j (1-x)^n \log(x) dx \\
&= \sum_{j=0}^{\nu} \binom{\nu}{j} (-\lambda)^{\nu-j} S(n, j), \qquad (2.4)
\end{aligned}
$$

where

$$
\begin{aligned}
S(n, j) &= -\int_0^1 x^j (1-x)^n \log(x) dx \\
&= -\int_0^1 x^j \sum_{k=0}^{n} \binom{n}{k} (-x)^k \log(x) dx \\
&= -\sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^1 x^{j+k} \log(x) dx. \qquad (2.5)
\end{aligned}
$$

Since

$$
\int_0^1 x^m \log(x) dx = -\frac{1}{(m+1)^2} \quad (0 \le m \in \mathbb{Z}), \qquad (2.6)
$$

we obtain

$$
S(n, j) = \sum_{k=0}^{n} \frac{(-1)^k}{(k+j+1)^2} \binom{n}{k}. \qquad (2.7)
$$

From (2.3), (2.4) and (2.7) we conclude that (1.18) holds.

(ii) Let $0 < t < 1$ and $0 < \mu < 2$. Since $-1 < 1 - \mu t < 1$, we obtain

$$
\frac{1}{t} = \frac{\mu}{1-(1-\mu t)} = \mu \sum_{\nu=0}^{\infty} (1-\mu t)^{\nu}. \qquad (2.8)
$$

It follows from (1.1) and (2.6) that

$$\begin{aligned}\frac{H_n}{n} &= -\int_0^1 \frac{t^n \log(1-t)}{t} dt \\ &= -\int_0^1 t^n \log(1-t) \mu \sum_{\nu=0}^{\infty} (1-\mu t)^{\nu} \\ &= -\mu \sum_{\nu=0}^{\infty} \int_0^1 t^n \log(1-t) \sum_{j=0}^{\nu} \binom{\nu}{j} (-\mu t)^j dt \\ &= -\mu \sum_{\nu=0}^{\infty} \sum_{j=0}^{\nu} (-\mu)^j \binom{\nu}{j} \int_0^1 t^{n+j} \log(1-t) dt \\ &= \sum_{\nu=0}^{\infty} \sum_{j=0}^{\nu} (-1)^j \mu^{j+1} \binom{\nu}{j} \frac{H_{n+j+1}}{n+j+1}. \qquad \square\end{aligned}$$

Proof of Theorem 5. (i) Using (1.1) and (2.2) gives

$$\begin{aligned}\sum_{k=1}^{n} \frac{H_k}{k} &= -\int_0^1 \log(1-t) \sum_{k=1}^{n} t^{k-1} dt \\ &= -\int_0^1 \frac{\log(1-t)(1-t^n)}{1-t} dt \\ &= -\int_0^1 \log(1-t) \frac{1-t^n}{1-\lambda} \sum_{\nu=0}^{\infty} \left(\frac{t-\lambda}{1-\lambda}\right)^{\nu} dt \\ &= -\sum_{\nu=0}^{\infty} \frac{1}{(1-\lambda)^{\nu+1}} \int_0^1 \log(1-t)(1-t^n)(t-\lambda)^{\nu} dt \\ &= \sum_{\nu=0}^{\infty} \frac{1}{(1-\lambda)^{\nu+1}} \sum_{j=0}^{\nu} \binom{\nu}{j} (-\lambda)^{\nu-j} \left(\int_0^1 t^{n+j} \log(1-t) dt - \int_0^1 t^j \log(1-t) dt\right) \\ &= \sum_{\nu=0}^{\infty} \frac{1}{(1-\lambda)^{\nu+1}} \sum_{j=0}^{\nu} \binom{\nu}{j} (-\lambda)^{\nu-j} \left(\frac{H_{j+1}}{j+1} - \frac{H_{n+j+1}}{n+j+1}\right).\end{aligned}$$

(ii) We apply (1.1) and (2.8). Then

$$\begin{aligned}\sum_{k=1}^{n}\frac{H_k}{k} &= -\int_0^1 \frac{(1-t^n)\log(1-t)}{1-t}dt \\ &= -\int_0^1 \frac{(1-(1-x)^n)\log(x)}{x}dx \\ &= -\int_0^1 \log(x)(1-(1-x)^n)\mu\sum_{\nu=0}^{\infty}(1-\mu x)^{\nu}dx \\ &= -\mu\sum_{\nu=0}^{\infty}\int_0^1 \log(x)(1-(1-x)^n)(1-\mu x)^{\nu}dx \\ &= -\mu\sum_{\nu=0}^{\infty}\sum_{j=0}^{\nu}\binom{\nu}{j}(-\mu)^j T(n,j) \qquad (2.9)\end{aligned}$$

with

$$T(n,j) = \int_0^1 \log(x)(1-(1-x)^n)x^j dx.$$

Using (2.5), (2.6) and (2.7) gives

$$T(n,j) = S(n,j) - \frac{1}{(j+1)^2} = \sum_{k=1}^{n}\frac{(-1)^k}{(k+j+1)^2}\binom{n}{k}. \qquad (2.10)$$

From (2.9) and (2.10) we conclude that (1.21) holds. □

References

[1] S. Ahlgren and K. Ono, A Gaussian hypergeometric series evaluation and Apéry number congruences, *J. Reine Angew. Math.* **518** (2000), 187-212.

[2] H. Alzer and S. Koumandos, Series representations for γ and other mathematical constants, *Anal. Math.* **34** (2008), 1-8.

[3] H. Alzer and K. C. Richards, Series representations for special functions and mathematical constants, *Ramanujan J.* **40** (2016), 291-310.

[4] N. Batir, S. Sorgun and S. Atpinar, On some binomial coefficient identities with applications, *J. Integer Seq.* **26** (2023), Article 23.4.4.

[5] B. Berndt, *Ramanujan's Notebooks: Part I*, Springer, New York, 1985.

[6] J. Choi, Summation formulas involving binomial coefficients, harmonic numbers, and generalized harmonic numbers, *Abstr. Appl. Anal.* **2014** (2014), Article ID 501906.

[7] O. Furdui, Series involving products of two harmonic numbers, *Math. Mag.* **84** (2011), 371-377.

[8] M. Hassani, 3-dimensional L-summing method, *RGMIA Research Report Collection* **8**(1) (2005), Article 10.

[9] P. Kórus, Some applications of Retkes' identity, *Arch. Math.* **104** (2015), 217-222.

[10] J. C. Lagarias, An elementary problem equivalent to the Riemann hypothesis, *Amer. Math. Monthly* **109** (2002), 534-543.

[11] A. Sofo, Identities for alternating inverse squared binomial and harmonic number sums, *Mediterr. J. Math.* **13** (2016), 1407-1418.

[12] P. Turán, Über die arithmetischen Mittel der Fourierreihe, *J. London Math. Soc.* **10** (1935), 277-280.