ON PROOFS OF GENERALIZED KNUTH'S OLD SUM

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Abstract

In this paper we provide combinatorial and computer-generated proofs using the Wilf-Zeilberger algorithm for generalized Knuth's old sum.

1. Introduction

For non-negative integers n, the Reed-Dawson identity is given by

$$\sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k \binom{n}{k}\binom{2k}{k} = \begin{cases} \frac{1}{2^n}\binom{n}{n/2} & \text{for } n \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

In the literature, this identity is also referred to as Knuth's old sum ([3]). A number of different proofs, generalizations, and extensions have been established (see [1], [3], [4], [6], [7], [8]). One of the most recent works in this direction is from Rathie et al., where they formulated and proved a generalized Reed-Dawson identity using analytic methods (see [9]). In this paper, we provide combinatorial proofs as well as computer-generated proofs using the Wilf-Zeilberger (WZ) method for the generalized Reed-Dawson identity given in [9]. This paper is organized as follows. In Section 2, we formulate the main results. In Section 3, we provide combinatorial proofs by counting words of certain properties and defining an appropriate sign reversing involution. In Section 4, we present computer-generated proofs of the main results. For an explanation of the WZ method, see the book, $A = B$, ([5]) which is devoted to this and other algorithmic proof methods. To generate the recurrence equations in the WZ proofs, one can use the built-in Maple package `SumTools` or the Zeilberger's MAPLE package `EKHAD`, which is available from [2].

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2. Main Results

Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. For $x \in \mathbb{R}$, $(x)_n$ denotes the rising factorial, defined by $(x)_n = x(x+1)\cdots(x+n-1)$ for $n \in \mathbb{N}_0 \backslash \{0\}$ and $(x)_0 = 1$. In the sequel, $\binom{a}{b}$ is zero if $a < b$, and a sum is zero if its range of summation is empty.

In 2022, Rathie et al. [9] generalized Knuth's old sum as follows. For $i \in \mathbb{N}_0$, the following identities hold true:

$$\sum_{k=0}^{2v}(-1)^k\binom{2v+i}{k+i}\binom{2k}{k}2^{-k} = \pi\,(2v+1)_i\frac{2^{2i}i!}{(2i)!}\sum_{r=0}^{i}\frac{2^{-r}\binom{i}{r}\left(\frac{1}{2}+\frac{1}{2}(i-r)\right)_v}{(i-r)!\Gamma^2\left(\frac{1}{2}+\frac{1}{2}(r-i)\right)\left(1+\frac{1}{2}(i-r)\right)_v} \quad (1)$$

and

$$\sum_{k=0}^{2v+1}(-1)^k\binom{2v+1+i}{k+i}\binom{2k}{k}2^{-k} = 2\pi\,(2v+2)_i\frac{2^{2i}i!}{(2i)!}\sum_{r=0}^{i}\frac{2^{-r}\binom{i}{r}\left(1+\frac{1}{2}(i-r)\right)_v}{(i-r+1)!\Gamma^2\left(\frac{1}{2}(r-i)\right)\left(\frac{3}{2}+\frac{1}{2}(i-r)\right)_v}. \quad (2)$$

Applying the basic properties of rising factorials, the Gamma function and the identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

Equations (1) and (2) can be formulated as follows.

Theorem 1. *For all $m, n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{2n}(-1)^k\binom{4m+2}{2m+1}\binom{2n+2m+1}{k+2m+1}\binom{2k}{k}2^{2n-k} = \sum_{i=0}^{m}\binom{2n+2m+1}{2n}\binom{2m+1}{2i+1}\binom{2n+2m-2i}{n+m-i}2^{2i+1}.$$

Theorem 2. *For all $m, n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{2n}(-1)^k\binom{4m}{2m}\binom{2n+2m}{k+2m}\binom{2k}{k}2^{2n-k} = \sum_{i=0}^{m}\binom{2n+2m}{2n}\binom{2m}{2i}\binom{2n+2m-2i}{n+m-i}2^{2i}.$$

Theorem 3. *For all $m, n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{2n+1}(-1)^k\binom{4m}{2m}\binom{2n+2m+1}{k+2m}\binom{2k}{k}2^{2n-k} = \sum_{i=0}^{m}\binom{2n+2m+1}{2n+1}\binom{2m}{2i+1}\binom{2n+2m-2i}{n+m-i}2^{2i}.$$

Theorem 4. *For all $m, n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{2n+1}(-1)^k\binom{4m+2}{2m+1}\binom{2n+2m+2}{k+2m+1}\binom{2k}{k}2^{2n+1-k} = \sum_{i=0}^{m}\binom{2n+2m+2}{2n+1}\binom{2m+1}{2i}\binom{2n+2m-2i+2}{n+m-i+1}2^{2i}.$$

3. Combinatorial Proofs

In this section, we give combinatorial proofs for Theorems 1 and 2.

Lemma 1. *For a fixed $n \in \mathbb{N}_0$, let S_n be the set of all words in the alphabet $\{a, b, c, d\}$ of length n such that the number of a's is equal to the number of b's. Then $|S_n| = \binom{2n}{n}$. Here $|A|$ denotes the cardinality of A.*

Proof. Let T_n be a set of bit strings of length $2n$ such that the number of 0's equals the number of 1's. Then, $|T_n| = \binom{2n}{n}$. Define a bijection f between S_n and T_n as follows. For w in S_n, read w from left to right and replace a by 00, b by 11, c by 01 and d by 10. Thus, the cardinality of S_n is $\binom{2n}{n}$. □

We are now able to prove Theorems 1 and 2.

Proof of Theorem 1. For fixed $n, m \in \mathbb{N}_0$, consider a set S of words in the alphabet

$$\{a, b, c, d, C, D, ?, ., !, *\}$$

such that w is in S if and only if the following conditions all hold: (i) the number of letters in w is $2n$; (ii) the number of a's in w is equal to the number of b's in w; (iii) the number of special characters in w is $2m+1$; (iv) the number of .'s in w is equal to the number of ?'s in w.

For $w \in S$, define the *weight of* w, $Wt(w)$, by

$$Wt(w) = (-1)^{L(w)},$$

where $L(w)$ represents the number of lower case letters in w. For $0 \leq k \leq 2n$, $L_k(w)$ represents the number of w's with k lowercase letters and $U_{2n-k}(w)$ represents the number of w's with $2n-k$ uppercase letters. Then

$$\sum_{w\in S} Wt(w) = \sum_{k=0}^{2n} L_k(w)\, Wt(w) = \sum_{k=0}^{2n} L_k(w)\,(-1)^k = \sum_{k=0}^{2n} U_{2n-k}(w)\,(-1)^k.$$

For any $w \in S$, by conditions (i) and (iii), the length of w is $2n+2m+1$. From these $2n+2m+1$ positions, select $2n-k$ positions ($0 \leq k \leq 2n$) for the uppercase letters. This is done in $\binom{2n+2m+1}{2n-k}$ ways. These $2n-k$ positions can be filled by the two uppercase letters in 2^{2n-k} ways. By condition (ii) and Lemma 1, there are $\binom{2k}{k}$ ways to fill k positions by lowercase letters. By conditions (iii), (iv) and Lemma 1, there are $\binom{2(2m+1)}{2m+1}$ ways to fill $2m+1$ positions by special characters. Thus, there are $\binom{2k}{k}\binom{2(2m+1)}{2m+1}$ ways to fill $k+2m+1$ positions. Hence,

$$\sum_{w\in S} Wt(w) \;=\; \sum_{k=0}^{2n} 2^{2n-k}\binom{2n+2m+1}{2n-k}\binom{2k}{k}\binom{2(2m+1)}{2m+1}(-1)^k.$$

Now define a sign reversing involution σ on S as follows: for w in S, read w from left to right until one of c, C, d, or D occurs. If we encounter a lowercase letter, we change it to its corresponding uppercase letter, and if we encounter an uppercase letter, we change it to its corresponding lowercase letter. This reverses the sign of $Wt(w)$ if w contains c, C, d, or D.

Let T be the set of all w in S such that w has only lowercase letters a, b, and the special characters $., ?, !, *$. Then,

$$\sum_{w\in S\setminus T} Wt(w) = 0.$$

Hence,

$$\sum_{w\in S} Wt(w) = \sum_{w\in T} Wt(w) + \sum_{w\in S\setminus T} Wt(w) = \sum_{w\in T} 1 = |T|.$$

To find $|T|$, we condition on the number of !'s and $*$'s in a word $w \in T$. Note that by conditions (iii) and (iv) the number of !'s and $*$' in a word $w \in T$ is odd. Thus, the number of !'s and $*$'s in a word $w \in T$ is $2i+1$ for some i, $0 \leq i \leq 2m+1$. First select $2m+1$ positions for the four special characters from the $2n+2m+1$ total positions. This can be done in $\binom{2n+2m+1}{2m+1}$ ways. Then select $2i+1$ positions for ! and $*$ from the $2m+1$ positions. This can be done in $\binom{2m+1}{2i+1}$ ways. Then the $2i+1$ positions are filled by ! or $*$ in 2^{2i+1} ways. The remaining $2n+2m-2i$ positions are filled by $a, b, ., ?$ where the number of a's is equal to that of b's (that is n), as well as the number of .'s is equal to that of ?'s (that is $m-i$). This can be done in $\binom{2(n+m-i)}{n+m-i}$ ways. Thus,

$$|T| \;=\; \sum_{i=0}^{m} \binom{2n+2m+1}{2m+1} \binom{2m+1}{2i+1} \binom{2(n+m-i)}{n+m-i} 2^{2i+1}.$$

This completes the proof. □

Proof of Theorem 2. Let S be the set of words defined as in the proof of Thereom 1 satisfying conditions (i), (ii), and (iv). Modify condition (iii) to: the number of special characters in w is $2m$. Let the *weight* of a word $w \in S$ be defined as in the proof of Theroem 1. Then

$$\sum_{w\in S} Wt(w) = \sum_{k=0}^{2n} L_k(w)\, Wt(w) = \sum_{k=0}^{2n} L_k\, (-1)^k = \sum_{k=0}^{2n} U_{2n-k}\, (-1)^k.$$

For any $w \in S$, by conditions (i) and (iii), the length of w is $2n+2m$. For $0 \le k \le 2n$, select $2n-k$ positions from $2n+2m$ positions for the uppercase letters. This is done in $\binom{2n+2m}{2n-k}$ ways. These $2n-k$ positions can be filled by the two uppercase letters in 2^{2n-k} ways. By condition (ii) and Lemma 1, there are $\binom{2k}{k}$ ways to fill k positions by lowercase letters. By conditions (iii), (iv), and Lemma 1, there are $\binom{2(2m)}{2m}$ ways to fill $2m$ positions by special characters. Thus, there are $\binom{2k}{k}\binom{2(2m)}{2m}$ ways to fill $k+2m$ positions. Hence,

$$\sum_{w\in S} Wt(w) \;=\; \sum_{k=0}^{2n} 2^{2n-k} \binom{2n+2m}{2n-k} \binom{2k}{k} \binom{4m}{2m} (-1)^k.$$

Define now a sign reversing involution σ on S as in the proof of Theorem 1. Let T be the set of all w in S such that w has only lowercase letters a, b, and the special characters $., ?, !, *$. Then, as in the proof of Theroem 1,

$$\sum_{w\in S\setminus T} Wt(w) = 0.$$

Hence,

$$\sum_{w\in S} Wt(w) = \sum_{w\in T} Wt(w) + \sum_{w\in S\setminus T} Wt(w) = \sum_{w\in T} 1 = |T|.$$

To find $|T|$, we condition on the number of !'s and $*$'s in a word $w \in T$. Note that by conditions (iii) and (iv) the number of !'s and $*$'s in a word $w \in T$ is even. Thus, the number of !'s and $*$'s in a word $w \in T$ is $2i$ for some i, $0 \le i \le 2m$. First select $2m$ positions for the four special characters from the $2n+2m$ total positions. This can be done in $\binom{2n+2m}{2m}$ ways. Then select $2i$ positions for the !'s and $*$'s from the $2m$ positions. This can be done in $\binom{2m}{2i}$ ways. Then the $2i$ positions are filled by ! or $*$ in 2^{2i} ways. The remaining $2n+2m-2i$ positions are filled by $a, b, ., ?$

where the number of a's is equal to that of b's (that is n), as well as the number of .'s is equal to that of ?'s (that is $m-i$). This can be done in $\binom{2(n+m-i)}{n+m-i}$ ways. Thus,

$$|T| \quad = \quad \sum_{i=0}^{m} \binom{2n+2m}{2m}\binom{2m}{2i}\binom{2(n+m-i)}{n+m-i}2^{2i}.$$

Hence, the proof is complete. □

4. Proofs by the WZ Method

In this section, we prove Theorems 3 and 4 using the WZ method.

Lemma 2. *For $m \in \mathbb{N}_0$,*

$$\sum_{i=0}^{m}(2m+1)\binom{2m}{2i+1}\binom{2m-2i}{m-i}2^{2i} = 2m\binom{4m}{2m}. \tag{3}$$

Proof. Clearly Equation (3) holds true for $m=0$. For $m>0$, dividing both sides of Equation (3) by the right side yields an equivalent identity:

$$\sum_{i=0}^{m}\frac{2m+1}{2m}\binom{4m}{2m}^{-1}\binom{2m}{2i+1}\binom{2m-2i}{m-i}2^{2i} = 1. \tag{4}$$

Since the summand is zero for all $i \geq m$, Equation (4) is equivalent to

$$\sum_{i=0}^{\infty}\frac{2m+1}{2m}\binom{4m}{2m}^{-1}\binom{2m}{2i+1}\binom{2m-2i}{m-i}2^{2i} = 1. \tag{5}$$

Let us denote the left side of Equation (5) by $S(m)$ and its summand by $F(m,i)$. Then applying the WZ algorithm on $F(m,i)$, we get the WZ-equation

$$F(m+1,i) - F(m,i) = G(m,i+1) - G(m,i), \tag{6}$$

where

$$G(m,i) = \binom{4m+2}{2m+1}^{-1}\binom{2m}{2i}\binom{2m-2i+1}{m-i}2^{2i+1}R(m,i),$$
$$R(m,i) = \frac{i(-6m^2+(4i-8)m+3i-3)}{m(4m+3)(2m-2i+1)}.$$

Summing both sides of Equation (6) with respect to i ($0 \leq i < \infty$) yields

$$S(m+1) - S(m) = 0.$$

Thus, $S(m)$ is a constant. Since $S(1)=1$, $S(m)=1$ for all positive integers m. □

Lemma 3. *For $m \in \mathbb{N}_0$,*

$$\sum_{i=0}^{m} \binom{2m+3}{3}\binom{2m}{2i+1}\binom{2m+2-2i}{m+1-i}2^{2i} = \binom{4m}{2m}\frac{4m(4m^2+6m+5)}{3}. \tag{7}$$

Proof. Note that Equation (7) holds true for $m = 0$. For $m > 0$, dividing both sides of Equation (7) by the right side yields

$$\sum_{i=0}^{m} \frac{3}{4m(4m^2+6m+5)}\binom{4m}{2m}^{-1}\binom{2m+3}{3}\binom{2m}{2i+1}\binom{2m+2-2i}{m+1-i}2^{2i} = 1. \tag{8}$$

Since the summand is zero for all $i \geq m$, Equation (8) is equivalent to

$$\sum_{i=0}^{\infty} \frac{3}{4m(4m^2+6m+5)}\binom{4m}{2m}^{-1}\binom{2m+3}{3}\binom{2m}{2i+1}\binom{2m+2-2i}{m+1-i}2^{2i} = 1. \tag{9}$$

Let us denote the left side of Equation (9) by $S(m)$ and its summand by $F(m, i)$. Then applying the WZ algorithm on $F(m, i)$ we get the WZ-equation

$$F(m+1, i) - F(m, i) = G(m, i+1) - G(m, i), \tag{10}$$

where

$$\begin{aligned}
G(m,i) &= -\binom{4m+3}{2m}^{-1}\binom{2m+1}{2i}\binom{2m+3-2i}{m+1-i}2^{2i-1}R(m,i),\\
R(m,i) &= \frac{ip(m,i)}{m(2m+3-2i)(4m^2+6m+5)(4m^2+14m+15)},\\
p(m,i) &= p_2(m)i^2 + p_1(m)i + p_0(m),\\
p_2(m) &= 2(3+4m)(4m^2+14m+15),\\
p_1(m) &= -(4m^2+14m+15)(20m^2+34m+17),\\
p_0(m) &= 48m^5 + 304m^4 + 784m^3 + 970m^2 + 570m + 150.
\end{aligned}$$

Summing both sides of Equation (10) with respect to i ($0 \leq i < \infty$) yields

$$S(m+1) - S(m) = 0,$$

which implies that $S(m)$ is a constant. Since $S(1) = 1$, $S(m) = 1$ for all positive integers m. □

Lemma 4. *For $m \in \mathbb{N}_0$,*

$$\sum_{i=0}^{m} (m+1)\binom{2m+1}{2i}\binom{2m-2i+2}{m-i+1}2^{2i} = 2(4m+1)\binom{4m}{2m}. \tag{11}$$

Proof. Let $m \geq 0$. Dividing both sides of Equation (11) by the right side yields

$$\sum_{i=0}^{m} \frac{m+1}{2(4m+1)} \binom{4m}{2m}^{-1} \binom{2m+1}{2i} \binom{2m-2i+2}{m-i+1} 2^{2i} = 1. \tag{12}$$

Since the summand is zero for all $i > m$, Equation (12) is equivalent to

$$\sum_{i=0}^{\infty} \frac{m+1}{2(4m+1)} \binom{4m}{2m}^{-1} \binom{2m+1}{2i} \binom{2m-2i+2}{m-i+1} 2^{2i} = 1. \tag{13}$$

Let us denote the left side of Equation (13) by $S(m)$ and its summand by $F(m,i)$. Then applying the WZ algorithm on $F(m,i)$, we get the WZ-equation

$$F(m+1,i) - F(m,i) = G(m,i+1) - G(m,i), \tag{14}$$

where

$$G(m,i) = \binom{4m}{2m}^{-1} \binom{2m+2}{2i} \binom{2m+3-2i}{m+1-i} 2^{2i-1} R(m,i),$$
$$R(m,i) = \frac{i(2i-1)(-6m^2 + (4i-16)m + 5i - 11)}{(2m+3-2i)(4m+5)(4m+3)(4m+1)}.$$

Summing both sides of Equation (14) with respect to i ($0 \leq i < \infty$), we get

$$S(m+1) - S(m) = 0.$$

Thus, $S(m)$ is a constant. Since $S(0) = 1$, $S(m) = 1$ for all integers $m \geq 0$. □

Lemma 5. *For $m \in \mathbb{N}_0$,*

$$\sum_{i=0}^{m} \binom{2m+4}{3} \binom{2m+1}{2i} \binom{2m+4-2i}{m+2-i} 2^{2i} = a(m), \tag{15}$$

where $a(m) = \binom{4m}{2m} \frac{8(4m+1)(4m^2+10m+9)}{3}$.

Proof. Let $m \geq 0$. Dividing both sides of Equation (15) by the right side yields

$$\sum_{i=0}^{m} \frac{1}{a(m)} \binom{2m+4}{3} \binom{2m+1}{2i} \binom{2m+4-2i}{m+2-i} 2^{2i} = 1. \tag{16}$$

Since the summand is zero for all $i > m$, Equation (16) is equivalent to

$$\sum_{i=0}^{\infty} \frac{1}{a(m)} \binom{2m+4}{3} \binom{2m+1}{2i} \binom{2m+4-2i}{m+2-i} 2^{2i} = 1. \tag{17}$$

Let us denote the left side of Equation (17) by $S(m)$ and its summand by $F(m,i)$. Then applying the WZ algorithm on $F(m,i)$ we get the WZ-equation

$$F(m+1,i) - F(m,i) = G(m,i+1) - G(m,i), \tag{18}$$

where

$$\begin{aligned}
G(m,i) &= -\binom{4m+5}{2m+1}^{-1}\binom{2m+2}{2i}\binom{2m+5-2i}{m+3-i}2^{2i-1}R(m,i),\\
R(m,i) &= \frac{i(2i-1)p(m,i)}{q(m,i)},\\
p(m,i) &= p_2(m)i^2 + p_1(m)i + p_0(m),\\
p_2(m) &= 2(4m+5)(4m^2+18m+23),\\
p_1(m) &= -(20m^2+62m+49)(4m^2+18m+23),\\
p_0(m) &= 48m^5+464m^4+1808m^3+3502m^2+3358m+1302,\\
q(m,i) &= q_2(m)i^2 + q_1(m)i + q_0(m),\\
q_2(m) &= (8m+4)(4m^2+18m+23)(4m^2+10m+9),\\
q_1(m) &= -8(2+m)(2m+1)(4m^2+18m+23)(4m^2+10m+9),\\
q_0(m) &= (2m+3)(2m+1)(2m+5)(4m^2+18m+23)(4m^2+10m+9).
\end{aligned}$$

Summing both sides of Equation (18) with respect to i ($0 \le i < \infty$) yields

$$S(m+1) - S(m) = 0.$$

Thus, $S(m)$ is a constant. Since $S(0) = 1$, $S(m) = 1$ for all integers $m \ge 0$. □

We are now able to prove Theorems 3 and 4.

Proof of Theorem 3. Let us denote the summands of the left and right sides of the equation in Theorem 3 by $F_1(n,m,k)$ and $F_2(n,m,i)$, respectively. Since $F_1(n,m,k) = 0$ for $k > 2n+1$ and $F_2(n,m,i) = 0$ for $i \ge m$, the equation in Theorem 3 is equivalent to

$$\sum_{k=0}^{\infty}(-1)^k\binom{4m}{2m}\binom{2n+2m+1}{k+2m}\binom{2k}{k}2^{2n-k} = \sum_{i=0}^{\infty}\binom{2n+2m+1}{2n+1}\binom{2m}{2i+1}\binom{2n+2m-2i}{n+m-i}2^{2i}. \tag{19}$$

Let us denote the left and right sides of Equation (19) by $S(n,m)$ and $T(n,m)$, respectively. To show that $S(n,m) = T(n,m,)$ for all $m,n \in \mathbb{N}_0$, it suffices to show that both $S(n,m)$ and $T(n,m)$ satisfy the same recurrence relation with the same initial conditions.

Applying the WZ algorithm on $F_1(n,m,k)$ we get the WZ-equation

$$p_2(n,m)F_1(n+2,m,k)+p_1(n,m)F_1(n+1,m,k)+p_0(n,m)F_1(n,m,k) \\ = G_1(n,m,k+1)-G_1(n,m,k), \quad (20)$$

where

$$\begin{aligned} p_2(m,n) &= 2n^2+9n+10, \\ p_1(n,m) &= -(16n^2+(16m+56)n+8m^2+28m+50), \\ p_0(n,m) &= 32n^2+(64m+80)n+32m^2+80m+48, \\ G_1(n,m,k) &= (-1)^k 2^{2n+2-k}\binom{4m}{2m}\binom{2n+2m+4}{k+2m-1}\binom{2k}{k}R_1(n,m,k), \\ R_1(n,m,k) &= \frac{k(k^2+(2m-4n-9)k+(-8m+2)n-18m+4)}{n+m+2}. \end{aligned}$$

Summing both sides of Equation (20) with respect to k $(0 \le k < \infty)$ yields

$$p_2(n,m)S(n+2,m)+p_1(n,m)S(n+1,m)+p_0(n,m)S(n,m)=0.$$

Moreover,

$$S(0,m) = 2m\binom{4m}{2m} \quad \text{and} \quad S(1,m) = \binom{4m}{2m}\frac{4m(4m^2+6m+5)}{3}.$$

By applying the WZ algorithm on $F_2(n,m,k)$ we get the WZ-equation

$$p_2(n,m)F_2(n+2,m,i)+p_1(n,m)F_2(n+1,m,i)+p_0(n,m)F_2(n,m,i) \\ = G_2(n,m,i+1)-G_2(n,m,i), \quad (21)$$

where

$$\begin{aligned} G_2(n,m,i) &= 2^{2i+1}\binom{2n+2m+3}{2n+3}\binom{2m}{2i+1}\binom{2n+2m+3-2i}{n+m+2-i}R_2(n,m,i), \\ R_2(n,m,i) &= \frac{i(2i+1)}{2n+2m-2i+3}. \end{aligned}$$

Summing both sides of Equation (21) with respect to i $(0 \le i < \infty)$, we get

$$p_2(n,m)T(n+2,m)+p_1(n,m)T(n+1,m)+p_0(n,m)T(n,m)=0.$$

Note that, by Lemmas 2 and 3,

$$T(0,m) = 2m\binom{4m}{2m} \quad \text{and} \quad T(1,m) = \binom{4m}{2m}\frac{4m(4m^2+6m+5)}{3}.$$

This completes the proof. □

Proof of Theorem 4. Denote the summands of the left and right sides of the equation in Theorem 4 by $F_1(n,m,k)$ and $F_2(n,m,i)$, respectively. Since $F_1(n,m,k)=0$ for $k>2n+1$ and $F_2(n,m,i)=0$ for $i>m$, the equation in Theorem 4 is equivalent to

$$\sum_{k=0}^{\infty}(-1)^k\binom{4m+2}{2m+1}\binom{2n+2m+2}{k+2m+1}\binom{2k}{k}2^{2n+1-k}$$
$$=\sum_{i=0}^{\infty}\binom{2n+2m+2}{2n+1}\binom{2m+1}{2i}\binom{2n+2m-2i+2}{n+m-i+1}2^{2i}. \quad (22)$$

Let us denote the left and right sides of Equation (22) by $S(n,m)$ and $T(n,m)$, respectively. We will show that $S(n,m)$ and $T(n,m)$ satisfy the same recurrence relation with the same initial conditions.

Applying the WZ algorithm on $F_1(n,m,k)$ we get the WZ-equation

$$p_2(n,m)F_1(n+2,m,k)+p_1(n,m)F_1(n+1,m,k)+p_0(n,m)F_1(n,m,k)$$
$$=G_1(n,m,k+1)-G_1(n,m,k), \quad (23)$$

where

$$p_2(m,n)=2n^2+9n+10,$$
$$p_1(n,m)=-(16n^2+(16m+64)n+8m^2+36m+66),$$
$$p_0(n,m)=32n^2+(64m+112)n+32m^2+112m+96,$$
$$G_1(n,m,k)=(-1)^k2^{2n+4-k}\binom{4m+2}{2m+1}\binom{2n+2m+5}{k+2m}\binom{2k}{k}R_1(n,m,k),$$
$$R_1(n,m,k)=\frac{k(k^2+(2m-4n-8)k-(8m+2)n-18m-5)}{2n+2m+5}.$$

Summing both sides of Equation (23) with respect to k ($0\le k<\infty$), we get

$$p_2(n,m)S(n+2,m)+p_1(n,m)S(n+1,m)+p_0(n,m)S(n,m)=0.$$

Moreover,

$$S(0,m)=4(4m+1)\binom{4m}{2m} \quad \text{and} \quad S(1,m)=\binom{4m}{2m}\frac{8(4m+1)(4m^2+10m+9)}{3}.$$

By applying the WZ algorithm on $F_2(n,m,k)$ we get the WZ-equation

$$p_2(n,m)F_2(n+2,m,i)+p_1(n,m)F_2(n+1,m,i)+p_0(n,m)F_2(n,m,i)$$
$$=G_2(n,m,i+1)-G_2(n,m,i), \quad (24)$$

where

$$G_2(n,m,i) = 2^{2i+1}\binom{2n+2m+4}{2n+3}\binom{2m+1}{2i}\binom{2n+2m+5-2i}{n+m+3-i}R_2(n,m,i),$$
$$R_2(n,m,i) = \frac{i(2i-1)}{2n+2m-2i+5}.$$

Summing both sides of Equation (24) with respect to i $(0 \le i < \infty)$, we get

$$p_2(n,m)T(n+2,m) + p_1(n,m)T(n+1,m) + p_0(n,m)T(n,m) = 0.$$

Note that, by Lemmas 4 and 5,

$$T(0,m) = 4(4m+1)\binom{4m}{2m} \quad \text{and} \quad T(1,m) = \binom{4m}{2m}\frac{8(4m+1)(4m^2+10m+9)}{3}.$$

Hence, the proof is complete. □

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