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DIRICHLET CONVOLUTION AND THE BINET FORMULA

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Abstract

The main aim of this note is to show that the set of closed triples of generalized Fibonacci arithmetic functions under the Dirichlet convolution is a singleton set. This unique Dirichlet convolution identity is the Binet Fibonacci number formula in terms of arithmetic functions and the Dirichlet convolution.

1. Introduction

Exactly 180 years ago, J.Ph.M. Binet published the formula bearing his name, which still attracts the interest of various authors. Binet's Fibonacci number formula (Equation (5) below; and in the convolution setting Equation (6) below) is really captivating. The presence of the golden ratio (4) in this formula creates a close connection between the Fibonacci sequence (3) and this irrational number. Among many formulas with Fibonacci numbers, the Binet formula has a central and unique position. Our present article, emphasizes unequivocally this position of Binet's formula and of the golden ratio.

The connection of this formula with the Dirichlet convolution (defined by Equation (1) below) becomes visible through generalized Fibonacci sequences (defined by Equation (8) below). Certain sequences uniquely generate prime-independent multiplicative arithmetic functions (see Equation (2) below). In this field the Dirichlet convolution acts with maximum efficiency. Binet's formula in this framework has again a central and unique position (see Theorem 4). The generalized Fibonacci arithmetic functions $\mathcal{G}_\alpha = \alpha^\Omega$ and $\mathcal{G}_\beta = \beta^\Omega$ (where $\alpha = \frac{1+\sqrt{5}}{2}$ is the golden ratio, $\beta = 1 - \alpha = -\frac{1}{\alpha}$, and $\Omega(n)$ is the number of prime factors of n, each being counted according to its multiplicity) and their Dirichlet inverses $\mathcal{G}_\alpha^{-1}$ and $\mathcal{G}_\beta^{-1}$ fit perfectly

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into the model created by the Dirichlet convolution. As for the unitary convolution, this describes the analog of Binet's Fibonacci formula for the Lucas arithmetic function (see Theorem 5).

By Theorem 2, the Fibonacci arithmetic function $\mathcal{F}$ is specially multiplicative (i.e., it is the Dirichlet convolution of two completely multiplicative arithmetic functions). Haukkanen [1], and McCarthy and Sivaramakrishnan [5] were the first to highlight the connection that exists between Fibonacci sequences and specially multiplicative arithmetic functions. In [9], Newton sequences are characterized using Dirichlet and unitary convolutions. There are many book-references for theory and basis of arithmetic functions and Fibonacci numbers. We referred to [4], [8] and [3].

Sections 2 and 3 contain several theoretical foundations. In Section 4, the main Theorem is presented, and the 5th Section also includes the unitary convolution in the discussed topic.

Throughout this article, p is a prime, m is a non-negative integer and n is a positive integer.

2. Prime-Independent and Multiplicative Arithmetic Functions

An arithmetic function is a complex-valued function defined on the set of positive integers. An arithmetic function f is said to be multiplicative if $f(1) = 1$ and $f(n_1 n_2) = f(n_1)f(n_2)$ whenever $(n_1, n_2) = 1$, where (n_1, n_2) is the g.c.d. of n_1 and n_2. The Dirichlet convolution $f * g$ of two arithmetic functions f and g (see [4, Chapter 1] and [8, Section 1.1 and Chapter 2]) is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}), \tag{1}$$

where the summation is over the positive divisors d of n. The set of all multiplicative functions with this operation is a commutative group denoted $(\mathfrak{M}, *)$. The identity ι of this group is given by

$$\iota(n) = \begin{cases} 1 & \text{if } \ n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Multiplicative arithmetic functions arise in the study of the prime factorization of an integer. First of all, we need to mention the fact that a multiplicative arithmetic function is completely determined by its values at the prime powers.

We will say that an arithmetic function f is *prime-independent* if

$$f(p^m) = f(q^m)$$

for all prime pairs (p, q) and $m \geq 0$. Some examples of prime-independent and multiplicative arithmetic functions are given below:

- the identity ι of the group $(\mathfrak{M}, *)$ defined above;
- the zeta function ζ defined by $\zeta(n) = 1$ for all $n \geq 1$;
- c^{Ω}, where $\Omega(n)$ is the number of prime factors of n, each being counted according to its multiplicity (with $\Omega(1) = 0$), and $c \neq 0$ is a complex constant;
- c^{ω}, where $\omega(n)$ is the number of distinct prime factors of n ($\omega(1) = 0$);
- $\tau(n)$ is the number of divisors of n;
- $\theta(n)$ is the number of square-free divisors of n $\ (\theta(n) = 2^{\omega(n)} = \sum_{d|n} |\mu(d)|)$;
- the Möbius function μ:

$$\mu(n) = \begin{cases} 1 & \text{if } \ n = 1 \\ 0 & \text{if } \ p^2 | n \text{ for some prime } p \\ (-1)^{\omega(n)} & \text{otherwise;} \end{cases}$$

- Liouville's Lambda function λ: $\lambda(n) = (-1)^{\Omega(n)}$;
- the Dirichlet inverse of Liouville's Lambda function:

$$\lambda^{-1}(n) = \lambda(n)\mu(n) = \begin{cases} 1 & \text{if } \ n \text{ is square-free} \\ 0 & \text{otherwise.} \end{cases}$$

Now, given a sequence $\{U_n\}_{n\geq 1}$ that begins with $U_1 = 1$,

$$U_1 = 1,\ U_2,\ U_3,\ U_4,\ \cdots$$

we define the prime-independent and multiplicative arithmetic function $\mathcal{U}$ by

$$\mathcal{U}(p^m) = U_{m+1} \qquad \text{for all primes } p \text{ and } m \geq 0, \tag{2}$$

that is,

$$\mathcal{U}(n) = \begin{cases} 1 & \text{if} \qquad n = 1 \\ \prod_p U_{n(p)+1} & \text{if} \quad n = \prod_p p^{n(p)} \text{ is the canonical factorization of } n. \end{cases}$$

We call $\{U_n\}_{n\geq 1}$ the *generating sequence* of the prime-independent and multiplicative arithmetic function $\mathcal{U}$, and we will say that $\mathcal{U}$ is the *associate prime-independent multipicative arithmetic function* of the sequence $\{U_n\}_{n\geq 1}$.

3. The Fibonacci Sequence and Generalized Fibonacci Sequences

The Fibonacci sequence $\{F_n\}_{n\geq 1}$ starts with $F_1 = F_2 = 1$ and each term is the sum of the two preceding ones,

$$F_{n+2} = F_{n+1} + F_n$$

for all $n > 0$. Thus the Fibonacci sequence begins:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots \tag{3}$$

Binet's Fibonacci numbers formula (published in 1843) expresses the n^{th} Fibonacci number F_n in terms of the golden ratio

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = 1 - \alpha = -\frac{1}{\alpha} = \frac{1-\sqrt{5}}{2}. \tag{4}$$

Theorem 1 ([3, Theorem 5.6]). *We have:*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{5}$$

This famous formula has the following expression in terms of arithmetic functions and Dirichlet convolution (see [6, Theorem 1] and [7, Theorem 2.1] in a larger context).

Theorem 2. *We have:*

$$\mathcal{F} = \alpha^{\Omega} * \beta^{\Omega}, \tag{6}$$

where $\mathcal{F}$ is the associate prime-independent multiplicative arithmetic function of the Fibonacci sequence $\{F_n\}_{n\geq 1}$.

Obviously,

$$\mathcal{F}(p^m) = F_{m+1} = \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} = \sum_{i=0}^{m} \alpha^i \beta^{m-i} = (\alpha^{\Omega} * \beta^{\Omega})(p^m),$$

for all primes p and $m \geq 0$.

Remark 1. This is an opportune moment to notice that the Dirichlet inverse $\mathcal{F}^{-1}$ of $\mathcal{F}$ (i.e., the inverse of $\mathcal{F}$ in the group $(\mathfrak{M}, *)$) is the prime-independent and multiplicative arithmetic function given by

$$\mathcal{F}^{-1}(p^m) = \begin{cases} 1 & \text{if} \quad m = 0 \\ -1 & \text{if} \quad m = 1, 2 \\ 0 & \text{if} \quad m > 2. \end{cases} \tag{7}$$

Indeed,

$$(\mathcal{F} * \mathcal{F}^{-1})(p^m) = \begin{cases} 1 & \text{if} \quad m = 0 \\ F_2 - F_1 & \text{if} \quad m = 1 \\ F_{m+1} - F_m - F_{m-1} & \text{if} \quad m > 1, \end{cases}$$

and therefore $\mathcal{F} * \mathcal{F}^{-1} = \mathcal{F}^{-1} * \mathcal{F} = \iota$.

The Fibonacci numbers have been generalized in many ways. Extremely extensive bibliography exists in this field. In what follows we will refer to Chapter 7 of Koshy's book [3] which contains a few pages on generalized Fibonacci numbers.

A sequence $\{G_n\}_{n\geq 1}$ with given first and second terms $G_1 = a$, $G_2 = b$, and

$$G_{n+2} = G_{n+1} + G_n \qquad (n \geq 1) \tag{8}$$

is called in [3] a generalized Fibonacci sequence. The first theorem in Chapter 7 of the book [3] provides a useful characterization of generalized Fibonacci sequences in terms of ordinary Fibonacci numbers.

Theorem 3 ([3, Theorem 7.1]). *Let $\{G_n\}_{n\geq 1}$ be a generalized Fibonacci sequence that begins with $G_1 = a$ and $G_2 = b$. Then*

$$G_{n+2} = aF_n + bF_{n+1} \tag{9}$$

for all $n \geq 1$.

Now, if $a = 1$ then the generalized Fibonacci sequence $\{G_n\}_{n\geq 1}$ begins with

$$1, b, 1+b, 1+2b, 2+3b, 3+5b, 5+8b, \cdots .$$

We will call the associate prime-independent multiplicative arithmetic function $\mathcal{G}_b$ of this sequence,

$$\mathcal{G}_b(p^m) = G_{m+1} = \begin{cases} 1 & \text{if} \quad m = 0 \\ b & \text{if} \quad m = 1 \\ F_{m-1} + bF_m & \text{if} \quad m > 1, \end{cases}$$

a generalized Fibonacci arithmetic function (GFAF in short).

A first example of a GFAF is $\mathcal{G}_1 = \mathcal{F}$. The GFAF $\mathcal{G}_3$ is the prime-independent and multiplicative arithmetic function defined by

$$\mathcal{G}_3(p^m) = L_{m+1},$$

for all primes p and $m \geq 0$, where

$$L_1 = 1,\ L_2 = 3,\ L_3 = 4,\ L_4 = 7,\ L_5 = 11,\ L_6 = 18, \cdots \quad (L_{n+2} = L_{n+1} + L_n)$$

is the Lucas sequence. Using Equation (9) it follows that

$$L_{n+1} = F_{n-1} + 3F_n = F_{n+1} + 2F_n = F_{n+2} + F_n \qquad (n \geq 1). \tag{10}$$

The GFAF $\mathcal{G}_2$ is the left shift Fibonacci multiplicative arithmetic function given by

$$\mathcal{G}_2(p^m) = F_{m+2},$$

for all primes p and $m \geq 0$. A routine check shows that

$$\mathcal{G}_2 = \lambda^{-1} * \mathcal{F}. \tag{11}$$

It is also straightforward to check that the set of all GFAF's is not closed under the Dirichlet convolution.

4. Closed Triples Under the Dirichlet Convolution and the Binet Formula

We will say that a triple $(\mathcal{G}_b, \mathcal{G}_c, \mathcal{G}_d)$ of GFAF's is closed under the Dirichlet convolution if

$$\mathcal{G}_b = \mathcal{G}_c * \mathcal{G}_d.$$

Since the Dirichlet convolution is commutative we will not consider the closed triples $(\mathcal{G}_b, \mathcal{G}_c, \mathcal{G}_d)$ and $(\mathcal{G}_b, \mathcal{G}_d, \mathcal{G}_c)$ different. In the convolution-identity (11) the triple $(\mathcal{G}_2, \lambda^{-1}, \mathcal{F})$ is not a closed triple under the Dirichlet convolution since λ^{-1} is not a GFAF.

Theorem 4. *We have:*

(*i*) *the set of all closed triples under the Dirichlet convolution is a singleton set;*

(*ii*) *the Binet formula*

$$\mathcal{F} = \alpha^{\Omega} * \beta^{\Omega}$$

is the only convolution-identity for which the Dirichlet convolution of two GFAF's is a GFAF.

Proof. (*i*). Suppose that $\mathcal{G}_b = \mathcal{G}_c * \mathcal{G}_d$ and $n \geq 1$. Then

$$\begin{aligned}
0 &= \mathcal{G}_b(p^{n+1}) - \mathcal{G}_b(p^n) - \mathcal{G}_b(p^{n-1}) \\
&= \sum_{i=0}^{n+1} \mathcal{G}_c(p^i)\mathcal{G}_d(p^{n+1-i}) - \sum_{i=0}^{n} \mathcal{G}_c(p^i)\mathcal{G}_d(p^{n-i}) - \sum_{i=0}^{n-1} \mathcal{G}_c(p^i)\mathcal{G}_d(p^{n-1-i}) \\
&= \mathcal{G}_c(p^{n+1})\mathcal{G}_d(p^0) + \mathcal{G}_c(p^n)\mathcal{G}_d(p) - \mathcal{G}_c(p^n)\mathcal{G}_d(p^0) \\
&\quad + \sum_{i=0}^{n-1} \mathcal{G}_c(p^i)(\mathcal{G}_d(p^{n+1-i}) - \mathcal{G}_d(p^{n-i}) - \mathcal{G}_d(p^{n-1-i})) \\
&= \mathcal{G}_c(p^{n+1})\mathcal{G}_d(p^0) + \mathcal{G}_c(p^n)\mathcal{G}_d(p) - \mathcal{G}_c(p^n)\mathcal{G}_d(p^0) \\
&= (\mathcal{G}_c(p^n) + \mathcal{G}_c(p^{n-1}))\mathcal{G}_d(p^0) + \mathcal{G}_c(p^n)\mathcal{G}_d(p) - \mathcal{G}_c(p^n)\mathcal{G}_d(p^0) \\
&= \mathcal{G}_c(p^{n-1}) + \mathcal{G}_c(p^n)\mathcal{G}_d(p).
\end{aligned}$$

We observe that for $n = 1$ and $n = 2$,

$$0 = \mathcal{G}_c(p^{n-1}) + \mathcal{G}_c(p^n)\mathcal{G}_d(p) = \begin{cases} 1 + \mathcal{G}_c(p)\mathcal{G}_d(p) & \text{if} \quad n = 1 \\ \mathcal{G}_c(p) + (1 + \mathcal{G}_c(p))\mathcal{G}_d(p) & \text{if} \quad n = 2. \end{cases}$$

Hence,

$$\begin{cases} \mathcal{G}_c(p)\mathcal{G}_d(p) = -1 \\ \mathcal{G}_c(p) + \mathcal{G}_d(p) = 1. \end{cases}$$

Thus, $\mathcal{G}_c(p) = \alpha$ and $\mathcal{G}_d(p) = \beta$, that is, $c = \alpha$ and $d = \beta$. Under these conditions, $\mathcal{G}_c(p^{n-1}) + \mathcal{G}_c(p^n)\mathcal{G}_d(p) = 0$ if $n > 2$. That is because

$$\mathcal{G}_\alpha(p^{n-1}) + \mathcal{G}_\alpha(p^n)\mathcal{G}_\beta(p) = F_{n-2} + \alpha F_{n-1} + (F_{n-1} + \alpha F_n)\beta = F_{n-2} + F_{n-1} - F_n.$$

The value of b remains to be found. We have

$$b = \mathcal{G}_b(p) = (\mathcal{G}_\alpha * \mathcal{G}_\beta)(p) = \alpha + \beta,$$

and therefore $b = 1$, that is, $\mathcal{G}_b = \mathcal{F}$. Thus,

$$\mathcal{F} = \mathcal{G}_\alpha * \mathcal{G}_\beta,$$

and $(\mathcal{G}_1, \mathcal{G}_\alpha, \mathcal{G}_\beta)$ is the unique closed triple under the Dirichlet convolution.

(ii)-(a). Since $\alpha^2 = \alpha + 1$, multiplying both sides by α^n we obtain $\alpha^{n+2} = \alpha^{n+1} + \alpha^n$, and similarly we get $\beta^{n+2} = \beta^{n+1} + \beta^n$. It follows that the sequences

$$1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \cdots$$

and

$$1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \cdots$$

are both generalized Fibonacci sequences starting with $1, \alpha$, and $1, \beta$, respectively. Thus, $\mathcal{G}_\alpha = \alpha^\Omega$ and $\mathcal{G}_\beta = \beta^\Omega$.

(ii)-(b) (A second proof of (ii)). In the group $(\mathfrak{M}, *)$ the equalities

$$\mathcal{F} = \mathcal{G}_\alpha * \mathcal{G}_\beta \quad \text{and} \quad \mathcal{F} = \alpha^\Omega * \beta^\Omega$$

imply that

$$\mathcal{G}_\alpha^{-1} = \mathcal{G}_\beta * \mathcal{F}^{-1} \quad \text{and} \quad (\alpha^\Omega)^{-1} = \beta^\Omega * \mathcal{F}^{-1}.$$

Thus, using Equation (7) we get:

$$\begin{aligned}
&\mathcal{G}_\alpha^{-1}(p^0) = 1, \qquad \mathcal{G}_\alpha^{-1}(p) = (\mathcal{G}_\beta * \mathcal{F}^{-1})(p) = \beta - 1,\\
&\mathcal{G}_\alpha^{-1}(p^n) = (\mathcal{G}_\beta * \mathcal{F}^{-1})(p^n)\\
&\qquad\qquad = \mathcal{G}_\beta(p^n) - \mathcal{G}_\beta(p^{n-1}) - \mathcal{G}_\beta(p^{n-2}) = 0 \ \ (\text{if } n \geq 2);
\end{aligned}$$

and

$$\begin{aligned}
&(\alpha^\Omega)^{-1}(p^0) = 1, \qquad (\alpha^\Omega)^{-1}(p) = (\beta^\Omega * \mathcal{F}^{-1})(p) = \beta - 1,\\
&(\alpha^\Omega)^{-1}(p^n) = (\beta^\Omega * \mathcal{F}^{-1})(p^n) = \beta^{\Omega(p^n)} - \beta^{\Omega(p^{n-1})} - \beta^{\Omega(p^{n-2})}\\
&\qquad\qquad = \beta^n - \beta^{n-1} - \beta^{n-2} = \beta^{n-2}(\beta^2 - \beta - 1) = 0 \ \ (\text{if } n \geq 2).
\end{aligned}$$

So, $\mathcal{G}_\alpha^{-1} = (\alpha^\Omega)^{-1}$, that is, $\mathcal{G}_\alpha = \alpha^\Omega,$ and analogously $\mathcal{G}_\beta = \beta^\Omega$. $\square$

Remark 2. The proof (*ii*)-(a) (the fact that α^Ω and β^Ω are GFAF's) leads us (using Equation (9)) to the formula (see [3, Lemma 5.1])

$$\alpha^{n+1} = F_n + \alpha F_{n+1} \qquad (\text{and } \beta^{n+1} = F_n + \beta F_{n+1}). \tag{12}$$

This identity allows any polynomial in the golden ratio α (respectively, β) to be reduced directly to a linear expression. In addition, using Equations (12) and (10) it follows that the analog of Binet's Fibonacci number formula (5) for the Lucas numbers is (see [3, Theorem 5.8])

$$\alpha^n + \beta^n = L_n. \tag{13}$$

Remark 3. The second proof (*ii*)-(b) allows for the extended expression of the Dirichlet inverses of $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ (of α^Ω and β^Ω):

$$\mathcal{G}_\alpha^{-1}(n) = \begin{cases} 1 & \text{if } \ n=1 \\ 0 & \text{if } \ p^2|n \text{ for some prime } p \\ (\beta-1)^{\omega(n)} & \text{otherwise,} \end{cases} \tag{14}$$

and

$$\mathcal{G}_\beta^{-1}(n) = \begin{cases} 1 & \text{if } \ n=1 \\ 0 & \text{if } \ p^2|n \text{ for some prime } p \\ (\alpha-1)^{\omega(n)} & \text{otherwise.} \end{cases} \tag{15}$$

Corollary 1. *For any positive integer n, the following assertions are true:*

$$\sum_{d|n} \mathcal{G}_\alpha^{-1}(d)\mathcal{G}_\beta^{-1}(\frac{n}{d}) = \begin{cases} 1 & \text{if } \ n=1 \\ 0 & \text{if } \ p^3|n \text{ for some prime } p \\ (-1)^{\omega(n)} & \text{otherwise,} \end{cases} \tag{16}$$

$$\sum_{d|n} \mathcal{G}_\alpha^{-1}(d) + \sum_{d|n} \mathcal{G}_\beta^{-1}(d) = L_{\omega(n)} \qquad (\text{with } L_0 = 2 \ - \ \text{if } n=1). \tag{17}$$

Proof. The Equation (16) follows from the convolution-identity $\mathcal{F}^{-1} = \mathcal{G}_\alpha^{-1} * \mathcal{G}_\beta^{-1}$.

For the Equation (17) we refer to the Equations (14) and (15). It is clear that for any prime p we have

$$(\mathcal{G}_\alpha^{-1} * \zeta)(p^0) = 1 = (\mathcal{G}_\beta^{-1} * \zeta)(p^0)$$

and

$$(\mathcal{G}_\alpha^{-1} * \zeta)(p^n) = \mathcal{G}_\alpha^{-1}(p) + \mathcal{G}_\alpha^{-1}(p^0) = \beta \ ; \quad (\mathcal{G}_\beta^{-1} * \zeta)(p^n) = \mathcal{G}_\beta^{-1}(p) + \mathcal{G}_\beta^{-1}(p^0) = \alpha$$

for all $n \geq 1$. Therefore,

$$(\mathcal{G}_\alpha^{-1} * \zeta)(n) = \beta^{\omega(n)} \quad \text{and} \quad (\mathcal{G}_\beta^{-1} * \zeta)(n) = \alpha^{\omega(n)}.$$

The proof is complete since $\alpha^{\omega(n)} + \beta^{\omega(n)} = L_{\omega(n)}$ (see Equation (13)). $\square$

5. Some Remarks on the Unitary Convolution

The unitary convolution $f \sqcup g$ of two arithmetic functions f and g is defined by

$$(f \sqcup g)(n) = \sum_{d \| n} f(d) g(\frac{n}{d}),$$

where the summation is over the unitary divisors d of n, i.e., those positive divisors d for which $(d, \frac{n}{d}) = 1$. The triple $(\mathfrak{M}, \sqcup, *)$ is a quasi-field in the sense of Kesava-Menon [2], i.e., $(\mathfrak{M}, \sqcup)$ and $(\mathfrak{M}, *)$ are commutative isomorphic groups (the identity element ι under multiplication $*$ also serves as the identity element under the addition $\sqcup$) with the following quasi-distributive law (see [8, Theorem 26]):

$$f * (g \sqcup h) \sqcup f = (f * g) \sqcup (f * h).$$

In what follows, we will denote by $\mathcal{L}$ the prime-independent multiplicative arithmetic function defined by

$$\mathcal{L}(p^m) = \begin{cases} 1 & \text{if} \quad m = 0 \\ L_m & \text{if} \quad m > 0, \end{cases}$$

and we will say that $\mathcal{L}$ is the *Lucas arithmetic function.*

Theorem 5. *The following assertions are true.*

(i) The analog of Binet's Fibonacci arithmetic function formula for the Lucas arithmetic function is given by:

$$\mathcal{L} = \mathcal{G}_\alpha \sqcup \mathcal{G}_\beta \qquad (i.e., \ \mathcal{L} = \alpha^\Omega \sqcup \beta^\Omega).$$

(ii) The Möbius function is the unitary convolution of the Dirichlet inverses of $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$:

$$\mu = \mathcal{G}_\alpha^{-1} \sqcup \mathcal{G}_\beta^{-1}.$$

(iii) The characteristic arithmetic function χ of the set of squares of square-free positive integers (n is square-free if and only if $\mu(n) \neq 0$) is a bridge between the Fibonacci and the Lucas arithmetic functions given by:

$$\mathcal{F} * \chi = \mathcal{L}.$$

Proof. Let p be a prime number and $n > 0$. Then

(i).

$$\begin{aligned}(\mathcal{G}_\alpha \sqcup \mathcal{G}_\beta)(p^n) &= \mathcal{G}_\alpha(p^n) + \mathcal{G}_\beta(p^n) = \alpha^{\Omega(p^n)} + \beta^{\Omega(p^n)} \\ &= \alpha^n + \beta^n = L_n = \mathcal{L}(p^n).\end{aligned}$$

Since $\mathcal{G}_\alpha \sqcup \mathcal{G}_\beta$ and $\mathcal{L}$ are prime-independent and multiplicative arithmetic functions, it follows that $\mathcal{L} = \mathcal{G}_\alpha \sqcup \mathcal{G}_\beta$.

(*ii*). Once again, we refer to the Equations (14) and (15). We have:

$$(\mathcal{G}_\alpha^{-1} \sqcup \mathcal{G}_\beta^{-1})(p) = \mathcal{G}_\alpha^{-1}(p) + \mathcal{G}_\beta^{-1}(p) = \beta - 1 + \alpha - 1 = -1,$$

and

$$(\mathcal{G}_\alpha^{-1} \sqcup \mathcal{G}_\beta^{-1})(p^n) = \mathcal{G}_\alpha^{-1}(p^n) + \mathcal{G}_\beta^{-1}(p^n) = 0 \quad \text{if } n > 1.$$

It follows that $\mathcal{G}_\alpha^{-1} \sqcup \mathcal{G}_\beta^{-1} = \mu$.

(*iii*). Since χ is the prime-independent, multiplicative arithmetic function defined by:

$$\chi(p^m) = \begin{cases} 1 & \text{if } \ m = 0; 2 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(\mathcal{F}^{-1} * \mathcal{L})(p^m) = \begin{cases} 1 & \text{if} \quad m = 0 \\ -1 + L_1 = 0 & \text{if} \quad m = 1 \\ -1 - L_1 + L_2 = 1 & \text{if} \quad m = 2 \\ -L_{m-2} - L_{m-1} + L_m = 0 & \text{if} \quad m > 2, \end{cases}$$

the proof is complete. □

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