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THE SPECTRUM OF NIM-VALUES FOR ACHIEVEMENT GAMES FOR GENERATING FINITE GROUPS

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Abstract

We study an impartial achievement game introduced by Anderson and Harary. The game is played by two players who alternately select previously unselected elements of a finite group. The game ends when the jointly selected elements generate the group. The last player able to make a move is the winner of the game. We prove that the spectrum of nim-values of these games is $\{0, 1, 2, 3, 4\}$. This positively answers two conjectures from a previous paper by the last two authors.

1. Introduction

Anderson and Harary [2] introduced two impartial games *Generate* and *Do Not Generate* in which two players alternately take turns selecting previously unselected elements of a finite group G. The first player who builds a generating set for the group from the jointly-selected elements wins the *achievement game* $\mathsf{GEN}(G)$. The first player who cannot select an element without building a generating set loses the *avoidance game* $\mathsf{DNG}(G)$. The outcomes of both games were studied for some of the more familiar finite groups, including abelian, dihedral, and symmetric groups in [2, 3].

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A fundamental problem in the theory of impartial combinatorial games [1, 10] is determining the nim-value of a game. The nim-value determines the outcome of the game, and it also allows for the easy calculation of the nim-values of game sums. In [8], Ernst and Sieben used structure digraphs for studying the nim-values of both the achievement and avoidance games, which they applied in the context of certain finite groups including cyclic, abelian, and dihedral. Loosely speaking, a structure digraph is a quotient of the game digraph by an equivalence relation called *structure equivalence*. Structure equivalence respects the nim-values of the positions of the game and drastically simplifies the calculation of the nim-values. The *type* of a structure class is a triple that encodes the nim-values of the positions. Ernst and Sieben [8, Proposition 3.20] determined the spectrum of types for the avoidance game $\mathsf{DNG}(G)$, which in turn allowed them to determine that the spectrum of nim-values for $\mathsf{DNG}(G)$ is $\{0, 1, 3\}$.

The goal of this paper is to determine the spectrum of nim-values for the achievement game $\mathsf{GEN}(G)$. Our approach is very similar to that of the avoidance game, but the required calculations are significantly more difficult for groups of even order. One reason for the increased difficulty is that the game digraph of the avoidance game is a subgraph of the the game digraph of the achievement game, and hence the achievement game has more positions than the avoidance game. As a result, the structure digraphs for achievement games can be more complex. Moreover, the types associated to structure classes no longer suffice since types contain insufficient information to be closed under type calculus. To overcome this apparent shortcoming, we introduce the *extended type* of a structure class, which adds a fourth component to the existing type. To analyze the behavior of the structure digraphs together with the associated extended types, we develop several type restrictions and then rely on computer calculations to handle the large number of cases. We prove that the spectrum of nim-values for the achievement game $\mathsf{GEN}(G)$ is $\{0, 1, 2, 3, 4\}$, which positively answers Conjectures 4.8 and 4.9 from [8].

The structure of the paper is as follows. We start with some preliminaries from [4, 5, 6, 8], and follow with a short characterization of the spectrum of $\mathsf{GEN}(G)$ for G of odd order. The bulk of the work is spent on characterizing the spectrum of $\mathsf{GEN}(G)$ for G of even order.

2. Preliminaries

We now give a more precise description of our game. We also recall some definitions and results from [6, 8]. The positions of $\mathsf{GEN}(G)$ are the possible sets of jointly selected elements. The starting position is the empty set. The options of a nonterminal position P are of the form $P \cup \{g\}$ for some $g \in G \setminus P$. The set of options of P is denoted by $\mathrm{Opt}(P)$.

The *nim-value* of a position P is recursively defined by

$$\mathrm{nim}(P) = \mathrm{mex}\{\mathrm{nim}(Q) \mid Q \in \mathrm{Opt}(P)\},$$

where the *minimum excludant* $\mathrm{mex}(S)$ is the smallest nonnegative integer missing from S. The terminal positions of the game have no options, and so their nim-value is $\mathrm{mex}(\emptyset) = 0$. The winning positions for the player who is about to move (N-positions) are those with nonzero nim-value. The winning strategy always moves the opponent into a position with zero nim-value.

2.1. Type Calculus

The set $\mathcal{M}$ of maximal subgroups of G plays an important role in this game. For a position P we let

$$\lceil P \rceil := \bigcap \{M \in \mathcal{M} \mid P \subseteq M\}.$$

We use the simplified notation $\lceil P, g_1, \ldots, g_n \rceil$ for $\lceil P \cup \{g_1, \ldots, g_n\} \rceil$. If P is a terminal position of the game, then P is a generating set of G, and so $\lceil P \rceil = \bigcap \emptyset = G$. Note that $\lceil \emptyset \rceil = \bigcap \mathcal{M}$ is the Frattini subgroup $\Phi(G)$.

Two positions P and Q are *structure equivalent* if $\lceil P \rceil = \lceil Q \rceil$. Structure equivalence is an equivalence relation. The maximum element of the equivalence class of P is $\lceil P \rceil$, so we denote the *structure class* of P by X_I where $I = \lceil P \rceil$. The set of equivalence classes is denoted by $\mathcal{D}$. The option relationship between positions is compatible with structure equivalence [8, Corollary 4.3], so we say X_J is an option of X_I if $Q \in \mathrm{Opt}(P)$ for some $P \in X_I$ and $Q \in X_J$. The set of options of X_I is denoted by $\mathrm{Opt}(X_I)$. The vertices of the *structure digraph* are the structure classes. The arrows of this digraph connect structure classes to their options.

The parity of an integer n is $\mathrm{pty}(n) := n \bmod 2$ in $\mathbb{Z}_2$. By [8, Proposition 4.4], two positions in a structure class of the same parity have the same nim-value. We can capture this information by defining the *type* of a structure class X_I to be

$$\mathrm{type}(X_I) := (\mathrm{pty}(|I|), \mathrm{nim}(P), \mathrm{nim}(Q)),$$

where $P, Q \in X_I$ with $\mathrm{pty}(|P|) = 0$ and $\mathrm{pty}(|Q|) = 1$. As shown in [8], each structure class contains positions of both odd and even paritities. Note that $\mathrm{type}(X_G) = (\mathrm{pty}(|G|), 0, 0)$ and $\mathrm{type}(X_I)$ is an element of $\mathbb{T} := \{0,1\} \times \mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the collection of nonnegative integers. Additionally, the second component of $\mathrm{type}(X_{\Phi(G)})$ is the nim-value of $\mathsf{GEN}(G)$, since the starting position $\emptyset$ is in $X_{\Phi(G)}$. We say that the *parity of the structure class* X_I is the parity of $|I|$. The sets of even and odd structure classes are denoted by $\mathcal{E}$ and $\mathcal{O}$, respectively. Thus, $\mathcal{D} = \mathcal{E} \dot{\cup} \mathcal{O}$.

Let $\pi_i : \mathbb{N}^n \to \mathbb{N}$ and $\tilde{\pi}_i : \mathbb{N}^n \to \mathbb{N}^{n-1}$ denote projection functions defined by

$$\pi_i(x_1, \ldots, x_n) := x_i,$$
$$\tilde{\pi}_i(x_1, \ldots, x_n) := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

We use the standard image notation $f(A) := \{f(a) \mid a \in A\}$ if A is a subset of the domain of f.

Definition 1. For $T \subseteq \mathbb{T}$, define $E_T := \pi_2(T)$, $O_T := \pi_3(T)$, $e_T := \text{mex}(O_T)$, and $o_T := \text{mex}(E_T)$. We also define

$$\begin{aligned}\text{mex}_0(T) &:= (0, e_T, \text{mex}(E_T \cup \{e_T\}))\\ \text{mex}_1(T) &:= (1, \text{mex}(O_T \cup \{o_T\}), o_T).\end{aligned}$$

We refer to this computation as *type calculus*.

The following consequence of [8, Corollary 4.3, Proposition 4.4] is our main tool to compute nim-values.

Proposition 1. *If $X_I \in \mathcal{D}$, then*

$$\text{type}(X_I) = \begin{cases}\text{mex}_0(\text{type}(\text{Opt}(X_I)), & |I| \textit{ is even}\\ \text{mex}_1(\text{type}(\text{Opt}(X_I)), & |I| \textit{ is odd.}\end{cases}$$

Example 1. Let I have odd order and X_I have options with types $(0, 1, 2)$ and $(1, 4, 3)$. Then $E_T = \{1, 4\}$ and $O_T = \{2, 3\}$. So

$$\text{type}(X_I) = \text{mex}_1(\{(0, 1, 2), (1, 4, 3)\}) = (1, 1, 0)$$

since the odd positions in X_I have nim-value $o_T = \text{mex}(\{1, 4\}) = 0$, while the even positions in X_I have nim-value $\text{mex}(\{2, 3, o_T\}) = 1$.

The *deficiency* of a subset P of G is the minimum size $\delta_G(P)$ of a subset Q of G such that $\langle P \cup Q\rangle = G$. Structure equivalent positions have equal deficiencies [6, Proposition 3.2]. We define

$$\begin{aligned}&\mathcal{D}_k := \{X_I \in \mathcal{D} \mid \delta_G(I) = k\}, && \mathcal{E}_k := \mathcal{E} \cap \mathcal{D}_k, && \mathcal{O}_k := \mathcal{O} \cap \mathcal{D}_k,\\ &\mathcal{D}_{\geq k} := \bigcup\{\mathcal{D}_i \mid i \geq k\}, && \mathcal{E}_{\geq k} := \mathcal{E} \cap \mathcal{D}_{\geq k}, && \mathcal{O}_{\geq k} := \mathcal{O} \cap \mathcal{D}_{\geq k}.\end{aligned}$$

We write $\mathcal{D}_k(G)$ when we want to emphasize the dependence on G. We recursively define

$$\begin{aligned}\mathcal{D}_{k,0} &:= \{X_I \in \mathcal{D}_k \mid \text{Opt}(X_I) \subseteq \mathcal{D}_{k-1}\}\\ \mathcal{D}_{k,l} &:= \{X_I \in \mathcal{D}_k \mid \text{Opt}(X_I) \subseteq \mathcal{D}_{k-1} \cup \mathcal{D}_{k,l-1}\}\end{aligned}$$

for $k, l \geq 1$. It is easy to check that the union of the nested collection $\mathcal{D}_{k,0} \subseteq \mathcal{D}_{k,1} \subseteq \mathcal{D}_{k,2} \subseteq \cdots$ is $\mathcal{D}_k$.

We visualize the structure digraph of $\mathsf{GEN}(G)$ with a structure diagram. In a *structure diagram*, vertices are denoted by triangles or circles. A structure class with even or odd parity is represented by a triangle with a flat bottom or flat top, respectively. A structure class with an unknown or unimportant parity is represented by a circle. We use several arrow types to indicate whether a change in deficiency occurs between a structure class and its option. A summary of these symbols is shown in Figure 1. Note that Proposition 2 justifies that no other arrow types are necessary.

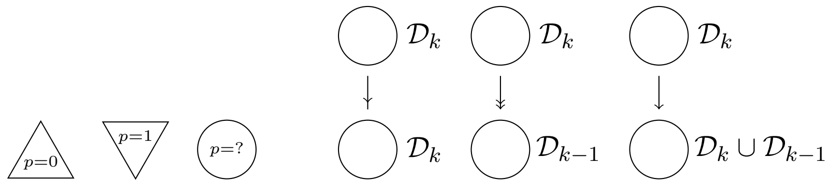


Figure 1: Structure diagram symbols with p denoting the parity of the structure class. The three different arrow types indicate whether the deficiency is unchanged, reduced by 1, or unspecified, respectively.

2.2. Extended Type Calculus

A further complication is that some of our restrictions require information about the even options of X_I, so we need to include this information in our type calculus. This motivates the following definition.

Definition 2. For $X_I \in \mathcal{D}_k$, the *smoothness* of X_I is

$$\operatorname{smo}(X_I) = \begin{cases} 2 & \text{if } \operatorname{pty}(X_I) = 0 \\ 1 & \text{if } \operatorname{pty}(X_I) = 1 \text{ and } \operatorname{Opt}(X_I) \cap \mathcal{E}_k \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

We say that X_I is *smooth* if $\operatorname{smo}(X_I) \geq 1$ and *rough* otherwise.

Note that an even structure class is always smooth, while the smoothness of an odd structure class depends on whether it has an even option with the same deficiency. The smoothness of an even structure class plays no role in our computations. We only define it to make the extended type in the next definition always a quadruple. This simplifies our formulas.

Definition 3. The *extended type* of X_I is $\operatorname{etype}(X_I) := (\operatorname{type}(X_I), \operatorname{smo}(X_I))$.

Note that $\operatorname{etype}(X_I)$ is an element of $\mathbb{E} := \mathbb{T} \times \{0, 1, 2\}$, although we will typically write extended types flattened as a quadruple (p, e, o, s).

In an *extended structure diagram*, we also indicate the smoothness of the structure classes. Smooth odd structure classes are drawn with a double solid boundary while rough odd structure classes are drawn with a single dotted boundary. A summary of these symbols is shown in Figure 2.

Definition 4. For $(A, B) \in \mathcal{P}(\mathbb{E}) \times \mathcal{P}(\mathbb{E})$ we define

$$\begin{aligned} \operatorname{emex}_0(A, B) &:= (\operatorname{mex}_0(\tilde{\pi}_4(A \cup B)), 2), \\ \operatorname{emex}_1(A, B) &:= (\operatorname{mex}_1(\tilde{\pi}_4(A \cup B)), 1 - \min(\pi_1(A))). \end{aligned}$$

We refer to this computation as *extended type calculus.*

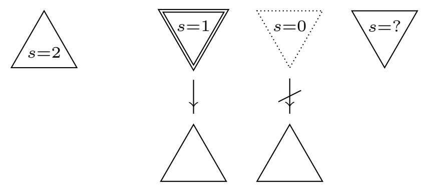


Figure 2: Extended structure diagram symbols for structure classes. For odd structure classes, we use a double solid boundary if X_I is smooth ($s = 1$), a single dotted boundary if X_I is rough ($s = 0$), and single solid boundary if the smoothness is unknown or unimportant.

We think of these two functions as ways of finding the extended type of $X_I \in \mathcal{D}_n$, either real or hypothetical. The first input A consists of the extended types of the options of X_I in $\mathcal{D}_n$, while the second input B consists of the extended types of the options of X_I in $\mathcal{D}_{n-1}$.

Extended type calculus allows us to recursively compute the extended types of every structure class, starting from the terminal structure class.

Example 2. Figure 3 depicts the extended structure diagram for $\mathsf{GEN}(\mathbb{Z}_6)$. The maximal subgroups are $\langle 2\rangle$ and $\langle 3\rangle$. The structure classes are $X_{\langle 1\rangle} \in \mathcal{E}_0$, $X_{\langle 3\rangle} \in \mathcal{E}_1$, and $X_{\langle 2\rangle}, X_{\langle 0\rangle} \in \mathcal{O}_1$. Note that $\mathcal{D}_{1,0} = \{X_{\langle 3\rangle}, X_{\langle 2\rangle}\}$ and $\mathcal{D}_{1,1} = \{X_{\langle 3\rangle}, X_{\langle 2\rangle}, X_{\langle 0\rangle}\}$. Extended type calculus can be used, for example, to compute

$$\begin{aligned}\operatorname{etype}(X_{\langle 0\rangle}) &= \operatorname{emex}_1(\operatorname{etype}(\{X_{\langle 2\rangle}, X_{\langle 3\rangle}\}), \operatorname{etype}(\{X_{\langle 1\rangle}\}))\\ &= \operatorname{emex}_1(\{(1,2,1,0),(0,1,2,2)\},\{(0,0,0,2)\})\\ &= (1,4,3,1).\end{aligned}$$

The structure class $X_{\langle 0\rangle}$ is smooth while $X_{\langle 2\rangle}$ is rough. The nim-value is

$$\begin{aligned}\operatorname{nim}(\mathsf{GEN}(\mathbb{Z}_6)) = \operatorname{nim}(\emptyset) &= \pi_2(\operatorname{etype}(X_{\lceil\emptyset\rceil}))\\ &= \pi_2(\operatorname{etype}(X_{\langle 0\rangle})) = \pi_2(1,4,3,1) = 4.\end{aligned}$$

2.3. Some Known Option-Type Restrictions

The following three results follow from [6, Proposition 3.8], Lagrange's Theorem, and [6, Proposition 3.9], respectively.

Proposition 2. *If $X_I \in \mathcal{D}_k$ for some $k \geq 1$, then $\operatorname{Opt}(X_I) \subseteq \mathcal{D}_{k-1} \cup \mathcal{D}_k$ and $\operatorname{Opt}(X_I) \cap \mathcal{D}_{k-1} \neq \emptyset$.*

The previous statement is depicted in Figure 1. It essentially restricts the possible arrow types between structure classes.

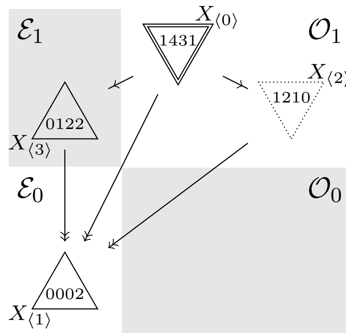


Figure 3: Extended structure diagram for $\mathsf{GEN}(\mathbb{Z}_6)$. The quadruples insides the triangles are the corresponding extended types.

Proposition 3. *If* $X_I \in \mathcal{E}$ *then* $\operatorname{Opt}(X_I) \subseteq \mathcal{E}$.

This means that an even structure class has only even options, as shown in Figure 5(a).

Proposition 4. *If G is a group of even order and X_I has an option, then X_I has an even option.*

The previous statement is depicted in Figure 5(b).

3. Groups of Odd Order

The type of a structure class can be determined relatively easily if G has odd order. The following theorem is an extension of [8, Theorem 4.7] and has a proof that is very similar to the proof of [6, Proposition 3.10]. Note that we implicitly use Proposition 1 in the following proof, as well as throughout the rest of the paper.

Proposition 5. *If G is a group of odd order, then*

$$\operatorname{type}(X_I) = \begin{cases} (1,0,0), & X_I \in \mathcal{O}_0 \\ (1,2,1), & X_I \in \mathcal{O}_1 \\ (1,2,0), & X_I \in \mathcal{O}_2 \\ (1,1,0), & X_I \in \mathcal{O}_{\geq 3}. \end{cases}$$

Proof. We will use structural induction on the structure classes. By Proposition 2 and Lagrange's Theorem, $X_I \in \mathcal{O}_m$ for $m \geq 1$ implies $\operatorname{Opt}(X_I) \subseteq \mathcal{O}_m \cup \mathcal{O}_{m-1}$ and $\mathcal{O}_{m-1} \cap \operatorname{Opt}(X_I) \neq \emptyset$.

If $X_I \in \mathcal{O}_0$, then $\text{type}(X_I) = (1,0,0)$ since $I = G$. If $X_I \in \mathcal{O}_1$, then $\text{type}(X_I) = (1,2,1)$ since

$$\text{type}(\text{Opt}(X_I)) = \begin{cases} \{(1,0,0)\} & \text{if } \text{Opt}(X_I) \subseteq \mathcal{O}_0 \\ \{(1,0,0),(1,2,1)\} & \text{otherwise} \end{cases}$$

by induction. If $X_I \in \mathcal{O}_2$, then $\text{type}(X_I) = (1,2,0)$ since

$$\text{type}(\text{Opt}(X_I)) = \begin{cases} \{(1,2,1)\} & \text{if } \text{Opt}(X_I) \subseteq \mathcal{O}_1 \\ \{(1,2,1),(1,2,0)\} & \text{otherwise} \end{cases}$$

by induction. If $X_I \in \mathcal{O}_3$, then $\text{type}(X_I) = (1,1,0)$ since

$$\text{type}(\text{Opt}(X_I)) = \begin{cases} \{(1,2,0)\} & \text{if } \text{Opt}(X_I) \subseteq \mathcal{O}_2 \\ \{(1,2,0),(1,1,0)\} & \text{otherwise} \end{cases}$$

by induction. If $X_I \in \mathcal{O}_{\geq 4}$, then $\text{type}(X_I) = (1,1,0)$, since every option of X_I has type $(1,1,0)$ by induction. □

4. Groups of Even Order

Our main goal in this section is to compute the possible nim-values of $\mathsf{GEN}(G)$ for a group G of even order. Our approach is similar to that of Proposition 5. We want to recursively build all possible types of structure classes with a given deficiency from the already-computed types with lower deficiency. Unfortunately this simple approach is not sufficient to complete this computation, because it quickly becomes unwieldy for groups of even order as it yields an infinite number of potential types. However, we can use group theory to impose restrictions on the type calculations, which will reduce the number of potential types by eliminating many types that are not possible. We already have three of these restrictions: Propositions 2, 3, and 4. In this section, we develop additional restrictions involving smoothness, which is the reason why we introduced extended types. We then use these restrictions to carry out the computation on extended types using the algorithm in Subsection 4.2.

4.1. Additional Option-Type Restrictions

In this subsection we present two option type restrictions that involve smoothness. A diagrammatic depiction of the statements are shown in Figures 5(c) and 5(d), respectively.

Proposition 6. *Let $X_I, X_J \in \mathcal{O}_n$ such that X_J is an option of X_I. If X_J is smooth, then so is X_I.*

Proof. Suppose that X_J has an option in $\mathcal{E}_n$, as shown in Figure 4(a). Then there is a $g \in G$ such that $X_{\lceil J,g \rceil} \in \mathcal{E}_n$. By Cauchy's Theorem, there is an element t in $\lceil J, g \rceil$ of order 2. Since $X_I \in \mathcal{O}_n$, $t \notin I$. Then $\lceil I, t \rceil$ has even order, so X_I has an option $X_{\lceil I,t \rceil}$ in $\mathcal{E}$. Since $I \leq \lceil I, t \rceil \leq \lceil J, t \rceil \leq \lceil J, g \rceil$ with both X_I and $X_{\lceil J,g \rceil}$ in $\mathcal{D}_n$, we conclude that $X_{\lceil I,t \rceil} \in \mathcal{E}_n$. Thus, X_I has an option $X_{\lceil I,t \rceil}$ in $\mathcal{E}_n$. □

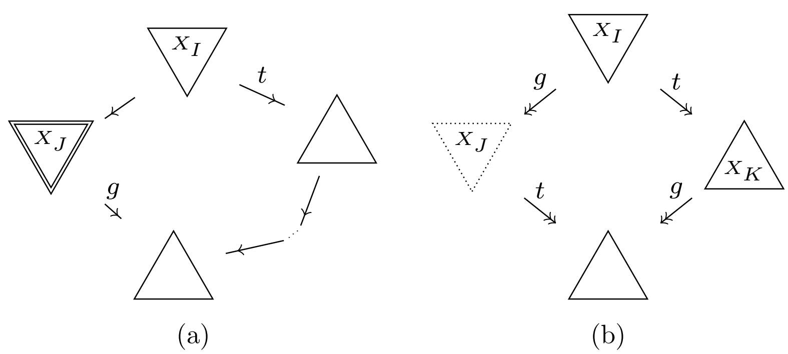


Figure 4: Figures for Propositions 6 and 7.

Proposition 7. *Let G be a group of even order and assume that $X_I \in \mathcal{O}_n$ and $X_J \in \mathcal{O}_{n-1}$ such that X_J is an option of X_I. If X_J is rough, then so is X_I.*

Proof. Assume X_L is an even option of X_I. We will show that X_L is in $\mathcal{E}_{n-1}$. Since L has even order, it contains an element t of even order. Let $K := \lceil I, t \rceil$, as shown in Figure 4(b). Note that $X_K \in \mathcal{E}_n \cup \mathcal{E}_{n-1}$ by Proposition 2 since t has even order and X_K is an option of X_I. By Lagrange's Theorem, $X_{\lceil J,t \rceil} \in \mathcal{E}$. Since X_J is rough, we have $X_{\lceil J,t \rceil} \notin \mathcal{E}_{n-1}$. Hence $X_{\lceil J,t \rceil} \in \mathcal{E}_{n-2}$ by Proposition 2. We have $J = \lceil I, g \rceil$ for some $g \in G$. Since $X_{\lceil K,g \rceil} = X_{\lceil I,g,t \rceil} = X_{\lceil J,t \rceil} \in \mathcal{E}_{n-2}$, we conclude that $X_K \in \mathcal{E}_{n-1}$ by Proposition 2. Since K is a subgroup of L, $X_L \in \mathcal{E}_{n-1}$, as well. □

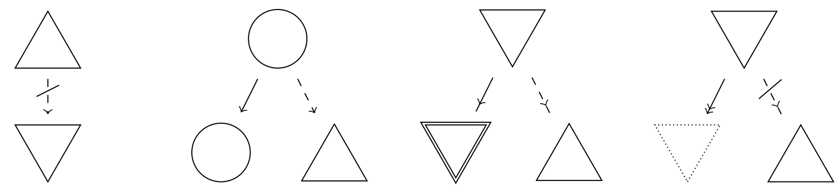
4.2. Spectrum of Extended Types

The next definition introduces the spectrum of extended types for groups of even order.

Definition 5. For $k, l \geq 0$ we let $E_k := \bigcup_l E_{k,l}$, where

$$E_{k,l} := \bigcup \{\text{etype}(\mathcal{D}_{k,l}(G)) \mid G \text{ is a group of even order}\},$$

so that $E_k = \bigcup \{\text{etype}(\mathcal{D}_k(G)) \mid G \text{ is a group of even order}\}$. We also define $E := \bigcup_k E_k$ to be the spectrum of extended types of groups of even order.



(a) Proposition 3 (b) Proposition 4 (c) Proposition 6 (d) Proposition 7

Figure 5: Diagrams for option type restrictions. As with commutative diagrams, solid arrows are assumed to exist, indicating premises, while the dashed arrows are guaranteed to exist, indicating conclusions. A crossed-out dashed arrow is guaranteed not to exist.

Determining E appears to be difficult, so we define the set of feasible extended types. The set of feasible extended types is easier to compute and turns out to be a superset of the spectrum of extended types. Recall that the nim-value of a game occurs as the second component of some extended type. This larger set of feasible extended types does not introduce extraneous nim-values because Example 4 demonstrates that we can find examples of groups with each of the nim-values.

The next definition reformulates Propositions 2 and 3 in the language of extended types.

Definition 6. A pair (A, B) in $\mathcal{P}(\mathbb{E})\times\mathcal{P}(\mathbb{E})$ is 0-*feasible* if $B \neq \emptyset$ and $1 \notin \pi_1(A\cup B)$.

The four criteria in the following definition are reformulations of Propositions 2, 4, 6, and 7, respectively.

Definition 7. A pair (A, B) in $\mathcal{P}(\mathbb{E}) \times \mathcal{P}(\mathbb{E})$ is 1-*feasible* if it satisfies the following conditions:

1. $B \neq \emptyset$;

2. $0 \in \pi_1(A \cup B)$;

3. $1 \in \pi_4(A)$ implies $0 \in \pi_1(A)$;

4. $0 \in \pi_4(B)$ implies $0 \notin \pi_1(A)$.

We are ready to define our approximation to the E_k.

Definition 8. We let $\bar{E}_0 := \{(0, 0, 0, 2)\}$. For $k, l \geq 1$, we recursively define

$$\begin{aligned}
\bar{E}_{k,0} &:= \{\mathrm{emex}_p(\emptyset, B) \mid p \in \mathbb{Z}_2, B \in \mathcal{P}(\bar{E}_{k-1}), (\emptyset, B) \text{ is } p\text{-feasible}\},\\
\bar{E}_{k,l} &:= \{\mathrm{emex}_p(A, B) \mid p \in \mathbb{Z}_2, (A, B) \in \mathcal{P}(\bar{E}_{k,l-1})\times\mathcal{P}(\bar{E}_{k-1}), (A, B) \text{ is } p\text{-feasible}\},\\
\bar{E}_k &:= \bigcup\{\bar{E}_{k,l} \mid l \in \mathbb{N}\}.
\end{aligned}$$

We also define $\bar{E} := \bigcup_k \bar{E}_k$ to be the *feasible spectrum* of extended types of groups of even order.

The reason why we are distinguishing between E_k and $\bar{E}_k$ is that $\bar{E}_k$ may contain extended types that cannot exist for an actual group.

Proposition 8. *For $k \geq 1$, $\bar{E}_{k,0} \subseteq \bar{E}_{k,1} \subseteq \bar{E}_{k,2} \subseteq \cdots \subseteq \bar{E}_k$.*

Proof. We will prove that $\bar{E}_{k,l} \subseteq \bar{E}_{k,l+1}$ by induction on l, and it is clear that $\bar{E}_{k,l} \subseteq \bar{E}_k$ by definition of $\bar{E}_k$.

Let $t \in \bar{E}_{k,0}$. Then $t = \mathrm{emex}_p(\emptyset, B)$ for some $p \in \{0,1\}$ and $B \in \mathcal{P}(\bar{E}_{k-1})$ such that $(\emptyset, B)$ is p-feasible. Since $\emptyset \in \mathcal{P}(\bar{E}_{k,0})$, we have $t = \mathrm{emex}_p(\emptyset, B) \in \bar{E}_{k,1}$.

Now suppose that $\bar{E}_{k,l-1} \subseteq \bar{E}_{k,l}$, and let $r \in \bar{E}_{k,l}$. Then $r = \mathrm{emex}_p(A,B)$ for some $p \in \{0,1\}$, $A \in \mathcal{P}(\bar{E}_{k,l-1})$, and $B \in \mathcal{P}(\bar{E}_{k-1})$ such that (A,B) is p-feasible. Since $\bar{E}_{k,l-1} \subseteq \bar{E}_{k,l}$ by induction, we have $\mathcal{P}(\bar{E}_{k,l-1}) \subseteq \mathcal{P}(\bar{E}_{k,l})$ and so $A \in \mathcal{P}(\bar{E}_{k,l})$. Then $r = \mathrm{emex}_p(A,B) \in \bar{E}_{k,l+1}$, and we conclude that $\bar{E}_{k,l} \subseteq \bar{E}_{k,l+1}$. □

The next result shows that every extended type that actually occurs in a group is a feasible extended type. This is no surprise. Both E_k and $\bar{E}_k$ are recursively computed in the same way with the emex function, although the construction of the extended types that actually occur may have additional restrictions on the input than the construction of the feasible extended types. Thus, the creation of E_k is a possibly more restrictive process, so it must be a subset of $\bar{E}_k$.

Proposition 9. *For all $k \in \mathbb{N}$, $E_k \subseteq \bar{E}_k$.*

Proof. For a contradiction, assume there is a least k such that $E_k \not\subseteq \bar{E}_k$. Since $E_0 = \{(0,0,0,2)\} = \bar{E}_0$, we may assume that $k \geq 1$. Then there must be a least l such that $E_{k,l} \not\subseteq \bar{E}_{k,l}$, and let $t \in E_{k,l} \setminus \bar{E}_{k,l}$. Since $t \in E_k$, there is a finite group G of even order and structure class $X_I \in \mathcal{D}_{k,l}(G)$ such that $t = \mathrm{etype}(X_I)$. Then $t = \mathrm{emex}_p(A,B)$ where $A := \mathrm{etype}(\mathrm{Opt}(X_I) \cap \mathcal{D}_{k,l-1}(G))$, $B := \mathrm{etype}(\mathrm{Opt}(X_I) \cap \mathcal{D}_{k-1})$, and $p := \mathrm{pty}(|I|)$. By Propositions 2, 3, 4, 6, and 7, we have that (A,B) is p-feasible. Additionally,

$$B \subseteq \mathrm{etype}(\mathcal{D}_{k-1}(G)) \subseteq E_{k-1} \subseteq \bar{E}_{k-1},$$

by the choice of k.

If $l = 0$ then $t = \mathrm{emex}_p(A,B) = \mathrm{emex}_p(\emptyset, B) \in \bar{E}_{k,0}$, a contradiction. Thus, we may assume that $l \geq 1$. Then

$$A = \mathrm{etype}(\mathrm{Opt}(X_I) \cap \mathcal{D}_{k,l-1}(G)) \subseteq \mathrm{etype}(\mathcal{D}_{k,l-1}(G)) \subseteq E_{k,l-1} \subseteq \bar{E}_{k,l-1},$$

by the choice of l. Thus, $t = \mathrm{etype}(X_I) = \mathrm{emex}_p(A,B) \in \bar{E}_{k,l}$, a contradiction. □

Proposition 10. *The elements of $\bar{E}$ are the extended types shown in Table 1.*

etype $\in \bar{E}_k$	k	etype $\in \bar{E}_k$	k	etype $\in \bar{E}_k$	k	etype $\in \bar{E}_k$	k
$(0,0,0,2)$	0	$(0,0,2,2)$	2	$(1,3,2,1)$	$\boxed{2}$	$(1,0,1,0)$	$3,4,\ldots$
$(0,1,2,2)$	1	$(1,0,1,1)$	$\boxed{2},3,\ldots$	$(1,4,0,0)$	2	$(1,0,2,0)$	$\boxed{3},4$
$(1,1,2,1)$	$1,2$	$(1,0,2,1)$	$2,3$	$(1,4,1,1)$	$\boxed{2}$	$(1,1,2,0)$	3
$(1,2,1,0)$	1	$(1,1,0,0)$	2	$(1,4,2,1)$	$\boxed{2}$	$(1,3,1,0)$	3
$(1,4,3,1)$	1	$(1,3,0,0)$	2	$(0,0,1,2)$	$3,4,\ldots$	$(1,3,2,0)$	3

Table 1: The elements of $\bar{E}_k$ for each deficiency k. We found instances of every extended type with each deficiency using a computer search except for the five with a $\boxed{\text{box}}$ around them.

Proof. Using Definition 8, we computed $\bar{E}$ using a GAP [9] program. The code and its output are available on the companion web page [7]. The results show that the computation of each $\bar{E}_k$ finishes in finitely many iterations. This is indicated by the equality of $\bar{E}_{k,l}$ and $\bar{E}_{k,l+1}$ for some l. The results also show that $\bar{E}_5 = \bar{E}_6$. Hence $\bar{E} = \bigcup_{k=0}^{5} \bar{E}_k$ and the whole computation finishes in finitely many steps. $\square$

Even though we computed $\bar{E}$ with a computer, we also verified the output by hand. Note that a human can eliminate many of the large number of cases that the computer checked.

Example 3. We demonstrate the computation of $\bar{E}_1 = \bar{E}_{1,1}$. We have

$$\begin{aligned}
\bar{E}_0 &= \{(0,0,0,2)\}, \\
\bar{E}_{1,0} &= \{(0,1,2,2),(1,2,1,0)\}, \\
\bar{E}_{1,1} &= \{(0,1,2,2),(1,1,2,1),(1,2,1,0),(1,4,3,1)\}, \\
\bar{E}_{1,2} &= \bar{E}_{1,1}.
\end{aligned}$$

For example $(0,1,2,2) = \mathrm{emex}_0(\emptyset, \bar{E}_0)$ and $(1,4,3,1) = \mathrm{emex}_1(\bar{E}_{1,0}, \bar{E}_0)$. Note that $(\emptyset, \bar{E}_0)$ is 0-feasible and $(\bar{E}_{1,0}, \bar{E}_0)$ is 1-feasible. Also, note that $\bar{E}_1 = \bar{E}_{1,1}$ since $\bar{E}_{1,2} = \bar{E}_{1,1}$.

Remark 1. We found examples of the extended types with every deficiency shown in Table 1 using a computer search except for the five listed with a box around them. For instance, it is possible that $(1,0,2,0) \in \bar{E}_3 \setminus E_3$, but we have not found such an example. However, we have verified that $(1,0,2,0) \in E_4$ by looking at subgroups of `SmallGroup(500,48)` in GAP's [9] SmallGroup database.

5. Spectrum of Nim-Values

We are now ready to determine the spectrum of nim-values of $\mathsf{GEN}(G)$. If the order of G is odd, then Proposition 5 implies that the spectrum of nim-values of

G	$\mathbb{Z}_1$	$\mathbb{Z}_3^3$	$\mathbb{Z}_3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	S_3	$\mathbb{Z}_6$
$\text{nim}(\mathsf{GEN}(G))$	0	1	2	0	1	2	3	4

Table 2: Examples of groups and nim-values.

$\mathsf{GEN}(G)$ is a subset of $\{0,1,2\}$. If the order of G is even, then Proposition 10 and the containment

$$\{\pi_2(\text{type}(X_{\Phi(G)})) \mid G \text{ is an even group}\} \subseteq \pi_2(E) \subseteq \pi_2(\bar{E}).$$

shows that the spectrum of nim-values of $\mathsf{GEN}(G)$ is a subset of $\{0,1,2,3,4\}$. The next example verifies that we have equality in both cases.

Example 4. The nim-values for the odd and even-ordered groups listed in Table 2 were computed in [8]. The groups listed in the table have the smallest possible order for the given parity and nim-value.

The discussion above together with Example 4 immediately implies the following result.

Proposition 11. *The spectrum of nim-values of* $\mathsf{GEN}(G)$ *for groups with odd order is* $\{0,1,2\}$*. The spectrum of nim-values of* $\mathsf{GEN}(G)$ *for groups with even order is* $\{0,1,2,3,4\}$*.*

Now we have our main result.

Theorem 1. *The spectrum of nim-values of the achievement game* $\mathsf{GEN}(G)$ *for a finite group* G *is* $\{0,1,2,3,4\}$*.*

6. Open Problems and Conjectures

We close with a handful of open problems and conjectures.

1. One can find examples of all of the extended types for each deficiency in Table 1 except for the boxed $(1,0,1,1)$, $(1,3,2,1)$, $(1,4,1,1)$, $(1,4,2,1)$ types from $\bar{E}_2$ and $(1,0,2,0)$ from $\bar{E}_3$. Do these five extended types actually occur with the appropriate deficiencies?

2. Computer experimentation shows that adding a type restriction corresponding to the following conjecture eliminates all but $(1,0,2,0)$ of the five boxed extended types from Table 1:

   Conjecture 1. *If* G *is even and* $k \geq 0$*, then every* $X_I \in \mathcal{O}_{k+1}$ *has an option* $X_L \in \mathcal{E}_k$*.*

In fact, proving this conjecture for the special case when $X_I \in \mathcal{O}_2$ would be sufficient since the remaining four boxed extended types are all in $\mathcal{O}_2$. However, we were not able to prove this conjecture. A natural idea for a proof of this conjecture would be to prove the stronger statement:

> *If G is even and $k \geq 0$, then for every $X_I \in \mathcal{O}_{k+1}$ there is a t of even order such that $X_{\lceil I,t \rceil} \in \mathcal{E}_k$.*

Unfortunately, this statement is not true. In private correspondence, Marsden Conder provided a counterexample: `SmallGroup(240,191)` in GAP's [9] SmallGroup database, which is isomorphic to $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_{15}$. However, this is not a counterexample for the original conjecture.

3. Does a type give algebraic information about the corresponding subgroup? For instance, does the type characterize what kind of maximal subgroups contain the subgroup?

4. In [5], the authors provide a checklist in terms of maximal subgroups for determining the nim-value of $\mathsf{DNG}(G)$. Is there an analogous set of criteria for determining the nim-value of $\mathsf{GEN}(G)$?

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