SOME VARIATIONS OF q-ANALOGUES OF SUN AND TAURASO'S CONGRUENCE

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Abstract

We establish some q-congruences related to Sun and Tauraso's congruence:

$$\sum_{k=0}^{p^r-1} \frac{1}{2^k}\binom{2k}{k} \equiv (-1)^{\frac{p^r-1}{2}} \pmod{p},$$

where p is an odd prime and r is a positive integer.

1. Introduction

Congruences involving central binomial coefficients $\binom{2n}{n}$ have attracted many experts' attention in recent years. In 2010, Sun and Tauraso [6] proved that for any odd prime p and positive integer r,

$$\sum_{k=0}^{p^r-1} \frac{1}{2^k}\binom{2k}{k} \equiv (-1)^{\frac{p^r-1}{2}} \pmod{p}. \tag{1}$$

Sun [5] showed that (1) also holds modulo p^2 in the same year.

Subsequently, Guo and Zeng [4, Corollary 4.2] first established a q-analogue of (1) as follows:

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k}{(q;q)_k} q^k \equiv (-1)^{\frac{n-1}{2}} q^{\frac{n^2-1}{4}} \pmod{\Phi_n(q)}, \tag{2}$$

where n is an odd positive integer. We remark that Guo [2] further showed that (2) holds modulo $\Phi_n(q)^2$. Here and throughout the paper, the q-shifted factorials are given by $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geq 1$ and $(a;q)_0 = 1$. The nth cyclotomic polynomial is given by

$$\Phi_n(q) = \prod_{\substack{1\leq k\leq n \\ (n,k)=1}} (q-\zeta^k),$$

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where ζ denotes a primitive nth root of unity.

For more q-congruences related to (1), we refer the interested reader to [1, 3, 7, 8]. It is worth mentioning that Guo [3, Theorem 1] recently established two new q-analogues of (1):

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k} q^{2k+1} \equiv (-1)^{\frac{n-1}{2}} \pmod{\Phi_n(q)}, \tag{3}$$

and

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{2k} \equiv (-1)^{\frac{n-1}{2}} \pmod{\Phi_n(q)}, \tag{4}$$

where n is an odd positive integer.

The first aim of the paper is to give two different q-analogues of (1) as follows.

Theorem 1. *For any positive odd integer n, we have*

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k} q^{4(k+1)} \equiv (-1)^{\frac{n-1}{2}}(1-q+q^2) \pmod{\Phi_n(q)}, \tag{5}$$

and

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{4k+1} \equiv (-1)^{\frac{n-1}{2}}(2-q) \pmod{\Phi_n(q)}. \tag{6}$$

Letting $q \to 1$ and $n = p^r$ in the Theorem 1, (5) and (6) reduce to (1). So (5) and (6) are q-analogues of(1).

Gu and Guo [1, Theorem 1] established q-analogues of the following two congruences related to (1):

$$\sum_{k=0}^{p^r-1} \frac{1}{2^k(2k-1)}\binom{2k}{k} \equiv (-1)^{\frac{p^r-1}{2}} \pmod{p}, \tag{7}$$

and

$$\sum_{k=0}^{p^r-1} \frac{2k+1}{2^k}\binom{2k}{k} \equiv (-1)^{\frac{p^r+1}{2}} \pmod{p}.$$

The second aim of the paper is to give two new q-analogues of (7) as follows.

Theorem 2. *For any positive odd integer $n \geq 3$, we have*

$$\sum_{k=0}^{n-1} \frac{(q^{-1};q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{2k} \equiv (-1)^{\frac{n+1}{2}} \pmod{\Phi_n(q)}, \tag{8}$$

and

$$\sum_{k=0}^{n-1} \frac{\left(q^{-1};q^2\right)_k \left(-1;q^2\right)_k}{(q^2;q^2)_k} q^{4k} \equiv (-1)^{\frac{n+1}{2}}(2q-1) \pmod{\Phi_n(q)}. \tag{9}$$

Next, we shall extend Theorems 1 and 2 through Carlitz's identity.

Theorem 3. *For any integer d and odd positive integer n with $n > 2 \mid d \mid -1$, we have*

$$\sum_{k=0}^{n-1} \frac{(q^{2d+1};q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{2k} \equiv (-1)^{\frac{n-1}{2}+d} \qquad \pmod{\Phi_n(q)}, \tag{10}$$

and

$$\sum_{k=0}^{n-1} \frac{(q^{2d+1};q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{4k} \equiv (-1)^{\frac{n-1}{2}+d} \frac{2-q^{2d+1}}{q^{2d+1}} \pmod{\Phi_n(q)}. \tag{11}$$

In fact, Theorem 2 is a special case of Theorem 3 for $d = -1$.

Theorem 4. *For any integer d and odd positive integer n with $n > 2 \mid d \mid -1$, we have*

$$\sum_{k=0}^{n-1} \frac{(q^{2d+1};q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k} q^{2k+1} \equiv (-1)^{\frac{n-1}{2}+d} q^{-2d} \qquad \pmod{\Phi_n(q)}, \tag{12}$$

and

$$\sum_{k=0}^{n-1} \frac{(q^{2d+1};q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k} q^{4(k+1)} \equiv (-1)^{\frac{n-1}{2}+d} \frac{q^2+1-q^{2d+1}}{q^{4d}} \qquad \pmod{\Phi_n(q)}. \tag{13}$$

Letting $q \to 1$ in Theorems 3 and 4 and using L'hospital rule, we are led to the following result.

Corollary 1. *For any integer d, positive integer r and odd prime p with $p > 2 \mid d \mid -1$, we have*

$$\sum_{k=0}^{p^r-1} \frac{2^k(d+\frac{1}{2})_k}{k!} \equiv (-1)^{\frac{p^r-1}{2}+d} \pmod{p}, \tag{14}$$

where $(x)_k = x(x+1)\cdots(x+k-1)$.

We remark that letting $d = 0$ and $n = p^r$ in (14) reduces to (1).

2. Proof of Theorem 1

Let

$$A_1(k,q)=\frac{\left(q;q^2\right)_k\left(-q^2;q^2\right)_k}{(q^2;q^2)_k}.$$

It is easy to check that

$$\left(1-q^{2k}\right)A_1(k,q)=\left(1-q^{2k-1}\right)\left(1+q^{2k}\right)A_1(k-1,q). \tag{15}$$

Summing both sides of (15) over k from 1 to $n-1$ gives

$$\sum_{k=1}^{n-1}\left(1-q^{2k}\right)A_1(k,q)=\sum_{k=1}^{n-1}\left(1-q^{2k-1}\right)\left(1+q^{2k}\right)A_1(k-1,q).$$

It follows that

$$\sum_{k=0}^{n-1}\left(1-q^{2k+1}\right)\left(1+q^{2k+2}\right)A_1(k,q)-\sum_{k=0}^{n-1}\left(1-q^{2k}\right)A_1(k,q)$$
$$=\left(1-q^{2n-1}\right)\left(1+q^{2n}\right)A_1(n-1,q). \tag{16}$$

We can rewrite (16) as

$$\frac{1-q+q^2}{q}\sum_{k=0}^{n-1}q^{2k+1}A_1(k,q)-\sum_{k=0}^{n-1}q^{4k+3}A_1(k,q)$$
$$=\left(1-q^{2n-1}+q^{2n}-q^{4n-1}\right)\frac{\left(q;q^2\right)_{n-1}\left(-q^2;q^2\right)_{n-1}}{(q;q)_{n-1}(-q;q)_{n-1}}. \tag{17}$$

Furthermore, note that (see [7, Lemma 2.2])

$$\frac{\left(q;q^2\right)_{n-1}}{(q;q)_{n-1}}\equiv 0 \pmod{\Phi_n(q)}, \tag{18}$$

for any positive odd integer n.

Finally, combining (3), (17) and (18), we arrive at

$$\sum_{k=0}^{n-1}\frac{\left(q;q^2\right)_k\left(-q^2;q^2\right)_k}{(q^2;q^2)_k}q^{4(k+1)}\equiv(-1)^{(n-1)/2}(1-q+q^2) \pmod{\Phi_n(q)},$$

which completes the proof of (5).

We shall prove (6) similarly. Let

$$A_2(k,q)=\frac{\left(q;q^2\right)_k\left(-1;q^2\right)_k}{(q^2;q^2)_k}.$$

It is easy to verify that

$$\left(1-q^{2k}\right) A_2(k,q)=\left(1-q^{2k-1}\right)\left(1+q^{2k-2}\right) A_2(k-1,q). \tag{19}$$

By using the same method as in the proof of (5), we can deduce from (19) that

$$\begin{aligned}
&(2-q)\sum_{k=0}^{n-1} q^{2k} A_2(k,q)-\sum_{k=0}^{n-1} q^{4k+1} A_2(k,q)\\
&\quad=\left(1-q^{2n-1}+q^{2n-2}-q^{4n-3}\right)\frac{\left(q;q^2\right)_{n-1}\left(-1;q^2\right)_{n-1}}{(q;q)_{n-1}(-q;q)_{n-1}}.
\end{aligned} \tag{20}$$

Combining (4), (18) and (20), we arrive at

$$\sum_{k=0}^{n-1}\frac{\left(q;q^2\right)_k\left(-1;q^2\right)_k}{\left(q^2;q^2\right)_k} q^{4k+1}\equiv(-1)^{(n-1)/2}(2-q) \pmod{\Phi_n(q)},$$

as desired. □

3. Proof of Theorem 2

Let

$$B_1(k,q)=\frac{\left(q^{-1};q^2\right)_k\left(-1;q^2\right)_k}{\left(q^2;q^2\right)_k}.$$

It is easy to verify that

$$\left(1-q^{2k}\right) B_1(k,q)=\left(1-q^{2k-3}\right)\left(1+q^{2k-2}\right) B_1(k-1,q).$$

It follows that

$$\begin{aligned}
&\sum_{k=0}^{n-1}\left(1-q^{2k-1}\right)\left(1+q^{2k}\right) B_1(k,q)-\sum_{k=0}^{n-1}\left(1-q^{2k}\right) B_1(k,q)\\
&\quad=\left(1-q^{2n-3}\right)\left(1+q^{2n-2}\right) B_1(n-1,q).
\end{aligned} \tag{21}$$

By (18) and (21), we have

$$\begin{aligned}
&(2-\frac{1}{q})\sum_{k=0}^{n-1} q^{2k} B_1(k,q)-\sum_{k=0}^{n-1} q^{4k-1} B_1(k,q)\\
&\quad=\left(1+q^{2n-2}\right)(1-q^{-1})\frac{\left(q;q^2\right)_{n-1}\left(-1;q^2\right)_{n-1}}{(q;q)_{n-1}(-q;q)_{n-1}}\\
&\quad\equiv 0 \pmod{\Phi_n(q)}.
\end{aligned} \tag{22}$$

On the other hand, we set

$$B_2(k,q)=\frac{\left(q^{-1};q^2\right)_k\left(-1;q^2\right)_k}{(q^2;q^2)_k}q^{2k}.$$

It is easy to check that

$$\left(1-q^{2k}\right)B_2(k,q)=\left(1-q^{2k-3}\right)\left(1+q^{2k-2}\right)q^2B_2(k-1,q).$$

It follows that

$$\sum_{k=0}^{n-1}\left(1-q^{2k-1}\right)\left(1+q^{2k}\right)q^2B_2(k,q)-\sum_{k=0}^{n-1}\left(1-q^{2k}\right)B_2(k,q)$$
$$=\left(1-q^{2n-3}\right)\left(1+q^{2n-2}\right)q^2B_2(n-1,q). \tag{23}$$

Note that

$$(1-q^{2k-1})B_2(k,q)=q^{2k}(1-q^{-1})A_2(k,q), \tag{24}$$

and

$$B_2(k,q)=q^{2k}B_1(k,q). \tag{25}$$

Substituting (24) and (25) into the left-hand side of (23) and using (18), we obtain

$$(q^2-q)\sum_{k=0}^{n-1}\left(q^{4k}+q^{2k}\right)A_2(k,q)-\sum_{k=0}^{n-1}\left(q^{2k}-q^{4k}\right)B_1(k,q)\equiv 0 \pmod{\Phi_n(q)}. \tag{26}$$

Furthermore, applying (4) and (6) to the left-hand side of (26) gives

$$\sum_{k=0}^{n-1}\left(q^{2k}-q^{4k}\right)B_1(k,q)\equiv 2(-1)^{(n-1)/2}(q-1) \pmod{\Phi_n(q)}. \tag{27}$$

Finally, combining (22) and (27), we arrive at

$$\sum_{k=0}^{n-1}q^{2k}B_1(k,q)\equiv(-1)^{(n+1)/2} \pmod{\Phi_n(q)},$$

and

$$\sum_{k=0}^{n-1}q^{4k}B_1(k,q)\equiv(-1)^{(n+1)/2}(2q-1) \pmod{\Phi_n(q)},$$

which completes the proof of (8) and (9). □

4. Proof of Theorem 3

Firstly, letting $q \to q^{-1}$ in (10), we find that (10) is equivalent to

$$\sum_{k=0}^{n-1} \frac{(q^{2d+1};q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{-k^2-2dk} \equiv (-1)^{\frac{n-1}{2}+d} \pmod{\Phi_n(q)}.$$

Recall the following Carlitz identity [9]:

$$\sum_{k=0}^{n} \frac{(a;q)_k(b;q)_k}{(q;q)_k}(-ab)^{n-k} q^{(n-k)(n+k-1)/2} = \sum_{k=0}^{n} \frac{(a;q)_{n+1}(-b)^k q^{\binom{k}{2}}}{(q;q)_k(q;q)_{n-k}(1-aq^{n-k})}. \quad (28)$$

Letting $q \to q^2$, $n \to n-1$, $a \to 2d+1$ and $b \to -1$, we get

$$\begin{aligned}\sum_{k=0}^{n-1} &\frac{(q^{2d+1};q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{(n-(1+k))(n+k+2d-1)} \\ &= \sum_{k=0}^{n-1} \frac{(q^{2d+1};q^2)_n q^{k(k-1)}}{(q^2;q^2)_k(q^2;q^2)_{n-k-1}(1-q^{2n+2d-2k-1})}.\end{aligned} \quad (29)$$

Note that $(q^{2d+1};q^2)_n$ has a factor $1-q^n$ for $n>2|d|-1$. Thus, except for $k = \frac{n+2d-1}{2}$, the right-hand side of (29) is congruent to 0 modulo $\Phi_n(q)$, and so

$$\begin{aligned}\sum_{k=0}^{n-1} &\frac{(q^{2d+1};q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{(n-(1+k))(n+k+2d-1)} \\ &\equiv \sum_{k=0}^{n-1} \frac{(q^{2d+1};q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{-k^2-2dk-2d+1} \\ &\equiv \frac{(q^{2d+1};q^2)_n q^{\frac{n+2d-1}{2}(\frac{n+2d-1}{2}-1)}}{(q^2;q^2)_{\frac{n+2d-1}{2}}(q^2;q^2)_{\frac{n-2d-1}{2}}(1-q^n)} \pmod{\Phi_n(q)}.\end{aligned} \quad (30)$$

For any integer s and n, we have

$$q^{sn} \equiv 1 \pmod{\Phi_n(q)}, \quad (31)$$

where the q-binomial coefficients are defined as:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Simplying the right-hand side of (30) gives

$$\begin{aligned}
&\frac{(q^{2d+1};q^2)_n}{(q^2;q^2)_{\frac{n+2d-1}{2}}}\frac{q^{\frac{n+2d-1}{2}(\frac{n+2d-1}{2}-1)}}{(q^2;q^2)_{\frac{n-2d-1}{2}}(1-q^n)} \\
&\qquad=\frac{q^{(n+2d-1)(n+2d-3)/4}}{1-q^n}\frac{(q;q^2)_n(1-q^{2n+1})(1-q^{2n+3})...(1-q^{2n+2d-1})}{(1-q)(1-q^3)...(1-q^{2d-1})} \\
&\qquad\quad\times\frac{(1-q^{n-2d+1})(1-q^{n-2d+3})...(1-q^{n-1})}{(q^2;q^2)^2_{\frac{n-1}{2}}(1-q^{n+1})(1-q^{n+3})...(1-q^{n+2d-1})} \\
&\qquad\equiv\begin{bmatrix} n-1 \\ \frac{n-1}{2}\end{bmatrix}_{q^2}\begin{bmatrix} 2n-1 \\ n-1\end{bmatrix}\frac{(-1)^d q^{(n+2d-1)(n+2d-3)/4-d^2}}{(-q;q)^2_{n-1}} \pmod{\Phi_n(q)}.
\end{aligned} \tag{32}$$

Note that (see [3, (2.5) and (2.6)])

$$\begin{bmatrix} n-1 \\ \frac{n-1}{2}\end{bmatrix}_{q^2} \equiv (-1)^{(n-1)/2}q^{(1-n^2)/4}(-q;q)^2 \pmod{\Phi_n(q)^2}, \tag{33}$$

and

$$\begin{bmatrix} 2n-1 \\ n-1\end{bmatrix} \equiv 1 \pmod{\Phi_n(q)}. \tag{34}$$

Substituting (33) and (34) into the right-hand side of (32), we get

$$\begin{bmatrix} n-1 \\ \frac{n-1}{2}\end{bmatrix}_{q^2}\begin{bmatrix} 2n-1 \\ n-1\end{bmatrix}\frac{(-1)^d q^{(n+2d-1)(n+2d-3)/4-d^2}}{(-q;q)^2_{n-1}} \equiv (-1)^{\frac{n-1}{2}+d}q^{1-2d} \pmod{\Phi_n(q)}. \tag{35}$$

Combining (30), (31), (32) and (35), we get

$$\sum_{k=0}^{n-1}\frac{(q^{2d+1};q^2)_k(-1;q^2)_k}{(q^2;q^2)_k}q^{-k^2-2dk-2d+1} \equiv (-1)^{\frac{n-1}{2}+d}q^{1-2d} \pmod{\Phi_n(q)},$$

as desired.

Next we shall prove (11). Let

$$C_1(k,q,d)=\frac{(q^{2d+1};q^2)_k(-1;q^2)_k}{(q^2;q^2)_k}.$$

It is easy to check that

$$(1-q^{2k})C_1(k,q,d)=(1-q^{2k+2d-1})(1+q^{2k-2})C_1(k-1,q,d). \tag{36}$$

Summing both side of (36) over k from 1 to $n-1$, we get

$$\sum_{k=1}^{n-1}(1-q^{2k})C_1(k,q,d)=\sum_{k=1}^{n-1}(1-q^{2k+2d-1})(1+q^{2k-2})C_1(k-1,q,d).$$

It follows that

$$\begin{aligned}(2-q^{2d+1})\sum_{k=0}^{n-1}\frac{(q^{2d+1};q^2)_k(-1;q^2)_k}{(q^2;q^2)_k}q^{2k}-q^{2d+1}\sum_{k=1}^{n-1}\frac{(q^{2d+1};q^2)_k(-1;q^2)_k}{(q^2;q^2)_k}q^{4k}\\ =(1-q^{2n+2d-1}+q^{2n-2}-q^{4n+2d-3})\frac{(q^{2d+1};q^2)_{n-1}(-1;q^2)_{n-1}}{(q^2;q^2)_{n-1}}.\end{aligned}\tag{37}$$

Note that

$$\begin{aligned}\frac{(q^{2d+1};q^2)_{n-1}}{(q^2;q^2)_{n-1}}&=\frac{(q;q^2)_{n-1}(1-q^{2n+1})(1-q^{2n+3})...(1-q^{2n+2d-3})}{(q^2;q^2)_{n-1}(1-q)(1-q^3)...(1-q^{2d-1})}\\ &\equiv\frac{(q;q^2)_{n-1}}{(q^2;q^2)_{n-1}(1-q^{2d-1})}\pmod{\Phi_n(q)}.\end{aligned}\tag{38}$$

By (18), we have

$$\frac{(q;q^2)_{n-1}}{(q^2;q^2)_{n-1}}\equiv 0\pmod{\Phi_n(q)},\tag{39}$$

for any integer $n\geq 3$.

Combining (31), (38) and (39), we find that the right-hand side of (37) is congruent to

$$(1-q^{2n+2d-1}+q^{2n-2}-q^{4n+2d-3})\frac{(q^{2d+1};q^2)_{n-1}(-1;q^2)_{n-1}}{(q^2;q^2)_{n-1}}\equiv 0\pmod{\Phi_n(q)}.\tag{40}$$

Using (10), (37) and (40), we arrive at

$$\sum_{k=1}^{n-1}\frac{(q^{2d+1};q^2)_k(-1;q^2)_k}{(q^2;q^2)_k}q^{4k}\equiv(-1)^{\frac{n-1}{2}+d}\frac{2-q^{2d+1}}{q^{2d+1}}\pmod{\Phi_n(q)}$$

as desired. □

5. Proof of Theorem 4

We first prove (12). Letting $q\to q^{-1}$ in (12), we find that (12) is equivalent to

$$\sum_{k=0}^{n-1}\frac{(q^{2d+1};q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k}q^{-(k+1)^2-2dk}\equiv(-1)^{\frac{n-1}{2}+d}q^{2d}\qquad\pmod{\Phi_n(q)}.$$

Letting $q \to q^2$, $n \to n-1$, $a \to 2d+1$ and $b \to -q^2$ in (28), we obtain

$$\begin{aligned}\sum_{k=0}^{n-1} &\frac{(q^{2d+1};q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k} q^{(n-(1+k))(n+k+2d+1)} \\ &= \sum_{k=0}^{n-1} \frac{(q^{2d+1};q^2)_n q^{k(k+1)}}{(q^2;q^2)_k(q^2;q^2)_{n-k-1}(1-q^{2n+2d-2k-1})}.\end{aligned} \tag{41}$$

The right-hand side of (41) has a factor $1-q^n$ for $n>2|d|-1$. Thus, for $k \neq \frac{n+2d-1}{2}$, we have

$$\begin{aligned}\sum_{k=0}^{n-1} &\frac{(q^{2d+1};q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k} q^{(n-(1+k))(n+k+2d+1)} \\ &\equiv \sum_{k=0}^{n-1} \frac{(q^{2d+1};q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k} q^{-(k+1)^2-2dk-2d} \\ &\equiv \frac{(q^{2d+1};q^2)_n q^{\frac{n+2d-1}{2}(\frac{n+2d-1}{2}+1)}}{(q^2;q^2)_{\frac{n+2d-1}{2}}(q^2;q^2)_{\frac{n-2d-1}{2}}(1-q^n)} \pmod{\Phi_n(q)}.\end{aligned} \tag{42}$$

Using the same method as in the proof of (32) and combining (31)–(34), we get

$$\frac{(q^{2d+1};q^2)_n q^{\frac{n+2d-1}{2}(\frac{n+2d+1}{2}+1)}}{(q^2;q^2)_{\frac{n+2d-1}{2}}(q^2;q^2)_{\frac{n-2d-1}{2}}(1-q^n)} \equiv (-1)^{\frac{n-1}{2}+d} \pmod{\Phi_n(q)}. \tag{43}$$

Combining (42) and (43), we have

$$\sum_{k=0}^{n-1} \frac{(q^{2d+1};q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k} q^{-(k+1)^2-2dk-2d} \equiv (-1)^{\frac{n-1}{2}+d} \pmod{\Phi_n(q)},$$

which completes the proof of (12).

Next we shall prove (13). Let

$$D_1(k,q,d) = \frac{(q^{2d+1};q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k}.$$

It is easy to check that

$$(1-q^{2k})D_1(k,q,d) = (1-q^{2k+2d-1})(1+q^{2k})D_1(k-1,q,d). \tag{44}$$

By using the same method as in the proof of (11), we can deduce from (44) that

$$\begin{aligned}\frac{q^2+1-q^{2d+1}}{q}&\sum_{k=0}^{n-1}\frac{(q^{2d+1};q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k}q^{2k+1}\\&-q^{2d-1}\sum_{k=1}^{n-1}\frac{(q^{2d+1};q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k}q^{4(k+1)}\\&=(1-q^{2n+2d-1}+q^{2n}-q^{4n+2d-1})\frac{(q^{2d+1};q^2)_{n-1}(-q^2;q^2)_{n-1}}{(q^2;q^2)_{n-1}}.\end{aligned} \tag{45}$$

Combining (31), (38) and (39), we find that the right-hand side of (45) is congruent to

$$(1-q^{2n+2d-1}+q^{2n}-q^{4n+2d-1})\frac{(q^{2d+1};q^2)_{n-1}(-q^2;q^2)_{n-1}}{(q^2;q^2)_{n-1}}\equiv 0 \pmod{\Phi_n(q)}. \tag{46}$$

Finally, combining (12), (45) and (46), we arrive at

$$\sum_{k=1}^{n-1}\frac{(q^{2d+1};q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k}q^{4(k+1)}\equiv(-1)^{\frac{n-1}{2}+d}\frac{q^2+1-q^{2d+1}}{q^{4d}} \pmod{\Phi_n(q)},$$

as desired. □

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