RESTRICTED NIM WITH A PASS

Ryohei Miyadera
Keimei Gakuin Junior and High School, Kobe City, Japan.
runnerskg@gmail.com

Hikaru Manabe
Keimei Gakuin Junior and High School, Kobe City, Japan.
urakihebanam@gmail.com

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Abstract

This paper presents a study of restricted Nim with a pass. In the restricted Nim considered in this study, two players take turns and remove stones from the piles. In each turn, when the number of stones is m, each player is allowed to remove at least one stone and at most $\lceil \frac{m}{2} \rceil$ stones from a pile of m stones. The standard rules of the game are modified to allow a one-time pass, that is, a pass move that may be used at most once in the game and not from a terminal position. Once either player has used a pass, it is no longer available. In classical Nim, the introduction of the pass alters the underlying structure of the game, significantly increasing its complexity. In the restricted Nim considered in this study, the pass move had minimal impact. There is a simple relationship between the Grundy numbers of restricted Nim and the Grundy numbers of restricted Nim with a pass, where the number of piles can be any natural number. Therefore, the authors address a longstanding open question in combinatorial game theory: the extent to which the introduction of a pass into a game affects its behavior. The game we developed is the first variant of Nim that is fully solvable when a pass is not allowed and remains fully solvable following the introduction of a pass move.

1. Introduction

In this study, restricted Nim and restricted Nim with a pass are examined. An interesting but difficult question in combinatorial game theory has been to determine what happens when standard game rules are modified to allow a *one-time pass*, a pass move that may be used at most once in the game and not from a terminal position. Once either player has used a pass, it is no longer available. In the case of classical Nim, the introduction of the pass alters the mathematical structure of

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the game, considerably increasing its complexity. The effect of a pass on classical Nim remains an important open question that has defied traditional approaches. The late mathematician David Gale offered a monetary prize to the first person to develop a solution for three-pile classical Nim with a pass.

In [4] (p. 370), Friedman and Landsberg conjectured that "solvable combinatorial games are structurally unstable to perturbations, while generic, complex games will be structurally stable." One way to introduce such a perturbation is to allow a pass. One of the authors of the present article reported a counterexample to this conjecture in [3]. The game used in [3] is solvable because there is a simple formula for the Grundy numbers, and even when we introduce a pass move to the game, there is a simple formula for $\mathcal{P}$-positions.

The restricted Nim considered in the present study is of the same type, but the introduction of a pass move has a minimal impact. There is a simple relationship between the Grundy numbers of the game and the Grundy numbers of the game with a pass move, and the number of piles can be any natural number. This result is stated in Theorem 4 of the present article.

Let $Z_{\geq 0}$ and N be sets of non-negative numbers and natural numbers, respectively. For completeness, we briefly review some of the necessary concepts of combinatorial game theory. Details are presented in [1] and [5].

Definition 1.1. Let x and y be non-negative integers. They are expressed in Base 2 as follows: $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$, with $x_i, y_i \in \{0, 1\}$. We define the *nim-sum* $x \oplus y$ as follows:

$$x \oplus y = \sum_{i=0}^{n} w_i 2^i,$$

where $w_i = x_i + y_i \pmod 2$.

For impartial games without drawings, there are only two outcome classes.

Definition 1.2. (i) A position is referred to as a $\mathcal{P}$-*position* if it is a winning position for the previous player (the player who has just moved), as long as he plays correctly at every stage.
(ii) A position is referred to as an $\mathcal{N}$-*position* if it is a winning position for the next player as long as he plays correctly at every stage.

Definition 1.3. (i) For any position $\mathbf{p}$ of game $\mathbf{G}$, there is a set of positions that can be reached by precisely one move in $\mathbf{G}$, which we denote as $move(\mathbf{p})$.
(ii) The *minimum excluded value* (*mex*) of a set S of non-negative integers is the smallest non-negative integer that is not in S.
(iii) Let $\mathbf{p}$ be the position of an impartial game. The associated *Grundy number* is denoted as $\mathcal{G}(\mathbf{p})$ and is recursively defined as follows: $\mathcal{G}(\mathbf{p}) = mex(\{\mathcal{G}(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\})$.

Definition 1.4. The *disjunctive sum* of two games, which is denoted as $\mathbf{G}+\mathbf{H}$, is a supergame in which a player may move in either $\mathbf{G}$ or $\mathbf{H}$ but not both.

Theorem 1. *Let $\mathbf{G}$ and $\mathbf{H}$ be impartial rulesets and $\mathcal{G}_{\mathbf{G}}$ and $\mathcal{G}_{\mathbf{H}}$ be the Grundy numbers of position $\mathbf{g}$ played under the rules of $\mathbf{G}$ and position $\mathbf{h}$ played under the rules of $\mathbf{H}$, respectively. We have the following:*
(i) For any position $\mathbf{g}$ of $\mathbf{G}$, $\mathcal{G}_{\mathbf{G}}(\mathbf{g})=0$ if and only if $\mathbf{g}$ is a $\mathcal{P}$ position.
(ii) The Grundy number of positions $\{\mathbf{g},\mathbf{h}\}$ in game $\mathbf{G}+\mathbf{H}$ is $\mathcal{G}_{\mathbf{G}}(\mathbf{g})\oplus\mathcal{G}_{\mathbf{H}}(\mathbf{h})$.

For the proof of this theorem, see [1].

Remark 1.1. With Theorem 1, we can find a $\mathcal{P}$-position by calculating the Grundy numbers and a $\mathcal{P}$-position of the sum of two games by calculating the Grundy numbers of two games. Therefore, Grundy numbers are an important research topic in combinatorial game theory.

2. Maximum Nim

In this section, we study maximum Nim, which is a game of restricted Nim.

Definition 2.1. If the sequence $f(m)$ for $m \in Z_{\geq 0}$ satisfies $0 \leq f(m)-f(m-1) \leq 1$ for any natural number m, it is called a *regular sequence.*

Definition 2.2. Let $f(m)$ be a regular sequence. Suppose there is a pile of n stones, and two players take turns removing stones from the pile. In each turn, the player is allowed to remove at least one stone and at most $f(m)$ stones, where m represents the number of stones. The player who removes the last stone is the winner. We refer to f as a *rule sequence.*

Lemma 1. *Let $\mathcal{G}$ represent the Grundy number of maximum Nim with the rule sequence $f(x)$. Then, we have the following:*
(i) If $f(x)=f(x-1)$, $\mathcal{G}(x)=\mathcal{G}(x-f(x)-1)$.
(ii) If $f(x)>f(x-1)$, $\mathcal{G}(x)=f(x)$.

Proof. Properties (i) and (ii) are proven in Lemma 2.1 of [2]. □

2.1. Maximum Nim Whose Rule Sequence Is $f(x)=\lceil\frac{x}{2}\rceil$

In this section, we let $f(x)=\lceil\frac{x}{2}\rceil$. Because $0 \leq f(m)-f(m-1) \leq 1$ for any $m \in N$, $f(m)$ for $m \in Z_{\geq 0}$ is a regular sequence. Here, we examine maximum Nim of Definition 2.2 for $f(x)$.

Another option is to use $f(x)=\lfloor\frac{x}{2}\rfloor$; however, this case produces similar results because $\lfloor\frac{x+1}{2}\rfloor=\lceil\frac{x}{2}\rceil$ for any $n \in Z_{\geq 0}$. Therefore, the case of $f(x)=\lfloor\frac{x}{2}\rfloor$ is omitted in this study.

Definition 2.3. We denote the pile of m stones as (m), which we call the position of the game.

Definition 2.4. The set of all the positions that can be reached from position (t) is defined as *move*(t). For any $t \in Z_{\geq 0}$, we have

$$move(t) = \{(t-v) : v \leq \left\lceil \frac{t}{2} \right\rceil \text{ and } v \in N\}.$$

Lemma 2. *Let $\mathcal{G}$ represent the Grundy number of the maximum Nim with the rule sequence $f(x) = \lceil \frac{x}{2} \rceil$. Then, we have the following:*
(i) If t is even and $t \geq 2$, $\mathcal{G}(t) = \mathcal{G}(\frac{t-2}{2})$.
(ii) If t is odd, $\mathcal{G}(t) = \frac{t+1}{2}$.
(iii) $\mathcal{G}(t) = 1$ if and only if $t = 2^{n-1}3 - 2$ for $n \in N$.
(iv) $\mathcal{G}(t) = 2$ if and only if $t = 2^{n-1}5 - 2$ for $n \in N$.
(v) If $\mathcal{G}(t) = 1$ for $t > 1$, there exists $u \in move(t)$ such that $\mathcal{G}(u) = 2$.

Proof. (i) If t is even, $\lceil \frac{t}{2} \rceil = \lceil \frac{t-1}{2} \rceil$. Therefore, according to (i) in Lemma 1, $\mathcal{G}(t) = \mathcal{G}(t - \lceil \frac{t}{2} \rceil - 1) = \mathcal{G}(\frac{t-2}{2})$.
(ii) If t is odd, $\lceil \frac{t}{2} \rceil > \lceil \frac{t-1}{2} \rceil$. Therefore, according to (ii) of Lemma 1, we obtain $\mathcal{G}(t) = \lceil \frac{t}{2} \rceil = \frac{t+1}{2}$.
Next, we prove (iii) and (iv). By (i) and (ii), $\mathcal{G}(1) = \frac{1+1}{2} = 1$, $\mathcal{G}(2) = \mathcal{G}(\frac{2-2}{2})$ $= \mathcal{G}(0) = 0$, and $\mathcal{G}(3) = \frac{3+1}{2} = 2$. By (i),

$$\mathcal{G}(2^{n-1}3 - 2) = \mathcal{G}\left(\frac{2^{n-1}3 - 2 - 2}{2}\right) = \mathcal{G}(2^{n-2}3 - 2) = \cdots = \mathcal{G}(1) = 1. \quad (1)$$

By (i),

$$\mathcal{G}(2^{n-1}5 - 2) = \mathcal{G}\left(\frac{2^{n-1}5 - 2 - 2}{2}\right) = \mathcal{G}(2^{n-2}5 - 2) = \cdots = \mathcal{G}(3) = 2. \quad (2)$$

Because *move*$(2^{n-1}3 - 2) = \{2^{n-2}3 - 1, \cdots, 2^{n-1}3 - 3\}$, we have $\mathcal{G}(x) \neq 1$ for $x \in \{2^{n-2}3 - 1, \cdots, 2^{n-1}3 - 3\}$. Therefore, by Equation (1), we have (iii). Because *move*$(2^{n-1}5 - 2) = \{2^{n-2}5 - 1, \cdots, 2^{n-1} - 3\}$, from the definition of the Grundy number, we have $\mathcal{G}(x) \neq 2$ for $x \in \{2^{n-2}5 - 1, \cdots, 2^{n-1} - 3\}$. Therefore, by Equation (2), we have (iv).
(v) Suppose that $\mathcal{G}(t) = 1$ for $t > 1$. Then, by (iii), $t = 2^{n-1}3 - 2$ with $n \geq 2$. Then $u = 2^{n-2}5 - 2 \in \{2^{n-2}3 - 1, \cdots, 2^{n-1}3 - 3\} = move(t)$. Using (iv), we prove (v). □

2.2. Three-Pile Maximum Nim

Definition 2.5. Suppose that there are three piles of stones and two players take turns removing stones from the piles. In each turn, the player chooses a pile and removes at least one stone and at most $f(x) = \lceil \frac{x}{2} \rceil$ stones, where x represents the number of stones. The player who removes the last stone is the winner. The position of the game is represented by three coordinates (s, t, u), where s, t, and u represent the numbers of stones in the first, second, and third piles, respectively.

According to the results presented in Section 2.1 and Theorem 1, we can calculate the Grundy numbers of the game in Definition 2.5.

Theorem 2. *Let $\mathcal{G}(t)$ be the Grundy number of the game in Subsection 2.1. Then, the Grundy number $\mathcal{G}(s, t, u)$ of the game of Definition 2.5 satisfies the following equation: $\mathcal{G}(s, t, u) = \mathcal{G}(s) \oplus \mathcal{G}(t) \oplus \mathcal{G}(u)$.*

Proof. This is directly from Theorem 1. □

3. Maximum Nim with a Pass

In Subsections 3.1 and 3.2, we modify the standard rules of the games to allow for a one-time pass, that is, a pass move that may be used at most once in the game and not from a terminal position. Once a pass has been used by either player, it is no longer available.

3.1. Maximum Nim with a Pass Whose Rule Sequence Is $f(x) = \lceil \frac{x}{2} \rceil$

The position of this game is represented by two coordinates (t, p), where t represents the number of stones in the pile. We define $p = 1$ if the pass is still available; otherwise, $p = 0$.

We define *move* in this game.

Definition 3.1. For any $t \in Z_{\geq 0}$, we define $move(t, p)$ as follows.
(i) If $p = 1$ and $t > 0$,

$$move(t, p) = \{(t - v, p) : v \leq \left\lceil \frac{t}{2} \right\rceil \text{ and } v \in N\} \cup \{(t, 0)\}.$$

(ii) If $p = 0$ or $t = 0$,

$$move(t, p) = \{(t - v, p) : u \leq \left\lceil \frac{t}{2} \right\rceil \text{ and } v \in N\}.$$

Remark 3.1. Note that a pass is unavailable from position $(t, 1)$ with $t = 0$, which is the terminal position. Apparently, $\mathcal{G}(t, 0)$ is identical to $\mathcal{G}(t)$ in Section 2.1.

According to Definitions 1.3 and 3.1, we define the Grundy number $\mathcal{G}(t, p)$ of the position (t, p).

p \ t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	0	1	0	2	1	3	0	4	2	5	1	6	3	7	0	8	4	9	2	10	5	11	1	12	6	13
1	0	2	1	0	2	4	1	3	0	6	2	5	4	8	1	7	3	10	0	9	6	12	2	11	5	14

Figure 1: Table of Grundy numbers $\mathcal{G}(t,p)$

Lemma 3. *Let S be a function such that $S(0)=1$, $S(1)=2$, $S(2)=0$, $S(2x)=2x-1$ for $x \in N$ and $x>1$, and $S(2x-1)=2x$ for $x \in N$. Then, we have the following:*
(i) We have $S(\{x : x \in Z_{\geq 0}$ and $x \leq n\}) = \{n+1\} \cup \{x : x \in Z_{\geq 0}$ and $x \leq n-1\}$ when n is odd with $n \geq 3$.
(ii) We have $S(\{x : x \in Z_{\geq 0}$ and $x \leq n\}) = \{x : x \in Z_{\geq 0}$ and $x \leq n\}$ when n is even and $n \geq 2$.

Proof. For $m \in N$ such that $m \geq 2$, we have

$$\begin{aligned}
S(\{x :& x \in Z_{\geq 0} \text{ and } x \leq 2m-1\})\\
&= S(\{0,1,2,3,4,\cdots,2m-5,2m-4,2m-3,2m-2,2m-1\})\\
&= \{1,2,0,4,3,\cdots,2m-4,2m-2,2m-3,2m\}\\
&= \{2m\} \cup \{x : x \in Z_{\geq 0} \text{ and } x \leq 2m-2\}.
\end{aligned}$$

Similarly, for $m \in N$,

$$\begin{aligned}
S(\{x :& x \in Z_{\geq 0} \text{ and } x \leq 2m\})\\
&= S(\{0,1,2,3,4,\cdots,2m-5,2m-4,2m-3,2m-2,2m-1\})\\
&= \{1,2,0,4,3,\cdots,2m-4,2m-5,2m-2,2m-3,2m,2m-1\}\\
&= \{x : x \in Z_{\geq 0} \text{ and } x \leq 2m\}.
\end{aligned}$$

□

Theorem 3. *Let $\mathcal{G}(s,p)$ be the Grundy number of position (s,p). Then, we obtain the following:*
(i) $\mathcal{G}(0,0)=0$ and $\mathcal{G}(0,1)=0$.
(ii) For $u \in N$, if $\mathcal{G}(u,0)=0$, then $\mathcal{G}(u,1)=1$.
(iii) For $u \in N$, if $\mathcal{G}(u,0)=2$, then $\mathcal{G}(u,1)=0$.
(iv) For $u,m \in N$ such that $m>1$, if $\mathcal{G}(u,0)=2m$, then $\mathcal{G}(u,1)=2m-1$.
(v) For $u,m \in N$, if $\mathcal{G}(u,0)=2m-1$, then $\mathcal{G}(u,1)=2m$.

Proof. (i) Because $(0,0)$ is the terminal position, $\mathcal{G}(0,0)=0$. We cannot move to any position or use a pass-move from position $(0,1)$. Hence, $(0,1)$ is also a terminal position. Therefore, $\mathcal{G}(0,1)=0$.

Next, we prove (ii), (iii), (iv), and (v) using mathematical induction. From (ii) of Lemma 2,

$$\mathcal{G}(1,0)=\mathcal{G}(1)=\frac{1+1}{2}=1. \tag{3}$$

As $move(1,1) = \{(0,1),(1,0)\}$, $\mathcal{G}(0,1) = 0$, and $\mathcal{G}(1,0) = 1$, by Equation (3) we have

$$\mathcal{G}(1,1) = mex(\{0,1\}) = 2. \tag{4}$$

Let $t \in N$. From Equations (3) and (4), we have only to prove the case such that

$$t \geq 2. \tag{5}$$

We suppose that (ii), (iii), (iv), and (v) are valid for $k \in Z_{\geq 0}$ such that $k < t$. From the inequality in (5), we have

$$(0,0) \notin move(t,0) = \{(t-1,0),\cdots,(t-\left\lceil\frac{t}{2}\right\rceil,0)\};$$

hence, we have

$$k \in N \tag{6}$$

when $(k,0) \in move(t,0)$.
(ii) Suppose that

$$\mathcal{G}(t,0) = 0 \tag{7}$$

and

$$\mathcal{G}(t,0) = mex(\{\mathcal{G}(k,0) : (k,0) \in move(t,0)\}).$$

Hence, according to the definition of the Grundy number in Definition 1.3,

$$\begin{aligned} 0 \notin &\{\mathcal{G}(k,0) : (k,0) \in move(t,0)\} \\ &= \{\mathcal{G}(t-1,0),\cdots,\mathcal{G}(t-\left\lceil\frac{t}{2}\right\rceil,0)\}. \end{aligned} \tag{8}$$

Suppose that $\mathcal{G}(x,1) = 1$ for some x such that $t - \lceil\frac{t}{2}\rceil \leq x \leq t-1$. Then, from Relation (6) and applying the mathematical induction hypothesis to (ii), (iii), (iv), and (v), we have $\mathcal{G}(x,0) = 0$, which contradicts Relation (8). Therefore we have

$$1 \notin \{\mathcal{G}(t-1,1),\cdots,\mathcal{G}(t-\left\lceil\frac{t}{2}\right\rceil,1)\}. \tag{9}$$

By the definition of the Grundy number,

$$\begin{aligned} \mathcal{G}(t,1) &= mex(\{\mathcal{G}(k,m) : (k,m) \in move(t,1)\}) \\ &= mex(\{\mathcal{G}(k,1) : (k,1) \in move(t,1)\} \cup \{\mathcal{G}(t,0)\}) \\ &= mex(\{\mathcal{G}(t-1,1),\cdots,\mathcal{G}(t-\left\lceil\frac{t}{2}\right\rceil,1)\} \cup \{\mathcal{G}(t,0)\}). \end{aligned}$$

Hence, from Equation (7), Relation (9), and the definition of the Grundy number in Definition 1.3, we have $\mathcal{G}(t,1) = 1$.
(iii) Suppose that

$$\mathcal{G}(t,0) = 2. \tag{10}$$

Then,

$$\begin{aligned} 2 &\notin \{\mathcal{G}(k,0) : (k,0) \in move(t,0)\} \\ &= \{\mathcal{G}(t-1,0), \cdots, \mathcal{G}(t - \left\lceil \frac{t}{2} \right\rceil, 0)\}. \end{aligned} \tag{11}$$

Next, we use a method similar to the one that yielded Relation (9) from Relation (8). According to Relation (6), and applying the mathematical induction hypothesis to (ii), (iii), (iv), and (v), we have

$$0 \notin \{\mathcal{G}(t-1,1), \cdots, \mathcal{G}(t - \left\lceil \frac{t}{2} \right\rceil, 1)\}. \tag{12}$$

For the remainder of this paper, we use the method of proof that has been demonstrated to yield Relation (9) from Relation (8), and Relation (12) from Relation (11), without describing the method. By the definition of the Grundy number,

$$\begin{aligned} \mathcal{G}(t,1) &= mex(\{\mathcal{G}(k,m) : (k,m) \in move(t,1)\}) \\ &= mex(\{\mathcal{G}(t-1,1), \cdots, \mathcal{G}(t - \left\lceil \frac{t}{2} \right\rceil, 1)\} \cup \{\mathcal{G}(t,0)\}). \end{aligned}$$

Hence, from Equation (10) and Relation (12), we have $\mathcal{G}(t,1) = 0$.
(iv) Suppose that

$$\mathcal{G}(t,0) = 2m \tag{13}$$

for a natural number m such that $m > 1$. We will prove that $\mathcal{G}(t,1) = 2m - 1$. Because

$$2m = \mathcal{G}(t,0) = mex(\{\mathcal{G}(k,0) : (k,0) \in move(t,0)\}),$$

we have

$$\{\mathcal{G}(k,0) : (k,0) \in move(t,0)\} \supset \{x : x \in Z_{\geq 0} \text{ and } x \leq 2m-1\} \tag{14}$$

and

$$2m \notin \{\mathcal{G}(k,0) : (k,0) \in move(t,0)\}. \tag{15}$$

From Lemma 3, the Relations (6), (14), and (15), and applying the mathematical induction hypothesis to (ii),(iii), (iv), and (v), we have

$$\{\mathcal{G}(k,1) : (k,1) \in move(t,1)\} \supset \{2m\} \cup \{x : x \in Z_{\geq 0} \text{ and } x \leq 2m-2\}$$

and

$$2m - 1 \notin \{\mathcal{G}(k,m) : (k,1) \in move(t,1)\}.$$

By the definition of the Grundy number,

$$\mathcal{G}(t,1) = mex(\{\mathcal{G}(k,1) : (k,1) \in move(t,1)\} \cup \{\mathcal{G}(t,0)\});$$

hence, according to Equation (13), $\mathcal{G}(t,1) = 2m-1$.
(v) Suppose that

$$\mathcal{G}(t,0) = 2m-1 \tag{16}$$

for a natural number m. Then, we have the following cases (v.1) and (v.2).
(v.1) Suppose that $m \geq 2$. Because

$$2m-1 = \mathcal{G}(t,0) = \textit{mex}(\{\mathcal{G}(k,0) : (k,0) \in \textit{move}(t,0)\}),$$

we have

$$\{\mathcal{G}(k,0) : (k,0) \in \textit{move}(t,0)\} \supset \{x : x \in Z_{\geq 0} \text{ and } x \leq 2m-2\} \tag{17}$$

and

$$2m-1 \notin \{\mathcal{G}(k,0) : (k,0) \in \textit{move}(t,0)\}. \tag{18}$$

From Lemma 3, Relations (6), (17), and (18), and by applying the mathematical induction hypothesis to (ii), (iii), (iv), and (v),

$$\{\mathcal{G}(k,1) : (k,1) \in \textit{move}(t,1)\} \supset \{x : x \in Z_{\geq 0} \text{ and } x \leq 2m-2\} \tag{19}$$

and

$$2m \notin \{\mathcal{G}(k,1) : (k,1) \in \textit{move}(t,1)\}. \tag{20}$$

Given that

$$\mathcal{G}(t,1) = \textit{mex}(\{\mathcal{G}(k,1) : (k,1) \in \textit{move}(t,1)\} \cup \{\mathcal{G}(t,0)\}),$$

according to Equation (16), and Relations (19) and (20), we have $\mathcal{G}(t,1) = 2m$.
(v.2) Suppose that $m = 1$. Because

$$1 = 2m-1 = \mathcal{G}(t,0) = \textit{mex}(\{\mathcal{G}(k,0) : (k,0) \in \textit{move}(t,0)\}),$$

we have

$$1 \notin \{\mathcal{G}(k,0) : (k,0) \in \textit{move}(t,0)\}.$$

By applying the mathematical induction hypothesis to (v),

$$2 \notin \{\mathcal{G}(k,1) : (k,1) \in \textit{move}(t,1)\}. \tag{21}$$

By (v) of Lemma 2,

$$2 \in \{\mathcal{G}(k,0) : (k,0) \in \textit{move}(t,0)\}.$$

By applying the mathematical induction hypothesis to (iii),

$$0 \in \{\mathcal{G}(k,1) : (k,1) \in \textit{move}(t,1)\}. \tag{22}$$

Because

$$\mathcal{G}(t,1) = \textit{mex}(\{\mathcal{G}(k,1) : (k,1) \in \textit{move}(t,1)\} \cup \{\mathcal{G}(t,0)\}),$$

according to Equation (16), Relations (21) and (22), we have $\mathcal{G}(t,1) = 2$. □

3.2. Three-Pile Maximum Nim with a Pass

Here, we study maximum Nim with three piles based on Definition 2.5 by modifying the standard rules of the games to allow a one-time pass. We consider only three-pile games, although generalization to the case of an arbitrary natural number of piles is straightforward.

We denote the position of the game with four coordinates (s, t, u, p), where s, t, and u represent the numbers of stones in the first, second, and third piles, respectively. Here, we define $p = 1$ if the pass is still available, and $p = 0$ otherwise.

We define a *move* in this game as follows.

Definition 3.2. For any $s, t, u \in Z_{\geq 0}$, we have (i) and (ii).
(i) If $p = 1$ and $s + t + u > 0$,

$$\begin{aligned} move(s,t,u,p) = \{(s-v,t,u,p) : v \leq \left\lceil \frac{s}{2} \right\rceil \text{ and } v \in N\} \\ \cup \{(s,t-v,u,p) : v \leq \left\lceil \frac{t}{2} \right\rceil \text{ and } v \in N\} \\ \cup \{(s,t,u-v,p) : v \leq \left\lceil \frac{u}{2} \right\rceil \text{ and } v \in N\} \cup \{(s,t,u,0)\}. \end{aligned}$$

(ii) If $p = 0$ or $s + t + u = 0$,

$$\begin{aligned} move(s,t,u,p) = \{(s-v,t,u,p) : v \leq \left\lceil \frac{s}{2} \right\rceil \text{ and } v \in N\} \\ \cup \{(s,t-v,u,p) : v \leq \left\lceil \frac{t}{2} \right\rceil \text{ and } v \in N\} \\ \cup \{(s,t,u-v,p) : v \leq \left\lceil \frac{u}{2} \right\rceil \text{ and } v \in N\}. \end{aligned}$$

According to Definitions 1.3 and 3.2, we define the Grundy number $\mathcal{G}(s, t, u, p)$ of the position (s, t, u, p). Because s, t, and u represent the numbers of stones in the first, second, and third piles, respectively, the value of $\mathcal{G}(s, t, u, p)$ does not depend on the order of the arguments s, t, u.

Remark 3.2. Note that a pass is unavailable from the position $(s, t, u, 1)$ with $s+t+u = 0$, which is the terminal position. It is clear that $\mathcal{G}(s, 0, 0, p)$, $\mathcal{G}(0, s, 0, p)$, and $\mathcal{G}(0, 0, s, p)$ are identical to $\mathcal{G}(s, p)$ in Section 3.1.

Lemma 4. *Let $\mathcal{G}(s, t, u, p)$ be the Grundy number of position (s, t, u, p). Then, we obtain the following:*
(i) $\mathcal{G}(1, 0, 0, 0) = \mathcal{G}(0, 1, 0, 0) = \mathcal{G}(0, 0, 1, 0) = 1$.
(ii) $\mathcal{G}(1, 1, 0, 0) = \mathcal{G}(0, 1, 1, 0) = \mathcal{G}(1, 0, 1, 0) = 0$.
(iii) $\mathcal{G}(1, 1, 1, 0) = 1$.
(iv) $\mathcal{G}(1, 0, 0, 1) = \mathcal{G}(0, 1, 0, 1) = \mathcal{G}(0, 0, 1, 1) = 2$.
(v) $\mathcal{G}(1, 1, 0, 1) = \mathcal{G}(0, 1, 1, 1) = \mathcal{G}(1, 0, 1, 1) = 1$.
(vi) $\mathcal{G}(1, 1, 1, 1) = 0$.

Proof. (i) From Lemma 2,

$$\mathcal{G}(1,0,0,0) = \mathcal{G}(0,1,0,0) = \mathcal{G}(0,0,1,0) = \mathcal{G}(1,0) = \mathcal{G}(1) = 1.$$

(ii) From (i) and Theorem 1, we have

$$\mathcal{G}(1,1,0,0) = \mathcal{G}(0,1,1,0) = \mathcal{G}(1,0,1,0) = \mathcal{G}(1,0,0,0) \oplus \mathcal{G}(0,0,1,0) = 1 \oplus 1 = 0.$$

(iii) From (i) and Theorem 1,

$$\mathcal{G}(1,1,1,0) = \mathcal{G}(1,0,0,0) \oplus \mathcal{G}(0,1,0,0) \oplus \mathcal{G}(0,0,1,0) = 1.$$

(iv)From (i) of Lemma 2 and (v) of Theorem 3,

$$\mathcal{G}(1,0,0,1) = \mathcal{G}(0,1,0,1) = \mathcal{G}(0,0,1,1) = \mathcal{G}(1,1) = \mathcal{G}(1,0) + 1 = 2.$$

(v) From (ii) and (iv),

$$\begin{aligned}\mathcal{G}(1,1,0,1) &= mex(\{\mathcal{G}(h,k,0,1) : (h,k,0,1) \in move(1,1,0,1)\} \cup \{\mathcal{G}(1,1,0,0)\}) \\ &= mex(\{\mathcal{G}(1,0,0,1), \mathcal{G}(0,1,0,1), \mathcal{G}(1,1,0,0)\}) \\ &= mex(\{2,2,0\}) = 1.\end{aligned}$$

(vi) From (iii) and (v),

$$\begin{aligned}\mathcal{G}(1,1,1,1) &= mex(\{\mathcal{G}(h,k,j,1) : (h,k,j,1) \in move(1,1,1,1)\} \cup \{\mathcal{G}(1,1,1,0)\}) \\ &= mex(\{\mathcal{G}(1,1,0,1), \mathcal{G}(1,0,1,1), \mathcal{G}(0,1,1,1), \mathcal{G}(1,1,1,0)\}) \\ &= mex(\{1,1,1,1\}) = 0.\end{aligned}$$

□

Lemma 5. *Let T be a function such that $T(2x) = 2x+1$ and $T(2x+1) = 2x$ for $x, y \in Z_{\geq 0}$. Then, we have the following:*
(i) $T(\{x : x \in Z_{\geq 0}$ and $x \leq n\}) = \{x : x \in Z_{\geq 0}$ and $x \leq n\}$ when n is odd;
(ii) $T(\{x : x \in Z_{\geq 0}$ and $x \leq n\}) = \{n+1\} \cup \{x : x \in Z_{\geq 0}$ and $x \leq n-1\}$ when n is even.

Proof. For $m \in Z_{\geq 0}$, we have

$$\begin{aligned}&T(\{x : x \in Z_{\geq 0} \text{ and } x \leq 2m+1\}) \\ &\quad = T(\{0,1,2,3,4,\cdots,2m-2,2m-1,2m,2m+1\}) \\ &\quad = \{1,0,3,2,5,\cdots,2m-1,2m-2,2m+1,2m\} \\ &\quad = \{x : x \in Z_{\geq 0} \text{ and } x \leq 2m+1\}.\end{aligned}$$

Similarly,

$$\begin{aligned}&T(\{x : x \in Z_{\geq 0} \text{ and } x \leq 2m\}) \\ &\quad = T(\{0,1,2,3,\cdots,2m-4,2m-3,2m-2,2m-1,2m\}) \\ &\quad = \{1,0,3,2,\cdots,2m-3,2m-4,2m-1,2m-2,2m+1\} \\ &\quad = \{2m+1\} \cup \{x : x \in Z_{\geq 0} \text{ and } x \leq 2m-1\}.\end{aligned}$$

□

Theorem 4. *We have the following for the Grundy numbers:*
(i) *For* $s \in Z_{\geq 0}$, $\mathcal{G}(s,0,0,1) = \mathcal{G}(0,s,0,1) = \mathcal{G}(0,0,s,1) = \mathcal{G}(s,1)$.
(ii) *We suppose that* $s,t > 0$, $t,u > 0$, *or* $u,s > 0$. *Thus, we have the following:*
$(ii.1)$ *For any* $m \in Z_{\geq 0}$, *if* $\mathcal{G}(s,t,u,0) = 2m$, *then* $\mathcal{G}(s,t,u,1) = 2m+1$.
$(ii.2)$ *For any* $m \in Z_{\geq 0}$, *if* $\mathcal{G}(s,t,u,0) = 2m+1$, *then* $\mathcal{G}(s,t,u,1) = 2m$.

Proof. (i) Let $(s,t,u,1)$ be the position of the game. The position $(s,0,0,1)$ is identical to the position $(s,1)$ in the game of Subsection 2.1; hence, $\mathcal{G}(s,0,0,1) = \mathcal{G}(s,1)$. Similarly, $\mathcal{G}(0,s,0,1) = \mathcal{G}(0,0,s,1) = \mathcal{G}(s,1)$, and thus we have proved (i).
(ii) We prove (ii) by using mathematical induction. From Lemma 4, $(ii.1)$ and $(ii.2)$ are valid for $s,t,u \in Z_{\geq 0}$ such that $s,t,u \leq 1$. Suppose that $(ii.1)$ and $(ii.2)$ hold for (h,k,j,p) when $h \leq s, k \leq t$, $j \leq u$, $h+k+j < s+t+u$, and $p = 0,1$.
We suppose that

$$\mathcal{G}(s,t,u,0) = n \tag{23}$$

for $n \in Z_{\geq 0}$.
(ii.1) Suppose that

$$move(s,t,u,0) \cap \{(s,0,0,0),(0,t,0,0),(0,0,u,0)\} = \emptyset. \tag{24}$$

According to Equation (23) and the definition of the Grundy number in Definition 1.3,

$$\{\mathcal{G}(h,k,j,0) : (h,k,j,0) \in move(s,t,u,0)\} \supset \{x : x \in Z_{\geq 0} \text{ and } x \leq n-1\} \tag{25}$$

and

$$n \notin \{\mathcal{G}(h,k,j,0) : (h,k,j,0) \in move(s,t,u,0)\}. \tag{26}$$

By Lemma 5 and applying the mathematical induction hypothesis to $(ii.1)$ and $(ii.2)$, along with Relations (25) and (24), we obtain

$$\{\mathcal{G}(h,k,j,1) : (h,k,j,1) \in move(s,t,u,1)\} \supset \{x : x \in Z_{\geq 0} \text{ and } x \leq n-1\} \tag{27}$$

when n is even, or

$$\{\mathcal{G}(h,k,j,1) : (h,k,j,1) \in move(s,t,u,1)\} \supset \{n\} \cup \{x : x \in Z_{\geq 0} \text{ and } x \leq n-2\} \tag{28}$$

when n is odd. By Relations (24) and (26), we have

$$n+1 \notin \{\mathcal{G}(h,k,j,1) : (h,k,j,1) \in move(s,t,u,1)\} \tag{29}$$

when n is even, and when n is odd, we have

$$n-1 \notin \{\mathcal{G}(h,k,j,1) : (h,k,j,1) \in move(s,t,u,1)\}. \tag{30}$$

As

$$\mathcal{G}(s,t,u,1) = mex(\{\mathcal{G}(h,k,j,1) : (h,k,j,1) \in move(s,t,u,1)\} \cup \{\mathcal{G}(s,t,u,0)\}),$$

according to Equation (23) and Relations (27), (28), (29), and (30), we have

$$\mathcal{G}(s,t,u,1) = n+1$$

when n is even, or

$$\mathcal{G}(s,t,u,1) = n-1$$

when n is odd. Therefore, we obtain $(ii.1)$ and $(ii.2)$.
(ii.2) Suppose that

$$move(s,t,u,0) \cap \{(s,0,0,0),(0,t,0,0),(0,0,u,0)\} \neq \emptyset.$$

Then, we have $(t,u) = (1,0)$, $(t,u) = (0,1)$, $(s,u) = (1,0)$, $(s,u) = (0,1)$, $(s,t) = (1,0)$, or $(s,t) = (0,1)$. Without loss of generality, we may assume that

$$(t,u) = (1,0). \tag{31}$$

Then by Equations (31) and (23), we assume the following:

$$\mathcal{G}(s,1,0,0) = n. \tag{32}$$

If $s = 1$, according to Lemma 4, we have $\mathcal{G}(1,1,0,0) = 0$ and $\mathcal{G}(1,1,0,1) = 1$ $= \mathcal{G}(1,1,0,0) + 1$. Then, we have proved $(ii.1)$.

Next, we assume that $s > 1$. Then, we have

$$move(s,1,0,0) = \{\{s-v,1,0,0\} : v \in N \text{ and } v \leq \left\lceil \frac{s}{2} \right\rceil\} \cup \{\{s,0,0,0\}\}, \tag{33}$$

where

$$s - v \geq s - \left\lceil \frac{s}{2} \right\rceil > 0 \tag{34}$$

for v such that $v \leq \lceil \frac{s}{2} \rceil$.

According to the definition of the Grundy number in Definition 1.3 we have

$$\begin{aligned} &\{\mathcal{G}(h,k,0,0) : (h,k,0,0) \in move(s,1,0,0)\} \\ &\quad = \{\mathcal{G}(h,1,0,0) : (h,1,0,0) \in move(s,1,0,0)\} \cup \{\mathcal{G}(s,0,0,0)\} \\ &\quad\quad \supset \{x : x \in Z_{\geq 0} \text{ and } 0 \leq x \leq n-1\} \end{aligned} \tag{35}$$

and

$$n \notin \{\mathcal{G}(h,1,0,0) : (h,1,0,0) \in move(s,1,0,0)\} \cup \{(s,0,0,0)\}. \tag{36}$$

By Lemma 4,

$$\mathcal{G}(s,1,0,0) = \mathcal{G}(s,0,0,0) \oplus \mathcal{G}(0,1,0,0) = \mathcal{G}(s,0,0,0) \oplus 1.$$

Hence, according to Equation (32), we have

$$\mathcal{G}(s,0,0,0) = n \oplus 1. \tag{37}$$

(ii.2.1) Suppose that n is even. By Equation (37),

$$\mathcal{G}(s,0,0,0) = n+1. \tag{38}$$

Then, by Relation (35) and Equation (38),

$$\{\mathcal{G}(h,1,0,0) : (h,1,0,0) \in move(s,1,0,0)\} \supset \{x : x \in Z_{\geq 0} \text{ and } x \leq n-1\}. \tag{39}$$

By applying the mathematical induction hypothesis to $(ii.1)$ and $(ii.2)$, along with Equation (33), Inequality (34), Relation (39), and Lemma 5, we obtain

$$\{\mathcal{G}(h,1,0,1) : (h,1,0,1) \in move(s,1,0,1)\} \supset \{x : x \in Z_{\geq 0} \text{ and } x \leq n-1\}. \tag{40}$$

By Relation (36), Equation (33), Inequality (34), and applying the mathematical induction hypothesis to $(ii.1)$ and $(ii.2)$, we have

$$n+1 \notin \{\mathcal{G}(h,1,0,1) : (h,1,0,1) \in move(s,1,0,1)\}. \tag{41}$$

By Equation (38) and Theorem 3 (v)

$$\mathcal{G}(s,0,0,1) = \mathcal{G}(s,1) = \mathcal{G}(s,0)+1 = \mathcal{G}(s,0,0,0)+1 = n+2. \tag{42}$$

By the definition of the Grundy number,

$$\begin{aligned}
&\mathcal{G}(s,1,0,1) = mex(\{\mathcal{G}(h,k,0,1) : (h,k,0,1) \in move(s,1,0,1)\} \cup \{\mathcal{G}(s,1,0,0)\}) \\
&= mex(\{\mathcal{G}(h,1,0,1) : (h,1,0,1) \in move(s,1,0,1)\} \cup \{\mathcal{G}(s,0,0,1)\} \cup \{\mathcal{G}(s,1,0,0)\}),
\end{aligned}$$

and hence by the Relations (40) and (41), and Equations (32) and (42), we have

$$\mathcal{G}(s,1,0,1) = n+1.$$

(ii.2.2) We assume that n is odd. By Equation (37),

$$\mathcal{G}(s,0,0,0) = n-1. \tag{43}$$

By Relation (35) and Equation (43)

$$\{\mathcal{G}(h,1,0,0) : (h,1,0,0) \in move(s,1,0,0)\} \supset \{x : x \in Z_{\geq 0} \text{ and } x \leq n-2),$$

and hence by Equation (33), Inequality (34), Lemma 5, and applying the mathematical induction hypothesis to $(ii.1)$ and $(ii.2)$, we have

$$\{\mathcal{G}(h,1,0,1) : (h,1,0,1) \in move(s,1,0,1)\} \supset \{x : x \in Z_{\geq 0} \text{ and } x \leq n-2\}. \tag{44}$$

From Relation (36), Equation (33), Inequality (34), and applying the mathematical induction hypothesis to $(ii.1)$ and $(ii.2)$, we have

$$n-1 \notin \{\mathcal{G}(h,1,0,1) : (h,1,0,1) \in move(s,1,0,1)\}. \tag{45}$$

Since n is odd, by Equation (43), we have $\mathcal{G}(s,0,0,0) = \mathcal{G}(s,0)$ is even. As $s > 1$, by (ii), (iii), (iv) of Theorem 3, $\mathcal{G}(s,0,0,1) = \mathcal{G}(s,1) = \mathcal{G}(s,0)+1 = n$, $\mathcal{G}(s,0)-2 = n-3$ or $\mathcal{G}(s,0)-1 = n-2$. Therefore we have

$$\mathcal{G}(s,0,0,1) \neq n-1. \tag{46}$$

As

$$\begin{aligned}\mathcal{G}(s,1,0,1) =& mex(\{\mathcal{G}(h,1,0,1) : (h,1,0,1) \in move(s,1,0,1)\} \\ & \cup \{\mathcal{G}(s,0,0,1)\} \cup \{\mathcal{G}(s,1,0,0)\}),\end{aligned}$$

according to Equation (32), Relations (44) and (45), and the inequality in (46), we have $\mathcal{G}(s,t,u,1) = n-1$. □

4. Conclusion and the Prospect of Further Research

In the restricted Nim considered in this study, there is a simple relationship between the Grundy numbers of restricted Nim and the Grundy numbers of restricted Nim with a pass-move, where the number of piles can be any natural number. Therefore, the restriction on the number of stones simplifies the mathematical structure of the game with a pass-move. Is there any way to make n-pile Nim with a pass-move simpler? The authors discovered Conjecture 1 via a computer calculation.

Definition 4.1. Suppose that there are three piles of stones and two players take turns removing stones from the piles. In each turn, the player chooses a pile and removes at least one stone and at most all the stones. The player who removes the last stone is the winner. The position of the game is represented by four coordinates (s,t,u,p), where s, t, and u represent the numbers of stones in the first, second, and third piles, respectively. Here, we define $p = 1$ if the pass-move is still available, and $p = 0$ otherwise. Let m be a fixed natural number, and a pass-move is allowed when the total number of stones $s+t+u \geq m$. Once a pass-move has been used by either player, it is no longer available.

Conjecture 1. Suppose that $m = 4k$ for a natural number k. Then in the game of Definition 4.1,

$$\begin{aligned}&\{(x,y,z,p) : (x,y,z,p) \text{ is a } \mathcal{P}\text{-position}\} \\ &\quad = \{(x,y,z,0) : x \oplus y \oplus z = 0\} \\ &\qquad \cup \{(x,y,z,1) : x+y+z < m \text{ and } x \oplus y \oplus z = 0\} \\ &\qquad \cup \{(x,y,z,1) : x+y+z \geq m \text{ and } x \oplus y \oplus z = 1\}.\end{aligned}$$

It seems that there is not a simple mathematical formula for the set of (s,t,u,p) being a $\mathcal{P}$-position when m is not a multiple of 4 for the game in Definition 4.1.

If $m = 1$, then the game defined in Definition 4.1 is the traditional three-pile Nim with a pass-move.

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