DISTRIBUTION OF 3-REGULAR AND 5-REGULAR PARTITIONS

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Abstract

In this paper, we study the functions $b_3(n)$ and $b_5(n)$, which denote the number of 3-regular partitions and 5-regular partitions of n, respectively. Using the theory of modular forms, we prove several arithmetic properties of $b_3(n)$ and $b_5(n)$ modulo primes greater than 3.

1. Introduction

The number of partitions of n in which no parts are multiples of k is denoted by $b_k(n)$; such partitions are known as *k-regular* partitions. It is also the number of partitions of n into at most $k-1$ copies of each part.

We define $b_3(0) = b_5(0) = 1$ for convenience. Moreover, let $b_3(n) = b_5(n) = 0$ if $n \notin \mathbb{Z}_{\geq 0}$. The k-regular partitions have a generating function as follows:

$$\sum_{n=0}^{\infty} b_k(n) q^n = \prod_{n=1}^{\infty} \frac{1-q^{kn}}{1-q^n}.$$

In 1919, Ramanujan found three remarkable congruences for the partition function $p(n)$ as follows:

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

In 2000, Ono [9] proved that for each prime number $m \geq 5$, there exist infinitely many arithmetic sequences $An+B$ such that

$$p(An+B) \equiv 0 \pmod{m}.$$

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We refer to such congruences as *Ramanujan-type congruences*. Subsequently, Lovejoy [6] obtained similar results for the function $Q(n)$, which counts the number of partitions of n into distinct parts. Following the strategies of Ono and Lovejoy, we prove the following theorem.

Theorem 1. *The following hold:*

- *For each prime $m \geq 5$, there are infinitely many Ramanujan-type congruences of $b_3(n)$ modulo m.*
- *For each prime $m \geq 7$, there are infinitely many Ramanujan-type congruences of $b_5(n)$ modulo m.*

Lovejoy and Penniston [7] studied the distribution of $b_3(n)$ modulo 3, while Keith and Zanello [5] studied the parity of $b_3(n)$. As for 5-regular partitions, Calkin et al. [1] as well as Hirschhorn and Sellers [4] studied the parity of $b_5(n)$. Gordon and Ono [3] analyzed the distribution of $b_5(n)$ modulo 5. Moreover, they proved that

$$b_5(5n+4) \equiv 0 \pmod{5}. \tag{1}$$

So far, we have only studied the distribution of $b_3(n)$ and $b_5(n)$ modulo the primes mentioned above. In this paper, we extend our investigation to the distribution of $b_3(n)$ and $b_5(n)$ modulo primes $m \geq 5$. It is worth noting that we still lack information about $b_5(n)$ modulo 3.

We can naturally derive a corollary from Theorem 1, which is as follows:

Corollary 1. *If $m \geq 5$ is a prime and $k \in \{3, 5\}$, then there are infinitely many positive integers n for which*

$$b_k(n) \equiv 0 \pmod{m}.$$

More precisely, we have

$$\#\{0 \leq n \leq X \ : \ b_k(n) \equiv 0 \pmod{m}\} \gg X.$$

For other residue classes $i \not\equiv 0 \pmod{m}$, we provide a useful criterion to verify whether there are infinitely many n such that $b_k(n) \equiv i \pmod{m}$.

Theorem 2. *If $m \geq 5$ is a prime and there exists an integer k such that*

$$b_3\left(mk + \frac{m^2-1}{12}\right) \equiv e \not\equiv 0 \pmod{m},$$

then for each $i = 1, 2, \cdots, m-1$, we have

$$\#\{0 \leq n \leq X \ : \ b_3(n) \equiv i \pmod{m}\} \gg \frac{X}{\log X}.$$

Moreover, if such a k exists, then $k < 18(m-1)$.

We obtain similar results for $b_5(n)$.

Theorem 3. *If $m \geq 5$ is a prime and there exists an integer k such that*

$$b_5\left(mk + \frac{m^2-1}{6}\right) \equiv e \not\equiv 0 \pmod{m},$$

then for each $i = 1, 2, \cdots, m-1$, we have

$$\#\{0 \leq n \leq X \mid b_5(n) \equiv i \pmod{m}\} \gg \frac{X}{\log X}.$$

Moreover, if such a k exists, then $k < 10(m-1)$.

Remark 1. The congruence (1) shows that our criterion is inapplicable for the case $m = 5$. Nevertheless, the case $m = 5$ is studied in [3].

2. Preliminaries on Modular Forms

First, we introduce the U *operator* on formal series. For a positive integer j, we define it as follows:

$$\left(\sum_{n=0}^{\infty} a(n)q^n\right) \mid U(j) := \sum_{n=0}^{\infty} a(jn)q^n.$$

Recalling that *Dedekind's eta function* is defined by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n),$$

where $q = e^{2\pi i z}$ and z is a complex number with $\operatorname{Im} z > 0$.

If m is a prime, let $M_k(\Gamma_0(N), \chi)_m$ (respectively $S_k(\Gamma_0(N), \chi)_m$) denote the $\mathbb{F}_m$-vector space of the reductions modulo m of the q-expansions of modular forms (respectively cusp forms) in $M_k(\Gamma_0(N), \chi)$ (respectively $S_k(\Gamma_0(N), \chi)$) with integer coefficients.

We need the following theorem to construct modular forms.

Theorem 4 ([2])**.** *Let*

$$f(z) = \prod_{\delta \mid N} \eta^{r_\delta}(\delta z)$$

be an η-quotient. If $f(z)$ has the additional properties that

(i)

$$\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24};$$

(ii)

$$\sum_{\delta|N} \frac{N r_\delta}{\delta} \equiv 0 \pmod{24};$$

(iii)

$$k := \frac{1}{2} \sum_{\delta|N} r_\delta \in \mathbb{Z},$$

then

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z),$$

for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. *Here the Dirichlet character* $\chi \pmod{N}$ *is defined by*

$$\chi(n) := \left(\frac{(-1)^k \prod_{\delta|N} \delta^{r_\delta}}{n}\right), \text{ if } n > 0 \text{ and } (n,6) = 1.$$

If $f(z)$ is holomorphic (respectively vanishes) at all cusps of $\Gamma_0(N)$, then $f(z) \in M_k(\Gamma_0(N), \chi)$ (respectively $S_k(\Gamma_0(N), \chi)$), since $\eta(z)$ never vanishes on $\mathcal{H}$. The following theorem provides a useful criterion for computing the orders of an η-quotient at all cusps of $\Gamma_0(N)$.

Theorem 5 ([8]). *Let c, d and N be positive integers with $d \mid N$ and $(c,d) = 1$. If $f(z)$ is an η-quotient satisfying the conditions of Theorem 4, then the order of vanishing of $f(z)$ at the cusp c/d is*

$$\frac{N}{24} \sum_{\delta|N} \frac{r_\delta (d^2, \delta^2)}{\delta (d^2, N)}.$$

3. Ramanujan-Type Congruences

In this section, we will prove Theorem 1 using the theory of modular forms. However, the generating function of the regular partition function is not a modular form. For primes $m \geq 5$, it turns out that for a properly chosen function $h_m(n)$,

$$\sum_{n=0}^{\infty} b_k(h_m(n)) q^n$$

is the Fourier expansion of a cusp form modulo m. In fact, we can state the following theorem.

Theorem 6. *If $m \geq 5$ is a prime, then*

$$\sum_{n=0}^{\infty} b_3\left(\frac{mn-1}{12}\right) q^n \in S_{3m-3}(\Gamma_0(432), \chi_{12})_m,$$

where $\chi_{12}(n) = \left(\frac{3}{n}\right)$.

Theorem 7. *If $m \geq 5$ is a prime, then*

$$\sum_{n=0}^{\infty} b_5\left(\frac{mn-1}{6}\right) q^n \in S_{2m-2}(\Gamma_0(180), \chi_5)_m,$$

where $\chi_5(n) = \left(\frac{n}{5}\right)$.

Proof of Theorem 6. We begin with an η-quotient defined as follows:

$$f(m;z) := \frac{\eta(3z)}{\eta(z)} \eta^a(3mz)\eta^b(mz),$$

where $m' := (m \bmod 12)$, $a := 9 - m'$, and $b := m' - 3$.

It is easy to verify that $f(m;z) \equiv_m \eta^{am+1}(3z)\eta^{bm-1}(z)$ satisfies the conditions of Theorem 4. Furthermore, by applying Theorem 5, one can compute that $\eta^{am+1}(3z)\eta^{bm-1}(z)$ has the minimal order of vanishing of $(m(3a+b)+2)/24$ at the cusp ∞ and $(m(a+3b)-2)/24$ at the cusp 0.

Since $(m(3a+b)+2)/24 = (m(12-m')+1)/12 > 0$ and $(m(a+3b)-2)/24 = (mm'-1)/12 > 0$, we can conclude that $\eta^{am+1}(3z)\eta^{bm-1}(z) \in S_{3m}(\Gamma_0(3), \chi_3)$, where $\chi_3(n) = \left(\frac{n}{3}\right)$. On the other hand,

$$f(m;z) = \sum_{n=0}^{\infty} b_3(n) q^{n+\frac{m(3a+b)+2}{24}} \cdot \prod_{n=1}^{\infty} (1-q^{3mn})^a (1-q^{mn})^b. \tag{2}$$

Applying the U operator to both sides of (2), we have

$$\begin{aligned} &\eta^{am+1}(3z)\eta^{bm-1}(z) \mid U(m) \\ &\qquad \equiv_m \left(\sum_{n=0}^{\infty} b_3(n) q^{n+\frac{m(3a+b)+2}{24}} \mid U(m) \right) \cdot \prod_{n=1}^{\infty} (1-q^{3n})^a (1-q^n)^b. \end{aligned} \tag{3}$$

As for the right-hand side of (3), we have

$$\sum_{n=0}^{\infty} b_3(n) q^{n+\frac{m(3a+b)+2}{24}} \mid U(m) = \sum_{n\geq 0}{}^{*} b_3(n) q^{\frac{24n+m(3a+b)+2}{24m}},$$

where $\sum^*$ denotes taking the integral power coefficients of q, i.e.

$$24n + m(3a+b) + 2 \equiv 0 \pmod{24m}.$$

It is easy to check that $24 \mid 24n + m(3a+b) + 2$. Therefore, the condition simplifies to $m \mid 12n+1$.

As for the left-hand side of (3), we have

$$\eta^{am+1}(3z)\eta^{bm-1}(z) \mid U(m) \equiv_m \eta^{am+1}(3z)\eta^{bm-1}(z) \mid T(m),$$

where $T(m)$ denotes the usual Hecke operator acting on $S_{3m}(\Gamma_0(3), \chi_3)$.

Now, let us analyze the η-product $\eta^6(z)\eta^6(3z)$. According to Theorem 4 and Theorem 5, we find that $\eta^6(z)\eta^6(3z)$ is a cusp form of weight 6 and level 3, with the minimal order of vanishing of 1 at the two cusps of $\Gamma_0(3)$. Since $\eta(z)$ never vanishes on $\mathcal{H}$, we can derive that $\eta^{am+1}(3z)\eta^{bm-1}(z) \mid T(m) = \eta^6(z)\eta^6(3z)g(m;z)$, where $g(m;z) \in M_{3m-6}(\Gamma_0(3), \chi_3)$.

In summary, we have

$$\sum_{\substack{n\geq 0\\ m\mid 12n+1}} b_3(n)q^{\frac{24n+m(3a+b)+2}{24m}} \equiv_m \frac{\eta^6(z)\eta^6(3z)g(m;z)}{\prod_{n=1}^{\infty}(1-q^{3n})^a(1-q^n)^b}. \tag{4}$$

By replacing q with q^{12} and then multiplying both sides of (4) by $q^{-(3a+b)/2}$, we obtain

$$\sum_{\substack{n\geq 0\\ m\mid 12n+1}} b_3(n)q^{\frac{12n+1}{m}} \equiv_m \eta^{6-a}(36z)\eta^{6-b}(12z)g(m;12z),$$

namely,

$$\sum_{n=0}^{\infty} b_3\left(\frac{mn-1}{12}\right)q^n \equiv_m \eta^{6-a}(36z)\eta^{6-b}(12z)g(m;12z).$$

Using Theorem 4 and Theorem 5 once again, one can verify that $\eta^{6-a}(36z)\eta^{6-b}(12z)$ is a cusp form of weight 3 and level 432. It has the minimal order of vanishing of m' at the cusps c/d if $d \in \{1,2,3,4,6,8,12,16,24,48\}$, and $12-m'$ at the cusp if d is any other divisor of 432.

Therefore, we obtain

$$\eta^{6-a}(36z)\eta^{6-b}(12z) \in S_3\left(\Gamma_0(432), \chi_4\right),$$

where $\chi_4(n) = \left(\frac{-1}{n}\right)$. Together with $g(m;12z) \in M_{3m-6}(\Gamma_0(36), \chi_3)$, we have

$$\sum_{n=0}^{\infty} b_3\left(\frac{mn-1}{12}\right)q^n \in S_{3m-3}(\Gamma_0(432), \chi_{12})_m.$$

□

Proof of Theorem 7. For a fixed prime m, let

$$f(m;z) := \frac{\eta(5z)}{\eta(z)}\eta^a(5mz)\eta^b(mz),$$

where $m' := (m \bmod 6)$, $a := 5 - m'$, and $b := m' - 1$. It is easy to show that $f(m; z) \equiv \eta^{am+1}(5z)\eta^{bm-1}(z) \pmod{m}$ and

$$\eta^{am+1}(5z)\eta^{bm-1}(z) \in S_{2m}(\Gamma_0(5), \chi_5),$$

where $\chi_5(n) = \left(\frac{n}{5}\right)$. On the other hand,

$$f(m; z) = \sum_{n=0}^{\infty} b_5(n) q^{\frac{24n+m(5a+b)+4}{24}} \cdot \prod_{n=1}^{\infty} (1 - q^{5mn})^a (1 - q^{mn})^b.$$

Having the operator $U(m)$ act on $f(m; z)$ and using the fact that $U(m) \equiv_m T(m)$, we obtain

$$\sum_{n=0}^{\infty} b_5(n) q^{\frac{24n+m(5a+b)+4}{24}} \mid U(m) \equiv \frac{\eta^{am+1}(5z)\eta^{bm-1}(z) \mid T(m)}{\prod_{n=1}^{\infty}(1 - q^{5n})^a (1 - q^n)^b} \pmod{m}, \qquad (5)$$

where $T(m)$ denotes the usual Hecke operator acting on $S_{2m}(\Gamma_0(5), \chi_5)$. As for the left-hand side of (5), we have

$$\sum_{n=0}^{\infty} b_5(n) q^{\frac{24n+m(5a+b)+4}{24}} \mid U(m) = \sum_{\substack{n=0 \\ m \mid 6n+1}}^{\infty} b_5(n) q^{\frac{24n+m(5a+b)+4}{24m}}.$$

Using Theorems 4 and 5, one can verify that $\eta^4(5z)\eta^4(z) \in S_4(\Gamma_0(5))$ and has order of 1 at all cusps. Thus, we can write $\eta^{am+1}(5z)\eta^{bm-1}(z) \mid T(m) = \eta^4(5z)\eta^4(z)g(m; z)$, where $g(m; z) \in M_{2m-4}(\Gamma_0(5), \chi_5)$. Hence

$$\sum_{\substack{n=0 \\ m \mid 6n+1}}^{\infty} b_5(n) q^{\frac{6n+1}{6m}} \equiv \eta^{4-a}(5z)\eta^{4-b}(z)g(m; z) \pmod{m}.$$

Replacing q with q^6 shows that

$$\sum_{\substack{n=0 \\ m \mid 6n+1}}^{\infty} b_5(n) q^{\frac{6n+1}{m}} \equiv \eta^{4-a}(30z)\eta^{4-b}(6z)g(m; 6z) \pmod{m}.$$

Since $b_5(n)$ vanishes for non-integer n, we have

$$\sum_{n=0}^{\infty} b_5\left(\frac{mn-1}{6}\right) q^n \equiv \eta^{4-a}(30z)\eta^{4-b}(6z)g(m; 6z) \pmod{m}.$$

Moreover, one can verify that $\eta^{4-a}(30z)\eta^{4-b}(6z) \in S_2(\Gamma_0(180))$. Together with the fact that $g(m; 6z) \in M_{2m-4}(\Gamma_0(30), \chi_5)$, we have

$$\sum_{n=0}^{\infty} b_5\left(\frac{mn-1}{6}\right) q^n \in S_{2m-2}(\Gamma_0(180), \chi_5)_m.$$

$\square$

We need some important results of Serre, which are critical factors in the existence of Ramanujan-type congruences.

Theorem 8 ([10]). *The set of primes $l \equiv -1 \pmod{Nm}$ such that*

$$f \mid T(l) \equiv 0 \pmod{m}$$

for each $f(z) \in S_k(\Gamma_0(N), \psi)_m$ has positive density, where $T(l)$ denotes the usual Hecke operator acting on $S_k(\Gamma_0(N), \psi)$.

Now, Theorem 1 is an immediately corollary of the next two theorems.

Theorem 9. *Let $m \geq 5$ be a prime. A positive density of the primes l has the property that*

$$b_3\left(\frac{mln-1}{12}\right) \equiv 0 \pmod{m}$$

for each nonnegative integer n coprime to l.

Theorem 10. *Let $m \geq 5$ be a prime. A positive density of the primes l has the property that*

$$b_5\left(\frac{mln-1}{6}\right) \equiv 0 \pmod{m}$$

for each nonnegative integer n coprime to l.

Proof of Theorem 9. Let

$$F(m;z) = \sum_{n=0}^{\infty} b_3\left(\frac{mn-1}{12}\right) q^n \in S_{3m-3}(\Gamma_0(432), \chi_{12})_m.$$

For a fixed prime $m \geq 5$, let $S(m)$ denote the set of primes l such that

$$f \mid T(l) \equiv 0 \pmod{m}$$

for each $f \in S_{3m-3}(\Gamma_0(432), \chi_{12})$. By Theorem 8, the set $S(m)$ contains a positive density of primes. Hence for $l \in S(m)$, we have

$$F(m;z) \mid T(l) \equiv 0 \pmod{m}.$$

Then by the theory of Hecke operators, we have

$$F(m;z) \mid T(l) = \sum_{n=0}^{\infty} \left(b_3\left(\frac{mln-1}{12}\right) + \left(\frac{3}{l}\right) l^{3m-4} b_3\left(\frac{mn/l-1}{12}\right)\right) q^n \equiv 0 \pmod{m}.$$

Since $b_3(n)$ vanishes when n is not an integer, we have $b_3\left((mn/l-1)/12\right) = 0$ for each n coprime to l. Thus

$$b_3\left(\frac{mln-1}{12}\right) \equiv 0 \pmod{m}$$

is satisfied for each integer n coprime to l. Moreover, the set of such primes l is of positive density. □

Proof of Theorem 10. Let

$$F(m;z)=\sum_{n=0}^{\infty} b_5\left(\frac{mn-1}{6}\right)q^n \in S_{2m-2}(\Gamma_0(180),\chi_5)_m.$$

According to Theorem 8, the set of primes l for which

$$F(m;z)\mid T(l)\equiv 0 \pmod{m}$$

has positive density, where $T(l)$ denotes the Hecke operator on $S_{2m-2}(\Gamma_0(180),\chi_5)$. Moreover, by the theory of Hecke operators, we have

$$\sum_{n=0}^{\infty} F(m;z)\mid T(l)=\sum_{n=0}^{\infty}\left(b_5\left(\frac{mln-1}{6}\right)+\left(\frac{l}{5}\right)l^{2m-3}b_5\left(\frac{mn/l-1}{6}\right)\right)q^n.$$

Since $b_5(n)$ vanishes for non-integer n, we have $b_5((mn/l-1)/6)=0$ when $(n,l)=1$. Thus

$$b_5\left(\frac{mln-1}{6}\right)\equiv 0 \pmod{m}$$

is satisfied for each integer n with $(n,l)=1$. Moreover, the set of such primes l is of positive density. □

Since the selections for l are infinite, let us choose $l>3$. By replacing n with $12nl+ml+12$, we then have $b_3(ml^2n+ml+(m^2l^2-1)/12)\equiv 0 \pmod{m}$ satisfied for each integer n. A similar approach can be applied to $b_5(n)$. Hence, we obtain Theorem 1. Moreover, using the fact that the selections for l are infinite, together with the Chinese Remainder Theorem and previous results, we obtain the following result.

Corollary 2. *If m is a squarefree integer, then there are infinitely many Ramanujan-type congruences of $b_3(n)$ modulo m; if k is a squarefree integer coprime to 3, then there are infinitely many Ramanujan-type congruences of $b_5(n)$ modulo k.*

4. Distribution on Nonzero Residues

Following Lovejoy [6], we need the following theorem of Serre.

Theorem 11 ([10])**.** *The set of primes $l\equiv 1 \pmod{Nm}$ such that*

$$a(nl^r)\equiv (r+1)a(n) \pmod{m}$$

for every $f(z)=\sum_{n=0}^{\infty} a(n)q^n\in S_k(\Gamma_0(N),\psi)_m$ has positive density, where r is a positive integer and n is coprime to l.

Here, we introduce a theorem of Sturm, which provides a useful criterion for determining when modular forms with integer coefficients become congruent to zero modulo a prime through finite computation.

Theorem 12 ([11]). *Let* $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)_m$ *be a modular form such that*

$$a(n) \equiv 0 \pmod{m}$$

for all $n \leq \frac{kN}{12} \prod_{p|N} \left(1 + \frac{1}{p}\right)$. *Then* $a(n) \equiv 0 \pmod{m}$ *for all* $n \in \mathbb{Z}$.

We next present the proof of Theorem 2. The proof of Theorem 3 is very similar, so we omit it.

Proof of Theorem 2. If there is one $k \in \mathbb{Z}$ such that

$$b_3\left(mk + \frac{m^2-1}{12}\right) \equiv e \not\equiv 0 \pmod{m},$$

let $s = 12k + m$. Since $b_3(n)$ vanishes for negative n, we have $mk + \frac{m^2-1}{12} \geq 0$. Hence $s = 12k + m > 0$ and

$$b_3\left(\frac{ms-1}{12}\right) = b_3\left(mk + \frac{m^2-1}{12}\right) \equiv e \pmod{m}.$$

For a fixed prime $m \geq 5$, let $R(m)$ denote the set of primes l such that

$$a(nl^r) \equiv (r+1)a(n) \pmod{m}$$

for each $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in S_{3m-3}(\Gamma_0(432), \chi_{12})_m$, where r is a positive integer, and n is coprime to l. By the proof of Theorem 9, we have $\sum_{n=0}^{\infty} b_3\left(\frac{mn-1}{12}\right) q^n \in S_{3m-3}(\Gamma_0(432), \chi_{12})_m$. Since $R(m)$ is infinite by Theorem 11, we can choose an $l \in R(m)$ such that $l > s$, then

$$b_3\left(\frac{ml^r s-1}{12}\right) \equiv (r+1)b_3\left(\frac{ms-1}{12}\right) \equiv (r+1)e \pmod{m}.$$

Now, we fix l and choose $\rho \in R(m)$ such that $\rho > l$, then

$$b_3\left(\frac{m\rho n-1}{12}\right) \equiv 2b_3\left(\frac{mn-1}{12}\right) \pmod{m} \tag{6}$$

is satisfied for each n coprime to ρ. For each $i = 1, 2, \cdots, m-1$, let $r_i \equiv i(2e)^{-1} - 1 \pmod{m}$ and $r_i > 0$. Let $n = l^{r_i}s$ in (6), to obtain

$$b_3\left(\frac{m\rho l^{r_i} s-1}{12}\right) \equiv 2b_3\left(\frac{ml^{r_i} s-1}{12}\right) \equiv 2(r_i+1)e \equiv i \pmod{m}.$$

Since the variables, except for ρ, are fixed, it suffices to prove that the estimate for the choices of $\rho \gg X/\log X$, which can be easily derived from Theorem 11 and the Prime Number Theorem.

Moreover, by Sturm's Theorem, if $b_3\left(\frac{mn-1}{12}\right) \equiv 0 \pmod m$ for each $n \leq 216(m-1)$, then $b_3\left(\frac{mn-1}{12}\right) \equiv 0 \pmod m$ for all $n \in \mathbb{Z}$. Since $b_3(n)$ vanishes if n is not an integer, it suffices to compute those n of the form $12j+m$ for $12j+m \leq 216(m-1)$. This implies that $j < 18(m-1)$. In addition,

$$b_3\left(\frac{m(12j+m)-1}{12}\right) = b_3\left(mj + \frac{m^2-1}{12}\right).$$

Thus, if such a k exists, then $k < 18(m-1)$. □

5. Examples of Ramanujan-Type Congruences

By Theorem 12, we find that

$$\sum_{n=0}^{\infty} b_3\left(\frac{mn-1}{12}\right) q^n \mid T(l) \equiv 0 \pmod m$$

for the pairs (m, l) listed in the following table.

m	l
5	61,79,97,181,211,233,283,383,401,439,449,463, 557,641,647,691,739,743,751,863,887,907,947, 953,977,983,997,1093,1097,1129,1153,1201
7	71,761,1321,1607,1657,2543,2617
11	12553

An elementary computation yields the following result.

Proposition 1. *For the pairs (m, l) listed above, we have*

$$b_3\left(ml(ln+j) + \frac{m^2l^2-1}{12}\right) \equiv 0 \pmod m$$

for each n and $1 \leq j \leq l-1$.

We have the following examples.

Example 1.

$$b_3(18605n + 127) \equiv 0 \pmod 5,$$

$$b_3(35287n + 207) \equiv 0 \pmod 7,$$

$$b_3(1733355899n + 126576) \equiv 0 \pmod{11}.$$

Our method is not applicable to the case $m = 3$. Nevertheless, it can be proven that there are infinitely many Ramanujan-type congruences modulo 3 through the results of Lovejoy and Penniston [7, Corollary 4].

Proposition 2. *If Q is a prime of the form $12k + 1$, then*

$$b_3\left(Q^3 n + \frac{Q^2 - 1}{12}\right) \equiv 0 \pmod{3}.$$

For example, we obtain

$$b_3\left(2197n + 14\right) \equiv 0 \pmod{3}.$$

As for $b_5(n)$, we compute that

$$\sum_{n=0}^{\infty} b_5\left(\frac{mn-1}{6}\right) q^n \mid T(l) \equiv 0 \pmod{m}$$

for the pairs (m, l) listed in the following table.

m	l
7	17,37,197,211,239,263,269,271,331, 397,457,503,563,569,587,811, 823,853,929,941,1049,1163
11	41,1553,1867,4021,4783,6947,7193,7559
13	16519

An elementary computation yields the following result.

Proposition 3. *For the pairs (m, l) listed above, we have*

$$b_5\left(ml(ln + j) + \frac{m^2 l^2 - 1}{6}\right) \equiv 0 \pmod{m}$$

for each n and $1 \leq j \leq l - 1$.

We have the following examples.

Example 2.

$$b_5(2023n + 99) \equiv 0 \pmod{7},$$
$$b_5(18491n + 75) \equiv 0 \pmod{11},$$
$$b_5(3547405693n + 35791) \equiv 0 \pmod{13}.$$

Moreover, the congruence $b_5(5n + 4) \equiv 0 \pmod{5}$ implies that

$$\sum_{n=0}^{\infty} b_5\left(\frac{5n-1}{6}\right) q^n \mid T(l) \equiv 0 \pmod{5}$$

for all primes l.

6. More on k-Regular Partitions

In this paper, we prove that for $b_3(n)$ (respectively $b_5(n)$) and each prime $m \geq 5$ (respectively $m \geq 7$), there are infinitely many Ramanujan-type congruences modulo m. However, the situation for $b_5(n)$ modulo 3 remains a topic of investigation.

Problem 1. Find a congruence modulo 3 for $b_5(n)$, or provide a proof that no such congruence exists.

We also propose the following conjecture, which is analogous to Newman's Conjecture.

Conjecture 1. If m is an integer and $k = 3, 5$, then for each residue class $r \pmod{m}$, there exist infinitely many integers n such that $b_k(n) \equiv r \pmod{m}$.

While Ramanujan-type congruences modulo primes $m \geq 5$ do exist, discovering them may require extensive computational efforts. We encourage interested readers to explore congruences modulo different primes and seek examples.

We can adapt our proof to derive partial results for $b_{11}(n)$, which counts the number of 11-regular partitions of n. Specifically, if p is a prime greater than 5 and satisfies $p \equiv 5, 7 \pmod{12}$, then we can discover infinitely many Ramanujan-type congruences for $b_{11}(n)$ modulo p. For instance, we obtain $b_{11}(43687n + 230) \equiv 0 \pmod{7}$. However, even more promisingly, the existence of such congruences is assured by [12]. We plan to explore these findings in a future paper.

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