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A FURTHER INVESTIGATION OF POSITIONS IN SYLVER COINAGE FOR WHICH FOUR HAS BEEN CHOSEN

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Abstract

We extend previous investigations into positions in the game of Sylver coinage for which the number four has been chosen. We employ tools involving numerical semigroups to prove results concerning winning and losing positions as well as strategies for playing the game from specific positions.

1. Introduction

In the game of Sylver coinage, two players alternate choosing positive integers. On each turn, all positive integers which can be expressed as a linear combination, with non-negative integer coefficients, of all numbers that have been chosen are removed from play. The player that must choose the number 1 loses the game. The game is named in honor of mathematician J.J. Sylvester (1814 - 1897) and is the topic of Chapter 18 of *Winning Ways for Your Mathematical Plays* [2] by Berlekamp, Conway, and Guy. The game is also studied by Sicherman [9] and is discussed in Chapter 6 of Michael [6]. In [5], Guy poses 20 questions concerning the game of Sylver coinage. More recently, Sylver coinage has been connected to numerical semigroups, providing new tools with which to study the game, including a proof of the Quiet End Theorem using results from numerical semigroups. See [3] for details.

Within the existing literature concerning Sylver coinage, considerable attention is devoted to determining if a given position is an $\mathcal{N}$*-position* (next player wins) or a $\mathcal{P}$*-position* (previous player wins). For example, it is known from [2] that if Player 1 begins the game by choosing a prime p with $p \geq 5$, then they are in a

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winning position because the position $\{p\}$ is a $\mathcal{P}$*-position*. Knowing this, it may seem like there is not much to investigate concerning Sylver coinage. However, we quickly realize that knowing we are in an $\mathcal{N}$*-position* does not reveal what number we ought to choose next. For example, if Player 1 starts the game by choosing 11 and Player 2 responds by choosing 103, then Player 1 knows they are in a winning position as $\{11, 103\}$ is known to be an $\mathcal{N}$*-position*. What should Player 1 choose next? It is not clear. In [2], we read, "… from any position in Sylver coinage there is a winning strategy for one of the two players but because of the infinite nature of the game we cannot work through all positions and guarantee to find winning strategies when they exist."

In [2], we find an examination of "good responses" (that is, appropriate next choices) for certain families of $\mathcal{N}$*-positions* with a focus on positions in which 4 or 5 has been chosen. We will continue the examination of positions in Sylver coinage for which 4 has been chosen but we will use the numerical semigroup tools established in [3] to extend both the scope and detail of the analysis. In Section 2, we establish the definitions and notations that we will use concerning the game of Sylver coinage and numerical semigroups. We also recall some key results proved in [3] that we will need here. In Section 3, we investigate positions in which 4 has been played, and establish bounds on the number of legal choices that are congruent to 1, 2, or 3 modulo 4. We also establish which positions correspond to symmetric or pseudo-symmetric numerical semigroups, both of which are known to correspond to $\mathcal{N}$*-positions*. We use these results to identify some choices that players should avoid. We briefly discuss Guy's fourteenth question concerning Sylver coinage (see [5]) in light of these results. In Section 4, we establish which positions are $\mathcal{P}$*-positions* when 6 has also been played, when 10 has also been played, and when 14 has also been played. The proofs of these results provide strategies for how to play the game from $\mathcal{N}$*-positions* meeting these conditions.

2. Definitions, Notation, and Previous Results

Definition 1. *Sylver coinage* is a two-player game played on the positive integers, $\mathbb{N}$. The rules of the game are:

1. The players alternate choosing available numbers.
2. If $a_1, \ldots, a_k$ represent the first k choices in the game, then the numbers available to choose on the next turn are the elements of the set

$$\mathbb{N} \backslash \{n_1 a_1 + \cdots + n_k a_k \mid n_i \in \mathbb{N} \cup \{0\}\}.$$

3. The player who chooses the number 1 loses the game.

Example 1. As an example of a game of Sylver coinage, say Player 1 begins the game by choosing the number 5. Player 2 can now choose any number that is not a multiple of 5. Say Player 2 responds by choosing 7. Next, Player 2 can choose any number that cannot be expressed as $5x + 7y$ for some $x, y \in \mathbb{N} \cup \{0\}$. The entire game might look like this:

Available Plays	Player 1 Choice	Player 2 Choice
$\mathbb{N}$	5	
$\mathbb{N} \backslash 5\mathbb{N}$		7
$\{1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23\}$	8	
$\{1, 2, 3, 4, 6, 9, 11\}$		4
$\{1, 2, 3, 6\}$	6	
$\{1, 2, 3\}$		3
$\{1, 2\}$	2	
$\{1\}$		1

We see that Player 2 was forced to choose 1 and loses the game.

Remark 1. The following facts about Sylver coinage can be found in [2] and [3]. We omit the proofs here.

(i) If $\{a_1, \ldots, a_k\}$ represents the numbers that have been chosen in a game of Sylver coinage and $\gcd\{a_1, \ldots, a_k\} = 1$, then the set of available plays is finite.

(ii) Every game of Sylver coinage involves a finite number of plays.

(iii) If both players play "intelligently," then the first player to choose 1, 2, or 3 will lose the game.

Concerning Remark 1 (iii), we will always assume that both players play "intelligently," meaning both players will always make the most advantageous choice at every turn in the game.

Definition 2. The following definitions and notations pertain to the game of Sylver coinage. These can be found in [2] and [9], as well as Section 1 of [3].

(i) A *position* in a game of Sylver coinage is a set $M = \{a_1, \ldots, a_k\} \subseteq \mathbb{N}$ of the numbers that have been chosen by the players.

(ii) The set of *legal plays for a position* $M = \{a_1, \ldots, a_k\}$, denoted by $L(M)$, is $L(M) = \mathbb{N} \backslash \{n_1 a_1 + \cdots + n_k a_k \mid n_i \in \mathbb{N} \cup \{0\}\}$.

(iii) Let $x, y \in L(M)$ for some position $M = \{a_1, \ldots, a_k\}$. We say x *eliminates* y provided $y \notin L(M \cup \{x\})$. That is, choosing x results in y no longer being a legal play. We say that x *quietly eliminates* y provided $y - x \notin L(M)$. That is, x eliminates y and $y = 1 \cdot x + n_1 a_1 + \cdots + n_k a_k$ for some $n_1, \ldots, n_k \in \mathbb{N} \cup \{0\}$.

(iv) For a position $M = \{a_1, \ldots, a_k\}$, we say that a_i is *superfluous* provided $a_i \notin L(M\backslash\{a_i\})$. Equivalently, a_i is superfluous provided $L(M) = L(M\backslash\{a_i\})$.

(v) We say that a position $M = \{a_1, \ldots, a_k\}$ is in *canonical form* provided none of $a_1, \ldots, a_k$ are superfluous. Unless otherwise stated, we will assume that position $M = \{a_1, \ldots, a_k\}$ is in canonical form with $a_1 < \cdots < a_k$.

(vi) We say that a position M is a *finite position* provided $L(M)$ is a finite set. For a finite position M, we define $F(M)$ to be $\max(L(M))$.

(vii) A position is called an $\mathcal{N}$*-position* provided that the next player to play can win the game starting at that position. A position that is not an $\mathcal{N}$*-position* is called a $\mathcal{P}$*-position*. We will sometimes refer to $\mathcal{N}$*-positions* as *winning positions* and $\mathcal{P}$*-positions* as *losing positions*.

(viii) Given a position M, we say that $x \in L(M)$ is an *end* provided that $L(M \cup \{x\}) = L(M)\backslash\{x\}$. That is, x does not eliminate any legal plays other than itself. We say that M is an *ender* provided $F(M)$ is the only end. We say that an ender M is a *quiet ender* provided that every $x \in L(M)$ quietly eliminates $F(M)$. An ender that is not a quiet ender is called an *unquiet ender*.

Example 2. The following examples relate to Definition 2.

(i) Consider the position $M = \{4, 7, 17\}$. We compute $L(M) = \{1, 2, 3, 5, 6, 9, 10, 13\}$. Thus, $F(M) = 13$ and both 10 and 13 are ends. Further, 3 and 6 both eliminate 13 but 6 quietly eliminates 13 whereas 3 does not quietly eliminate 13.

(ii) Consider the position $M = \{4, 9\}$. We compute

$$L(M) = \{1, 2, 3, 5, 6, 7, 10, 11, 14, 15, 19, 23\}.$$

Thus, $F(M) = 23$ and this is the only end. We quickly check that all elements of $L(M)$ quietly eliminate 23 and hence M is a quiet ender.

We note that the notation "$F(M)$," representing the largest legal play for a finite position, is different from the notation "$t(M)$" that is usually found in the literature. The motivation is to match the notation to the Frobenius number of a numerical semigroup which we will define shortly. The next two propositions are mentioned in [2] and [9]. Formal proofs of these propositions can be found in [3].

Proposition 1. *If M is an ender, then for all $x \in L(M)$ we have that x eliminates $F(M)$.*

Proposition 2. *If M is an ender with $F(M) > 1$, then M is an $\mathcal{N}$-position.*

It is important to note that the converse of Proposition 2 is not true as shown by the position $M = \{4, 6, 11, 13\}$. The details are left to the reader.

Next, we establish the necessary definitions and notations pertaining to numerical semigroups. For additional material on numerical semigroups beyond what is contained here, the reader should see [1], [4], and [8]. The following definitions are the same as in [3].

Definition 3. A *numerical semigroup* is a subset S of $\mathbb{N} \cup \{0\}$ satisfying the following three conditions:

1. $0 \in S$;
2. S is closed under addition;
3. $\mathbb{N} \backslash S$ is a finite set.

The smallest positive element of S is called the *multiplicity* of S and is denoted by $m(S)$. The largest element of $\mathbb{N} \backslash S$ is called the *Frobenius number* of S and is denoted by $F(S)$. The set $\mathbb{N} \backslash S$ is called the *gaps* of S and the cardinality of this set, $|\mathbb{N} \backslash S|$ is called the *genus* of S and is denoted by $g(S)$.

Example 3. Let $S = \{0, 4, 7, 8, 11, 12\} \cup \{n \in \mathbb{N} |\ n \geq 14\}$. Then S is a numerical semigroup with multiplicity $m(S) = 4$, Frobenius number $F(S) = 13$, and genus $g(S) = 8$.

From [8], we know that for every numerical semigroup S, there exists a unique finite subset $\{a_1, \ldots, a_k\} \subset S$ which is minimal with respect to containment and such that for all $s \in S$, there exist $n_1, \ldots, n_k \in \mathbb{N} \cup \{0\}$ such that $s = n_1 a_1 + \cdots + n_k a_k$. In this case, the set $\{a_1, \ldots, a_k\}$ is called *the minimal system of generators* for S and is denoted by $msg(S)$. It is also common to write $S = \langle a_1, \ldots, a_k \rangle$. The number of elements in the minimal system of generators is called the *embedding dimension* of S and is denoted by $e(S)$. For the numerical semigroup in Example 3, we have $S = \langle 4, 7, 17 \rangle$ and $e(S) = 3$.

Of particular interest to the examination of Sylver coinage are numerical semigroups whose genus is as small as possible. From the definition of the Frobenius number, we know that if $x \in S$, then $F(S) - x \notin S$. Thus, $g(S) \geq \frac{F(S)+1}{2}$ when $F(S)$ is odd, and $g(S) \geq \frac{F(S)+2}{2}$ when $F(S)$ is even.

Definition 4. We say that a numerical semigroup S is *symmetric* provided $F(S)$ is odd and $g(S) = \frac{F(S)+1}{2}$. We say that a numerical semigroup is *pseudo-symmetric* provided $F(S)$ is even and $g(S) = \frac{F(S)+2}{2}$.

Example 4. The following examples relate to Definitions 3 and 4.

(i) From Example 3, we see that if $S = \langle 4, 7, 17 \rangle$, then $F(S) = 13$ and $g(S) = 8$. We conclude that $S = \langle 4, 7, 17 \rangle$ is not symmetric.

(ii) For $S = \langle 6, 9, 11\rangle = \{0, 6, 9, 11, 12, 15, 17, 18, 20, 21, 22, 23, 24\} \cup \{n \in \mathbb{N}|\ n \geq 26\}$ we have $F(S) = 25$ and $g(S) = 13$. We conclude that $S = \langle 6, 9, 11\rangle$ is symmetric.

(iii) For $S = \langle 4, 7, 9\rangle = \{0, 4, 7, 8, 9\} \cup \{n \in \mathbb{N}|\ n \geq 11\}$ we have $F(S) = 10$ and $g(S) = 6$. We conclude that $S = \langle 4, 7, 9\rangle$ is pseudo-symmetric.

There are several conditions that imply a numerical semigroup is symmetric or pseudo-symmetric. The next proposition reviews some of these.

Proposition 3. *For a numerical semigroup S, we have*

(i) *S is symmetric or pseudo-symmetric if and only if S is maximal with respect to containment among all numerical semigroups with Frobenius number equal to $F(S)$;*

(ii) *S is symmetric if and only if both $F(S)$ is odd and $x \in \mathbb{N}\backslash S$ implies that $F(S) - x \in S$;*

(iii) *S is pseudo-symmetric if and only if both $F(S)$ is even and $x \in \mathbb{N}\backslash S$ implies that either $F(S) - x \in S$ or $x = \frac{F(S)}{2}$;*

(iv) *If $e(S) = 2$, then S is symmetric;*

(v) *If S is a numerical semigroup with $m(S) = 4$ and $e(S) = 3$, then S is pseudo-symmetric if and only if $S = \langle 4, k, k+2\rangle$ where k is odd and $k \geq 3$.*

Statements (i) - (iv) of Proposition 3 are from [8] and statement (v) is from [7]. The reader is encouraged to see these resources for additional information on symmetric and pseudo-symmetric numerical semigroups.

Example 5. The following examples relate to Proposition 3.

(i) From Proposition 3 (iv), we know $S = \langle 4, 9\rangle$ is symmetric.

(ii) From Proposition 3 (v), we know $S = \langle 4, 7, 9\rangle$ is pseudo-symmetric.

(iii) A quick check shows that $F(\langle 4, 7, 9\rangle) = 10$. Proposition 3 (i) tells us that if T is any numerical semigroup such that $\langle 4, 7, 9\rangle \subsetneq T$, then $F(T) < 10$.

Now that we have established the definitions and notations for Sylver coinage and for numerical semigroups, we recall the following results from Section 4 of [3] connecting these topics. The proofs are omitted.

Proposition 4. *Let $M = \{a_1, \ldots, a_k\}$ be a finite position in a game of Sylver coinage written in canonical form. Define $S(M) = (\mathbb{N} \cup \{0\}) - L(M)$. Then,*

(i) *$S(M)$ is a numerical semigroup with $msg(S(M)) = \{a_1, \ldots, a_k\}$;*

(ii) *M is an ender if and only if $S(M)$ is symmetric or pseudo-symmetric;*

(iii) *M is a quiet ender if and only if $S(M)$ is symmetric;*

(iv) *M is an unquiet ender if and only if $S(M)$ is pseudo-symmetric.*

Example 6. The following examples relate to Propositions 3 and 4.

(i) The position $M = \{5, 7\}$ is an $\mathcal{N}$*-position* because $S(M) = \langle 5, 7\rangle$ is symmetric by Proposition 3 (iv) and hence M is an ender.

(ii) The position $M = \{4, 7, 9\}$ is an $\mathcal{N}$*-position* because $S(M) = \langle 4, 7, 9\rangle$ is pseudo-symmetric by Proposition 3 (v) and hence M is an unquiet ender.

We will make repeated use of the results in this section as we investigate games of Sylver coinage for which 4 has been chosen.

3. Games for Which 4 Has Been Chosen

We now focus our attention on games of Sylver coinage for which the number 4 has been chosen. Our goal is to utilize numerical semigroup tools to extend the scope of the investigation in [2] and provide additional details.

Definition 5. The following definitions and notations will be used throughout the remainder of this investigation.

(i) Let $SC(4)$ denote the set of all finite positions of Sylver coinage for which the number 4 is the smallest number that has been chosen. That is, $SC(4)$ contains all positions M such that all of the following are true:

(a) $4 \in M$;

(b) $\{1, 2, 3\} \cap M = \emptyset$;

(c) there exists $x \in M$ such that x is odd and $x \geq 5$.

(ii) If $M \in SC(4)$, then the elements of $L(M)$ that are greater than 3 can be partitioned into subsets M_1, M_2, and M_3 such that

$$M_i = \{x \in L(M) \mid x \equiv i \pmod 4 \text{ and } x > 3\} \text{ for } i = 1, 2, 3.$$

We will use the notation $m_i = |M_i|$ for $i = 1, 2, 3$.

Example 7. Using the notation from Definition 5 for position $M = \{4, 11, 14\} \in SC(4)$, we have

$$\begin{aligned} M_1 &= \{5, 9, 13, 17, 21\} \\ M_2 &= \{6, 10\} \\ M_3 &= \{7\}. \end{aligned}$$

Thus, $m_1 = 5$, $m_2 = 2$, and $m_3 = 1$. Also, $S(M) = \langle 4, 11, 14\rangle$ is symmetric by Proposition 3 (ii) and hence M is an $\mathcal{N}$*-position* by Proposition 4 (ii) and Proposition 2.

Remark 2. Let $M \in SC(4)$.

(i) We can write $M = \{4, 4m_1+5, 4m_2+6, 4m_3+7\}$, not necessarily in canonical form.

(ii) We can write $L(M) = \{1, 2, 3\} \cup M_1 \cup M_2 \cup M_3$ where

$$\begin{aligned} M_1 &= \{5, 9, \ldots, 4m_1 + 1\} \\ M_2 &= \{6, 10, \ldots, 4m_2 + 2\} \\ M_3 &= \{7, 11, \ldots, 4m_3 + 3\}. \end{aligned}$$

(iii) The set $\{4, 4m_1+5, 4m_2+6, 4m_3+7\}$ is a system of generators for $S(M)$ but it may not the minimal system of generators.

(iv) By the definition of Frobenius number, we have $F(S(M)) = \max\{4m_1 + 1, 4m_2 + 2, 4m_3 + 3\}$.

(v) By the definition of genus, we have $g(S(M)) = |L(M)| = m_1 + m_2 + m_3 + 3$.

We take a moment to examine which elements of the set $L(M)$ are eliminated when a player makes their choice. Assume $M \in SC(4)$ and that $x \in L(M)$ is the next number chosen. Let $\overline{M} = M \cup \{x\}$ and let $\overline{M}_1, \overline{M}_2, \overline{M}_3, \overline{m}_1, \overline{m}_2, \overline{m}_3$ be defined as in Definition 5.

Consider the case in which $x \in M_1$. Then we know we can write $x = 4(m_1-k)+1$ for some $0 \le k \le m_1 - 1$. Since $4 \in M$, we know $x, x+4, \ldots, 4m_1+1$ are all eliminated by choosing x. Thus,

$$\max(\overline{M}_1) = x - 4 = 4(m_1 - k - 1) + 1$$

and hence $\overline{m}_1 = m_1 - k - 1$.

Next, note that since $x, 4m_2 + 6 \in S(\overline{M})$, we have $2x, 3x, x + 4m_2 + 6 \in S(\overline{M})$. Since $2x \equiv 2 \pmod 4$, we have

$$\max(\overline{M}_2) = \min\{4m_2 + 2, 2x - 4 = 4(2m_1 - 2k - 1) + 2\}$$

and hence $\overline{m}_2 = \min\{m_2, 2m_1 - 2k - 1\}$.

Finally, since $3x \equiv x + 4m_2 + 6 \equiv 3 \pmod 4$, we have

$$\max(\overline{M}_3) = min\{4m_3+3, x+4m_2+2 = 4(m_1+m_2-k)+3, 3x-4 = 4(3m_1-3k-1)+3\}$$

and hence $\overline{m}_3 = \min\{m_3, m_1 + m_2 - k, 3m_1 - 3k - 1\}$.

Similar results hold in the cases $x = 4(m_2 - k) + 2 \in M_2$ and $x = 4(m_3 - k) + 3 \in M_3$. We summarize these in the following proposition.

Proposition 5. *Let $M \in SC(4)$ and let $x \in L(M)$ be the next number chosen. Let $\overline{M} = M \cup \{x\}$ and let $\overline{M}_1, \overline{M}_2, \overline{M}_3, \overline{m}_1, \overline{m}_2, \overline{m}_3$ be defined as in Definition 5.*

(i) *If $x = 4(m_1 - k) + 1 \in M_1$ for some $0 \leq k \leq m_1 - 1$, then*

- *(a)* $\max(\overline{M}_1) = 4(m_1 - k - 1) + 1$ *and* $\overline{m}_1 = m_1 - k - 1$*;*
- *(b)* $\max(\overline{M}_2) = \min\{4m_2 + 2, 4(2m_1 - 2k - 1) + 2\}$ *and* $\overline{m}_2 = \min\{m_2, 2m_1 - 2k - 1\}$*;*
- *(c)* $\max(\overline{M}_3) = \min\{4m_3 + 3, 4(m_1 + m_2 - k) + 3, 4(3m_1 - 3k - 1) + 3\}$ *and* $\overline{m}_3 = \min\{m_3, m_1 + m_2 - k, 3m_1 - 3k - 1\}$*.*

(ii) *If $x = 4(m_2 - k) + 2 \in M_2$ for some $0 \leq k \leq m_2 - 1$, then*

- *(a)* $\max(\overline{M}_1) = \min\{4m_1 + 1, 4(m_2 + m_3 - k + 1) + 1\}$ *and* $\overline{m}_1 = \min\{m_1, m_2 + m_3 - k + 1\}$*;*
- *(b)* $\max(\overline{M}_2) = 4(m_2 - k - 1) + 2$ *and* $\overline{m}_2 = m_2 - k - 1$*;*
- *(c)* $\max(\overline{M}_3) = \min\{4m_3 + 3, 4(m_1 + m_2 - k) + 3\}$ *and* $\overline{m}_3 = \min\{m_3, m_1 + m_2 - k\}$*.*

(iii) *If $x = 4(m_3 - k) + 3 \in M_3$ for some $0 \leq k \leq m_3 - 1$, then*

- *(a)* $\max(\overline{M}_1) = \min\{4m_1 + 1, 4(m_2 + m_3 - k + 1) + 1, 4(3m_3 - 3k + 1) + 1\}$ *and* $\overline{m}_1 = \min\{m_1, m_2 + m_3 - k + 1, 3m_3 - 3k + 1\}$*;*
- *(b)* $\max(\overline{M}_2) = \min\{4m_2 + 3, 4(2m_3 - 2k) + 2\}$ *and* $\overline{m}_2 = \min\{m_2, 2m_3 - 2k\}$*;*
- *(c)* $\max(\overline{M}_3) = 4(m_3 - k - 1) + 3$ *and* $\overline{m}_3 = m_3 - k - 1$*.*

Example 8. Let $M \in SC(4)$ with $m_1 = 4$ and $m_2 = 3$ and $m_3 = 6$. Thus,

$$\begin{aligned} M_1 &= \{5, 9, 13, 17\} \\ M_2 &= \{6, 10, 14\} \\ M_3 &= \{7, 11, 15, 19, 23, 27\}. \end{aligned}$$

Choosing $x = 9 = 4(m_1 - 2) + 1$ results in

$$\begin{aligned} \overline{M}_1 &= \{5\} \text{ and } \overline{m}_1 = 1. \\ \overline{M}_2 &= \{6, 10, 14\} \text{ and } \overline{m}_2 = 3. \\ \overline{M}_3 &= \{7, 11, 15, 19, 23\} \text{ and } \overline{m}_3 = 5. \end{aligned}$$

The values m_1, m_2, and m_3 offer a convenient alternative notation to express positions in $SC(4)$. Given $M \in SC(4)$, we will often write $M = [m_1, m_2, m_3]$

instead of the set notation $M = \{4, 4m_1 + 5, 4m_2 + 6, 4m_3 + 7\}$. For example, the position $M = \{4, 11, 14\}$ can be expressed as $M = [5, 2, 1]$ instead.

It is important to note that while every $M \in SC(4)$ has a unique expression of the from $[m_1, m_2, m_3]$ where $m_i \in \mathbb{N} \cup \{0\}$, not every such ordered triple corresponds to an element of $SC(4)$. Indeed, $[1, 3, 7]$ would correspond to position M such that

$$L(M) = \{1, 2, 3\} \cup \{5\} \cup \{6, 10, 14\} \cup \{7, 11, 15, 19, 23, 27, 31\}.$$

However, this means that 9 has been chosen at some point in this game of Sylver coinage so 27 and 31 are no longer legal plays. We conclude that $[1, 3, 7]$ does not correspond to an element of $SC(4)$.

Proposition 6. *Let $M = [m_1, m_2, m_3] \in SC(4)$. Then m_1, m_2, and m_3 must satisfy all of the following:*

(i) $m_1 \leq m_2 + m_3 + 2$;

(ii) $m_3 \leq m_1 + m_2 + 1$;

(iii) $m_2 \leq 2m_1 + 1$;

(iv) $m_2 \leq 2m_3 + 2$.

Proof. Recall from Remark 2 (i) that $M = \{4, 4m_1 + 5, 4m_2 + 6, 4m_3 + 7\}$. To prove the inequality in (i), assume by contradiction that $m_1 > m_2 + m_3 + 2$. Multiplying by 4 and adding 5 to both sides we have

$$4m_1 + 5 > (4m_2 + 6) + (4m_3 + 7).$$

Now, $4m_1 + 5 \equiv (4m_2 + 6) + (4m_3 + 7) \pmod 4$ and hence

$$4m_1 + 5 = (4m_2 + 6) + (4m_3 + 7) + 4k$$

for some $k \geq 1$. Subtracting 4 from both sides we have,

$$4m_1 + 1 = (4m_2 + 6) + (4m_3 + 7) + 4(k - 1).$$

However, $(4m_2 + 6) + (4m_3 + 7) + 4(k - 1) \in S(M)$ for $k \geq 1$ by Remark 2 (iii) and hence $4m_1 + 1 \in S(M)$. Therefore, $4m_1 + 1 \notin L(M)$. This is a contradiction. We conclude that $m_1 \leq m_2 + m_3 + 2$. The proofs of the inequalities in (ii) - (iv) are similar. □

Proposition 7. *Let $M = [m_1, m_2, m_3] \in SC(4)$. Then $S(M)$ is symmetric if and only if either $m_1 = m_2 + m_3 + 2$ or $m_3 = m_1 + m_2 + 1$.*

Proof. First assume that the numerical semigroup $S(M)$ is symmetric. Then we know $F(S(M))$ is odd and hence $\max\{4m_1+1, 4m_2+2, 4m_3+3\}$ is equal to $4m_1+1$ or $4m_3+3$. In the case $\max\{4m_1+1, 4m_2+2, 4m_3+3\} = 4m_1+1$ we know $F(S(M)) = 4m_1+1$ and since $S(M)$ is symmetric we know $g(S(M)) = \frac{F(S(M))+1}{2}$. By Remark 2 (v), we see

$$m_1 + m_2 + m_3 + 3 = \frac{4m_1+2}{2}.$$

Thus, $m_1 = m_2 + m_3 + 2$. The proof for the case $\max\{4m_1+1, 4m_2+2, 4m_3+3\} = 4m_3+3$ is similar.

Next assume that $m_1 = m_2 + m_3 + 2$. Then $\max\{4m_1+1, 4m_2+2, 4m_3+3\} = 4m_1+1$ and we have $F(S(M)) = 4m_1+1$ which is odd. Further, $m_1 = m_2+m_3+2$ implies $m_1+m_2+m_3+3 = \frac{4m_1+2}{2}$ from which we conclude that $g(S(M)) = \frac{F(S(M))+1}{2}$. Therefore, $S(M)$ is symmetric by definition. Assuming $m_3 = m_1+m_2+1$ leads to the same conclusion by a similar argument. □

Proposition 8. *Let $M = [m_1, m_2, m_3] \in SC(4)$. Then $S(M)$ is pseudo-symmetric if and only if either $m_2 = 2m_1 = 2m_3+2$ or $m_2 = 2m_1+1 = 2m_3+1$.*

Proof. First assume that $S(M)$ is pseudo-symmetric. By Proposition 3 (v), we know that $S(M) = \langle 4, k, k+2\rangle$ for some odd $k \geq 5$. By Remark 2 (iii), we know that either $k = 4m_1+5$ or $k = 4m_3+7$.

In the case $k = 4m_1+5$, we have $4m_1+7 = 4m_3+7$ and hence $m_1 = m_3$. Further, since $e(S(M)) = 3$, we have that either $4m_2+6 = 2k$ or $4m_2+6 = 2(k+2))$. In the latter case we see that

$$4m_2 + 6 = 2(4m_1+7) = 8m_1 + 14$$

and it follows that $m_2 = 2m_1+2$, which contradicts Proposition 6 (iii). Thus, we know

$$4m_2 + 6 = 2k = 8m_1 + 10.$$

It follows that $m_2 = 2m_1+1 = 2m_3+1$. The case $k = 4m_3+7$ leads to the conclusion $m_2 = 2m_1 = 2m_3+2$ by a similar argument.

For the converse, first consider the case $m_2 = 2m_1 = 2m_3+2$. In this case we have

$$4m_2 + 6 = 8m_3 + 14 = 2(4m_3+7) \quad \text{and} \quad 4m_1 + 5 = 4m_3 + 9$$

and hence $msg(S(M)) = \{4, 4m_3+7, 4m_3+9\}$ by Remark 2 (iii). By Proposition 3 (vi), we conclude $S(M)$ is pseudo-symmetric. The case $m_2 = 2m_1+1 = 2m_3+1$ also leads to the conclusion that $S(M)$ is pseudo-symmetric by a similar argument. □

The results of this section provide some guidance for playing Sylver coinage for positions in $SC(4)$. Specifically, they yield some choices that players should avoid.

Indeed, let $M = [m_1, m_2, m_3] \in SC(4)$ and let $x \in L(M)$ be the next number chosen. As in the discussion leading up to Proposition 5, we first consider the case $x = 4(m_1 - k) + 1 \in M_1$. Further, assume that $k > \max\{m_1 - \frac{m_2+1}{2}, m_1 - \frac{m_3+1}{3}\}$. Since $k > m_1 - \frac{m_2+1}{2}$, we conclude that $m_2 > 2m_1 - 2k - 1$. Multiplying by 4 and adding 2, we have

$$4m_2 + 2 > 4(2m_1 - 2k - 1) + 2. \tag{1}$$

By Proposition 5 (i)(b), we conclude that $\overline{m}_2 = 2m_1 - 2k - 1$.

Next, from Equation (1) we have $x + 4m_2 + 2 > x + 4(2m_1 - 2k - 1) + 2$ and hence

$$4(m_1 + m_2 - k) + 3 > 4(3m_1 - 3k - 1) + 3.$$

Further, the assumption $k > m_1 - \frac{m_3+1}{3}$ tells us $m_3 > 3m_1 - 3k - 1$. Multiplying by 4 and adding 3 yields

$$4m_3 + 3 > 4(3m_1 - 3k - 1) + 3.$$

By Proposition 5 (i)(c), we conclude that $\overline{m}_3 = 3m_1 - 3k - 1$. Now,

$$\overline{m}_3 = \overline{m}_1 + \overline{m}_2 + 1$$

and we conclude that $S(\overline{M})$ is symmetric by Proposition 7 and hence $\overline{M}$ is an $\mathcal{N}$*-position* by Proposition 2 and Proposition 4.

This discussion shows that choosing $x = 4(m_1 - k) + 1 \in M_1$ with $k > \max\{m_1 - \frac{m_2+1}{2}, m_1 - \frac{m_3+1}{3}\}$ results in an $\mathcal{N}$*-position* for the next player. Thus, such a play should be avoided. Similar results can be derived from other parts of Proposition 7 and from Proposition 8. We summarize these results in the following proposition.

Proposition 9. *Let* $M = [m_1, m_2, m_3] \in SC(4)$ *and let* $x \in L(M)$ *be the next number chosen. Let* $\overline{M} = M \cup \{x\}$. *All of the following choices will result in* $S(\overline{M})$ *being symmetric or pseudo-symmetric and hence* $\overline{M}$ *being an* $\mathcal{N}$*-position:*

(i) $x = 4(m_1 - k) + 1$ *assuming* $k > \max\{m_1 - \frac{m_2+1}{2}, m_1 - \frac{m_3+1}{3}\}$*;*

(ii) $x = 4(m_3 - k) + 3$ *assuming* $k > \max\{m_3 - \frac{m_2}{2}, m_3 - \frac{m_1-1}{3}\}$*;*

(iii) $x = 4m_1 + 3$ *assuming* $m_2 = 2m_1$*;*

(iv) $x = 4m_3 + 9$ *assuming* $m_2 = 2m_3 + 2$*;*

(v) $x = 4m_1 + 7$ *assuming* $m_2 = 2m_1 + 1$*;*

(vi) $x = 4m_3 + 5$ *assuming* $m_2 = 2m_3 + 1$.

Example 9. Let $M = [5, 11, 8] \in SC(4)$ Thus,

$$\begin{array}{rcl} M_1 & = & \{5, 9, 13, 17, 21\} \\ M_2 & = & \{6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46\} \\ M_3 & = & \{7, 11, 15, 19, 23, 27, 31, 35\}. \end{array}$$

Let x be the next number chosen and let $\overline{M} = M \cup \{x\}$.

(i) Referring to Proposition 9, we compute $\max\{m_1 - \frac{m_2+1}{2}, m_1 - \frac{m_3+1}{3}\} = 2$. Thus, $x \in \{5, 9\}$ will result in $S(\overline{M})$ being symmetric.

(ii) Similarly, we compute $\max\{m_3 - \frac{m_2}{2}, m_3 - \frac{m_1-1}{3}\} = \frac{20}{3}$. Thus, $x = 7$ will result in $S(\overline{M})$ being symmetric.

(iii) Since $m_2 = 2m_1 + 1$, we know $x = 4m_1 + 7 = 27$ will result in $S(\overline{M})$ being pseudo-symmetric.

We conclude that the next player should avoid playing $x \in \{5, 7, 9, 27\}$ as any of these choices will result in an $\mathcal{N}$*-position* for their opponent.

We finish this section by offering a thought about one of Richard Guy's *Twenty questions concerning Conway's Sylver coinage*, [5]. Guy defines a 4*-pair* to be an ordered pair (a, b) such that ab is odd and $M = \{4, a, 2a, b\}$ is a $\mathcal{P}$*-position*. The fourteenth question in [5] asks

Does $\frac{b}{a}$ tend to a limit as $a \to \infty$ while (a, b) remains a 4-pair?

Using the notation $M = [m_1, m_2, m_3]$ and Remark 2 (iii), we know that if $M = \{4, a, 2a, b\}$ corresponds to a 4-pair, then either

(i) $4m_2 + 6 = 2(4m_1 + 5)$ which yields $m_2 = 2m_1 + 1$, or

(ii) $4m_2 + 6 = 2(4m_3 + 7)$ which yields $m_2 = 2m_3 + 2$.

Further, since M is a $\mathcal{P}$*-position* we know $S(M)$ is not symmetric or pseudo-symmetric, so Propositions 7 and 8 yield

(iii) $m_2 < 2m_3 + 1$ (if m_2 is odd);

(iv) $m_2 < 2m_1$ (if m_2 is even);

(v) $m_1 < m_2 + m_3 + 2$;

(vi) $m_3 < m_1 + m_2 + 1$.

In [5], we see $(5, 11)$, $(7, 13)$, $(9, 19)$, $(15, 33)$, $(17, 43)$, $(21, 51)$, and $(23, 57)$ are all examples of 4-pairs and all 4-pairs up to $a \leq 707$ are known. Translating the 4-pairs in this list to the notation in this investigation we have the following:

m_2 odd			m_2 even		
$(5, 11)$	$\to$	$[0, 1, 1]$	$(7, 13)$	$\to$	$[2, 2, 0]$
$(9, 19)$	$\to$	$[1, 3, 3]$	$(15, 33)$	$\to$	$[7, 6, 2]$
$(17, 43)$	$\to$	$[3, 7, 9]$	$(23, 57)$	$\to$	$[13, 10, 4]$
$(21, 51)$	$\to$	$[4, 9, 11]$			

Perhaps it is possible to express m_3 as a function of m_1 when m_2 is odd, and to express m_1 as a function of m_3 when m_2 is even. If so, then 4-pairs can be completely characterized which would significantly contribute to resolving the fourteenth question of Sylver coinage.

4. Positions in $SC(4)$ For Which 6 Or 10 Or 14 Has Been Chosen

In this section we examine positions in $SC(4)$ with $m_2 = 0$ (that is, 6 has been chosen) and with $m_2 = 1$ (that is, 10 has been chosen). We will completely determine which positions are $\mathcal{N}$ and which are $\mathcal{P}$. At the end of the section, we offer some discussion of positions with $m_3 = 3$ (that is, 14 has been chosen) and $m_2 \geq 4$.

Proposition 10. *Let $M = [m_1, 0, m_3] \in SC(4)$. If $m_1 = m_3$, then M is a $\mathcal{P}$-position.*

Proof. Consider the position $[m, 0, m]$ for some $m \geq 0$. We proceed by induction. For the base case, we note that $M = [0, 0, 0]$ is a $\mathcal{P}$*-position* by Remark 1 (iii) since $L(M) = \{1, 2, 3\}$.

For the inductive step, assume $[n, 0, n]$ is a $\mathcal{P}$*-position* for all $0 \leq n \leq m - 1$ and consider $M = [m, 0, m]$. Then,

$$M_1 = \{5, 9, \ldots, 4m + 1\}$$

and

$$M_3 = \{7, 11, \ldots, 4m + 3\}.$$

Assume that the next player is Player 1 and they choose $x = 4(m - k) + 1 \in M_1$ for some $0 \leq k \leq m - 1$ resulting in position $\overline{M} = [m - k - 1, 0, \overline{m}_3]$ where $\overline{m}_3 = \min\{m, 2m - k, 3m - 3k - 1\}$ by Proposition 5 (i). In any of these three cases we note that $\overline{m}_3 \geq m - k$, so Player 2 can choose $y = 4(m - k) + 3$, resulting in the position $[m - k - 1, 0, m - k - 1]$ which is a $\mathcal{P}$*-position* by the induction hypothesis. Similarly, if Player 1 chooses $x = 4(m - k) + 3 \in M_3$, then Player 2 can respond by choosing $y = 4(m - k) + 1$ resulting in the position $[m - k - 1, 0, m - k - 1]$.

We conclude that for any number chosen by Player 1 in the position $M = [m, 0, m]$, Player 2 can respond with a choice that returns a $\mathcal{P}$*-position* back to Player 1. Thus, $M = [m, 0, m]$ is a $\mathcal{P}$*-position* for all $m \geq 0$. □

The proof of Proposition 10 reveals the $\mathcal{N}$*-positions* for $m_2 = 0$ as well.

Corollary 1. *The position $[m_1, 0, m_3]$ for $m_1 \neq m_3$ is an $\mathcal{N}$-position.*

The following also results from Proposition 10.

Corollary 2. *If $M = [m_1, m_2, m_3] \in SC(4)$ and $m_1 = m_3$, then M is an $\mathcal{N}$-position.*

Proof. Assume that $m_1 = m_3 = m$ and assume let $x = 6 = 4(m_2 - (m_2 - 1)) + 2$ is the next number chosen. By Proposition 5 (2), we have

$$\overline{m}_1 = \min\{m, m_2 + m - (m_2 - 1) + 1\} = \min\{m, m + 2\} = m.$$

Also,

$$\overline{m}_3 = \min\{m, m + m_2 - (m_2 - 1)\} = \min\{m, m + 1\} = m.$$

Thus, choosing $x = 6$ results in the position $\overline{M} = [m, 0, m]$, which is a $\mathcal{P}$-*position* by Proposition 10. □

Next, we examine positions with $m_2 = 1$.

Proposition 11. *Let $M = [m_1, 1, m_3] \in SC(4)$. If $\max\{m_1, m_3\}$ is odd and $|m_1 - m_3| = 1$, then M is a $\mathcal{P}$-position.*

Proof. Let $M = [m_1, 1, m_3] \in SC(4)$. Assume $\max\{m_1, m_3\}$ is odd and $|m_1 - m_3| = 1$. Thus $M = [2n, 1, 2n + 1]$ or $M = [2n + 1, 1, 2n]$ for some $n \geq 0$. We proceed by induction on n. For the base case, it is quick to confirm that the positions $[0, 1, 1]$ and $[1, 1, 0]$ are both $\mathcal{P}$-*positions*.

For the inductive step, assume that $[2n, 1, 2n + 1]$ and $[2n + 1, 1, 2n]$ are $\mathcal{P}$-*positions* for $0 \leq n \leq m - 1$. First consider the position $M = [2m, 1, 2m + 1]$. In this case we have

$$M_1 = \{5, 9, \ldots, 4(2m) + 1\}$$

and

$$M_3 = \{7, 11, \ldots, 4(2m + 1) + 3\}.$$

Assume that the next player is Player 1 and they choose $x = 4(2m - k) + 1 \in M_1$ for some $0 \leq k \leq 2m - 1$. By Proposition 5 (i), we have $\overline{m}_1 = 2m - k - 1$ and $\overline{m}_3 = \min\{2m + 1, 2m + 1 - k, 3(2m) - 3k - 1\}$. It is quick to check that in all three of these cases we have $\overline{m}_3 \geq \overline{m}_1$ and hence $\overline{m}_3 \geq 2m - k$.

Now, if $2m - k - 1$ is odd, then $k \leq 2m - 2$ so Player 2 can choose $y = 4(2m - k - 1) + 3$, resulting in the position $[2m - k - 1, 1, 2m - k - 2]$ which is a $\mathcal{P}$-*position* by the induction hypothesis. On the other hand, if $2m - k - 1$ is even, then Player 2 can choose $y = 4(2m - k + 1) + 3$, resulting in the position $[2m - k - 1, 1, 2m - k]$ which is also a $\mathcal{P}$-*position* by the induction hypothesis. We conclude that if Player 1 chooses an element of M_1, then Player 2 can respond with a choice that returns a $\mathcal{P}$-*position* back to Player 1.

A similar argument will show that if Player 1 chooses an element of M_3, then Player 2 can again respond with a choice that returns a $\mathcal{P}$-*position* back to Player 2. We conclude that $M = [2m, 1, 2m+1]$ is a $\mathcal{P}$-*position*.

Using a similar approach, we can show that $M = [2m+1, 1, 2m]$ is also a $\mathcal{P}$-*position*. This completes the induction. □

Corollary 3. *If $M = [m_1, 1, m_3]$ and $|m_1 - m_3| > 1$, then M is an $\mathcal{N}$-position.*

Proof. Let $M = [m_1, 1, m_3]$ and assume $m_1 - m_3 \geq 2$. Assume the next player is Player 1. If m_3 is odd, then Player 1 can choose $x = 4m_3 + 1 \in M_1$, resulting in the position $[m_3 - 1, 1, m_3]$ which is a $\mathcal{P}$-*position* by Proposition 11. On the other hand, if m_3 is even, then Player 1 can choose $x = 4(m_3 + 2) + 1 \in M_1$, resulting in the position $[m_3 + 1, 1, m_3]$ which is also a $\mathcal{P}$-*position* by Proposition 11. We conclude that M is an $\mathcal{N}$-*position* in the case $m_1 - m_3 \geq 2$.

A similar argument holds in the situation $m_3 - m_1 \geq 2$. □

Corollary 4. *If $M = [2k, m_2, 2k+1]$ or $M = [2k+1, m_2, 2k]$ for some $k \geq 0$ and for some $m_2 \geq 2$, then M is an $\mathcal{N}$-position.*

Proof. Assume $M = [2k, m_2, 2k+1]$. Choosing $x = 10 = 4(m_2 - (m_2 - 2)) + 2$, Proposition 5 (ii) yields

$$\overline{m}_1 = \min\{2k, m_2 + (2k+1) - (m_2 - 2) + 1\} = \min\{2k, 2k+4\} = 2k.$$

Further,

$$\overline{m}_3 = \min\{2k+1, 2k + m_2 - (m_2 - 2)\} = \min\{2k+1, 2k+2\} = 2k+1.$$

Thus, $\overline{M} = [2k, 1, 2k+1]$ which is a $\mathcal{P}$-*position* by Proposition 11. A similar argument holds for the case $M = [2k+1, m_2, 2k]$. □

The next logical step is to examine positions $M \in SC(4)$ such that $m_2 = 2$. Indeed, such an investigation reveals that the $\mathcal{P}$-*positions* in this family are $[2k+1, 2, 2k+2]$ and $[2k+2, 2, 2k+1]$ for $k \geq 0$ with the single exception being $[2, 2, 1]$. We discover that $[2, 2, 1]$ is actually an $\mathcal{N}$-*position*, since choosing the number 7 results in the position $\overline{M} = [2, 2, 0]$ which can be quickly shown to be a $\mathcal{P}$-*position*. The above description of $\mathcal{P}$-*positions* in $SC(4)$ with $m_2 = 2$ can be proved using induction in a manner similar to Propositions 10 and 11, with the single exception $[2, 2, 1]$ handled separately. We leave this proof to the reader.

Our examination of positions in $SC(4)$ with $m_2 \geq 4$ has revealed that describing the $\mathcal{P}$-*positions* becomes more complicated and involves an increasing number of exceptions as m_2 increases. Perhaps there is an effective method for characterizing the $\mathcal{P}$-*positions* in $SC(4)$ for all values of m_2, but so far, such a method has eluded us.

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