ON THE SPRAGUE-GRUNDY FUNCTION OF COMPOUND GAMES

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Abstract

The classical game of Nim can be naturally extended and played on an arbitrary hypergraph $\mathcal{H} \subseteq 2^V \setminus \{\emptyset\}$ whose vertices $V = \{1, \dots, n\}$ correspond to piles of stones. By one move, a player chooses an edge H of $\mathcal{H}$ and arbitrarily reduces all piles $i \in H$. In 1901 Bouton solved the classical Nim for which $\mathcal{H} = \{\{1\}, \dots, \{n\}\}$. In 1910 Moore introduced and solved a more general game, k-Nim, for which $\mathcal{H} = \{H \subseteq V \mid |H| \leq k\}$, where $1 \leq k < n$. In 1980, Jenkyns and Mayberry obtained an explicit formula for the Sprague-Grundy function of Moore's Nim for the case $k = n - 1$. Recently it was shown that the same formula works for a large class of hypergraphs. In this paper we study combinatorial properties of these hypergraphs and obtain explicit formulas for the Sprague-Grundy functions of the conjunctive and selective compounds of the corresponding hypergraph Nim games.

1. Introduction

In the classical game of Nim there are n piles of stones and two players move alternately. A move consists of choosing a nonempty pile and taking some positive

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number of stones from it. The player who must but cannot move is the loser. Bouton [10] analyzed this game and described the winning strategy for it.

In this paper we consider the following generalization of NIM. For a positive integer n, let us denote by $V = \{1, ..., n\}$ a set of n piles of stones. Let $\mathbb{Z}_+$ denote the set of nonnegative integers. We use $x \in \mathbb{Z}_+^V$ to describe a position, where coordinate x_i denotes the number of stones of pile $i \in V$. Given a hypergraph $\mathcal{H} \subseteq 2^V$, a move from a position $x \in \mathbb{Z}_+^V$ consists in choosing an edge $H \in \mathcal{H}$ and strictly decreasing all values x_i for $i \in H$. The game starts in an initial position $x \in \mathbb{Z}_+^V$ and involves two players who alternate in making moves. Similar to NIM, the player who must but cannot move is the loser. This happens when every edge $H \in \mathcal{H}$ has an empty pile, $x_i = 0$ for some $i \in H$. Such games were considered in [6, 7, 9] and called hypergraph NIM. We denote by $\text{NIM}_\mathcal{H}$ an instance of this family. We assume in this paper that $V = \bigcup_{H \in \mathcal{H}} H$ and $\emptyset \notin \mathcal{H}$ for all considered hypergraphs $\mathcal{H} \subseteq 2^V$. In other words, every move strictly decreases some of the piles, and passes are not allowed.

Hypergraph NIM games are impartial. In this paper we do not need to immerse ourselves in the theory of impartial games. We recall only a few basic facts to explain and motivate our research. We refer the reader to [1, 3, 18] for more details.

A position of an impartial game is called *winning*, or an $\mathcal{N}$-*position*, if, starting from it, the first player can win, no matter what the second player does. The remaining positions are called *losing*, or $\mathcal{P}$-*positions*. It is known that every move from a $\mathcal{P}$-position goes to an $\mathcal{N}$-position, while from any $\mathcal{N}$-position there always exists a move to a $\mathcal{P}$-position. The so-called Sprague-Grundy (SG) function $\mathcal{G}_\Gamma$ of an impartial game Γ is a refinement of the above $\mathcal{P}$-$\mathcal{N}$ partition; see Section 3 for the definition. Namely, $\mathcal{G}_\Gamma(x) = 0$ if and only if x is a $\mathcal{P}$-position. The notion of the SG function was introduced independently by Sprague and Grundy [13, 20, 21] and it plays a fundamental role in solving *disjunctive compounds* of impartial games, defined later in this section.

Finding a formula for the SG function of an impartial game remains a challenge. Closed form descriptions are known only for some special cases. We recall below some known results. The purpose of our research is to extend these results and to describe classes of hypergraphs for which we can provide a closed formula for the SG function of $\text{NIM}_\mathcal{H}$.

The game $\text{NIM}_\mathcal{H}$ is a common generalization of several families of impartial games from the literature. Given a subset $S \subseteq V$ and a positive integer $k \leq |S|$, let

$$\binom{S}{k} = \{H \subseteq S \mid |H| = k\}.$$

For instance, if $\mathcal{H} = \binom{V}{1}$ then $\text{NIM}_\mathcal{H}$ is the classical NIM. The case of $\mathcal{H} = \bigcup_{j=1}^{k} \binom{V}{j}$, where $k < n$, was considered by Moore [17]. He characterized for these games the set of $\mathcal{P}$-positions, that is, those with SG value 0. Jenkyns and Mayberry [16] described also the set of positions in which the SG value is 1 and provided an explicit formula for the SG function in case of $k = n - 1$. This result was extended in [5]. In [6] the game $\text{NIM}_{\mathcal{H}}$ was considered in the case of $\mathcal{H} = \binom{V}{k}$ and the corresponding SG function was determined when $2k \geq n$. Further examples, such as matroid, 2-uniform (graph), symmetric, and hereditary hypergraph NIM games were considered in [7, 8, 9, 12]. Surprisingly, for many of these examples the SG function is described by the same formula, a special case of which was introduced by Jenkyns and Mayberry [16]. In honor of their contribution we called that formula *JM* in [9]. A hypergraph and the corresponding hypergraph NIM game are both called *JM* if the JM-formula describes its SG function.

In this paper, we study compositions of JM games and their SG functions. There are three basic types of compounds considered in the literature. Given two games Γ_1 and Γ_2 with disjoint sets of positions X_1 and X_2, the compound game Γ has the set of positions $X = X_1 \times X_2$, while the set of its moves can be introduced in three different ways as follows.

Disjunctive compound $\Gamma_1 \oplus \Gamma_2$: a player makes a move in exactly one of the two games: either in Γ_1 or in Γ_2.

Conjunctive compound $\Gamma_1 \otimes \Gamma_2$: a player makes a move in both games: one in Γ_1 and one in Γ_2.

Selective compound $\Gamma_1 \boxplus \Gamma_2$: a player makes a move either in one of the two games or in both.

All three operations $\oplus$, $\otimes$, and $\boxplus$ are associative and commutative, and hence, all three compounds are well-defined not only for two, but for any number of compound games. The disjunctive compound was introduced by Sprague and Grundy [13, 20, 21]; the conjunctive and selective ones were added by Smith and Conway [19, 11]. In [7] a concept of hypergraph compound of games, which generalizes all three above concepts, was introduced.

In this paper, we introduce a special subfamily of JM games, called *JM+* games, that includes JM matroid and graph NIM games [8]. Our main results are the following:

(i) We show that the family of JM+ games is closed under conjunctive compound.

(ii) We provide closed formulas for the SG functions of selective compounds of JM+ games, which are analogous to the JM formula.

Let us note that the selective compound of JM+ games is not JM in general. Although the above three compound operations always yield hypergraph NIM games, yet, explicit formulas for their SG functions are known only in some special cases, extended now by the results mentioned above.

Let us add that the explicit formula for the SG function of a game may be difficult to determine even in very simple looking cases. In [4, 5] the combined compound $\Gamma = \Gamma_1 \boxplus (\Gamma_2 \oplus \Gamma_3)$ was studied, where Γ_i are single pile NIM games for $i = 1, 2, 3$. It is easy to see that this game is the hypergraph NIM on $\mathcal{H} = \{\{1,2\},\{1,3\},\{1\},\{2\},\{3\}\}$. It appears that the SG function $\mathcal{G}_\Gamma(\mathbf{x})$ of this game behaves in a chaotic way when $\mathbf{x}$ is small and becomes more regular only for large $\mathbf{x}$. Yet, no explicit formula for $\mathcal{G}_\Gamma(\mathbf{x})$ is known.

The rest of the paper is organized as follows. In Section 2 we introduce JM formulas and games, and define JM and JM+ hypergraphs. In Section 3 we provide several examples illustrating these definitions. In Section 4 we prove several technical lemmas. In Section 5 we prove that the family of JM+ games is closed under conjunctive compound. Although selective compound does not share this property, in Section 6 we give an explicit formula for the SG function in this case. In Section 7 we study combinatorial properties of JM+ hypergraphs. In Section 8 we formulate some open problems.

2. JM+ Hypergraphs and Main Results

To three integers m, y and h, let us associate the following quantities:

$$v(m,y) \;=\; \binom{y+1}{2} + \left(\left(m - \binom{y+1}{2} - 1\right) \mod (y+1)\right) \tag{1}$$

and

$$\mathcal{U}(m,y,h) = \begin{cases} h \quad \text{if } m \le \binom{y+1}{2}, & \text{(2a)} \\ v(m,y) \quad \text{otherwise.} & \text{(2b)} \end{cases}$$

Given a hypergraph $\mathcal{H} \subseteq 2^V$, the *height* $h_\mathcal{H}(\mathbf{x})$ of a position $\mathbf{x} \in \mathbb{Z}_+^V$ is defined as the maximum number of consecutive moves that the players can make in NIM$_\mathcal{H}$ starting from $\mathbf{x}$. Furthermore, for a position $\mathbf{x} \in \mathbb{Z}_+^V$ of NIM$_\mathcal{H}$ we define

$$m(\mathbf{x}) = \min_{i \in V} x_i, \tag{3a}$$

$$y_\mathcal{H}(\mathbf{x}) = h_\mathcal{H}(\mathbf{x} - m(\mathbf{x})\mathbf{e}). \tag{3b}$$

where $\mathbf{e} = (1,1,...,1)$ is the n-vector of ones. A position $\mathbf{x}$ is called *long* if $m(\mathbf{x}) \le \binom{y_\mathcal{H}(\mathbf{x})+1}{2}$ and it is called *short* otherwise.

The expression $\mathcal{U}(\mathbf{x}) = \mathcal{U}(m(\mathbf{x}), y_\mathcal{H}(\mathbf{x}), h_\mathcal{H}(\mathbf{x}))$ for a position $\mathbf{x} \in \mathbb{Z}_+^V$ is called the *JM formula*. We call the hypergraph $\mathcal{H}$ *JM* if the JM formula represents the SG function of NIM$_\mathcal{H}$.

To introduce a subfamily JM+ of JM hypergraphs, we will add to JM the following properties. For a hypergraph $\mathcal{H} \subseteq 2^V$, an edge $H \in \mathcal{H}$ is called a *transversal edge* if it intersects every edge of the hypergraph, that is, if $H \cap H' \neq \emptyset$ for all $H' \in \mathcal{H}$. A hypergraph with no transversal edge is called *transversal free*. For a subset $S \subseteq V$ we denote by $\mathcal{H}_S = \{H \in \mathcal{H} \mid H \subseteq S\}$ the induced subhypergraph. A hypergraph $\mathcal{H}$ is called *minimal transversal free* if it is transversal free, but any proper induced subhypergraph of it has a transversal edge. For example, $\mathcal{H} = \{\{1,2\},\{2,3\},\{3,4\}\}$ contains a transversal edge $\{2,3\}$, while $\mathcal{G} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$ has no transversal edge, and hence, it is transversal free. It is easy to see that $\mathcal{G}$ is minimal transversal free. Let us call a hypergraph $\mathcal{H}$ *minimum-decreasing* if for every position $\mathbf{x} \in \mathbb{Z}_+^V$ of $\text{N}\textsc{im}_{\mathcal{H}}$ there exists a move $\mathbf{x} \to \mathbf{x}'$ such that $m(\mathbf{x}') < m(\mathbf{x})$, $h_{\mathcal{H}}(\mathbf{x}') = h_{\mathcal{H}}(\mathbf{x}) - 1$, and $x_i - x_i' \leq 1$ for all $i \in V$. A sequence of edges $H_0, H_1, \ldots, H_q$ in $\mathcal{H}$ is called a *chain*, if $H_{i+1} \cap H_i \neq \emptyset$, $|H_{i+1} \setminus H_i| = 1$, and $H_i \subseteq H_0 \cup H_q$ for all $i = 0, 1, \ldots, q-1$. We say that a hypergraph $\mathcal{H}$ has the *chain-property* if for any two distinct edges $H, H' \in \mathcal{H}$ there exists a chain $H_0, \ldots, H_q$ in $\mathcal{H}$ such that $H = H_0$ and $H' = H_q$.

We are now ready to define JM+ hypergraphs. A hypergraph is called *JM+* if it satisfies the following three properties:

(MTF) it is minimal transversal free,

(MD) it is minimum-decreasing, and

(C) it has the chain-property.

It was shown in [8] that every JM+ hypergraph is JM, and that property (MTF) is necessary for a hypergraph to be JM. We show in Section 7 that no two of the above three properties imply that the hypergraph is JM, and in particular, they do not imply the third property. We also show that, unlike (MTF), property (C) is not necessary for a hypergraph to be JM. It remains an open question if (MD) is necessary.

Let us note that JM+ is a proper subfamily of JM, since, for instance, some of the symmetric JM hypergraphs constructed in [9] do not belong to JM+. On the other hand, JM+ contains all JM hypergraphs described in [8], including JM matroids and JM graphs. It is a challenging open problem to find a combinatorial characterization of JM hypergraphs.

Let us consider hypergraphs $\mathcal{H}_i \subseteq 2^{V_i}$, $i = 1, \ldots, p$, where the sets V_i, $i = 1, ..., p$, are pairwise disjoint, and define

$$\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_p = \left\{ \bigcup_{i=1}^{p} H^i \,\middle|\, H^i \in \mathcal{H}_i,\ i = 1, ..., p \right\}, \tag{4}$$

$$\mathcal{H}_1 \boxplus \cdots \boxplus \mathcal{H}_p = \left\{ \bigcup_{i=1}^{p} H^i \,\middle|\, H^i \in \mathcal{H}_i \cup \{\emptyset\},\ i = 1, ..., p \right\} \setminus \{\emptyset\}. \tag{5}$$

We call (4) and (5) the *conjunctive and selective compounds of the hypergraphs* $\mathcal{H}_i$, $i = 1, \ldots, p$. Furthermore, we call the corresponding two games, $\text{NIM}_{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_p}$ and $\text{NIM}_{\mathcal{H}_1 \boxplus \cdots \boxplus \mathcal{H}_p}$, respectively, the *conjunctive and selective compounds* of the p component games $\text{NIM}_{\mathcal{H}_i}$. In cases when these p games (and hypergraphs) are JM+, we obtain explicit formulas for the SG functions of the compound games (4) and (5). Our main results are the next two theorems.

JM+ hypergraphs are closed under conjunctive compound. In fact, we prove the following slightly more general statement.

Theorem 1. *The conjunctive compound $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_p$ is JM+ if $p \geq 2$ and, for all $i = 1, \ldots, p$, either $\mathcal{H}_i$ is JM+ or $\mathcal{H}_i = \binom{[2]}{1}$.*

Although the selective compound $\mathcal{H}$ of JM+ hypergraphs is not JM in general, in some cases we can describe the SG function of game $\text{NIM}_{\mathcal{H}}$ as follows. For a position $\mathbf{x} = (\mathbf{x}^1, ..., \mathbf{x}^p) \in \mathbb{Z}_+^{V_1} \times \cdots \times \mathbb{Z}_+^{V_p}$ of $\text{NIM}_{\mathcal{H}}$, we define

$$\begin{aligned} M(\mathbf{x}) &= m(\mathbf{x}^1) + \cdots + m(\mathbf{x}^p), \\ Y(\mathbf{x}) &= y_{\mathcal{H}_1}(\mathbf{x}^1) + \cdots + y_{\mathcal{H}_p}(\mathbf{x}^p), \\ h_{\mathcal{H}}(\mathbf{x}) &= h_{\mathcal{H}_1}(\mathbf{x}^1) + \cdots + h_{\mathcal{H}_p}(\mathbf{x}^p). \end{aligned} \tag{6}$$

Theorem 2. *If p hypergraphs $\mathcal{H}_i \subseteq 2^{V_i}$, $i = 1, ..., p$, are JM+, then the SG function of their selective compound $\mathcal{H} = \mathcal{H}_1 \boxplus \cdots \boxplus \mathcal{H}_p$ is given by*

$$\mathcal{G}_{\text{NIM}_{\mathcal{H}}}(\mathbf{x}) \;=\; \mathcal{U}(M(\mathbf{x}), Y(\mathbf{x}), h_{\mathcal{H}}(\mathbf{x})),$$

where $M(\mathbf{x})$, $Y(\mathbf{x})$, and $h_{\mathcal{H}}(\mathbf{x})$ are defined by (6).

Note that this is the JM formula if (and only if) $p = 1$. Note also that for both theorems it is an open question if JM+ can be replaced by JM.

3. Illustrative Examples

Recall that a function $g : \mathbb{Z}_+^V \to \mathbb{Z}_+$ is the *SG function* of $\text{NIM}_{\mathcal{H}}$ if and only if the following two conditions hold [13, 20, 21]:

- $g(\mathbf{x}) \neq g(\mathbf{x}')$ for any move $\mathbf{x} \to \mathbf{x}'$;
- for every integer z such that $0 \leq z < g(\mathbf{x})$ there exists a move $\mathbf{x} \to \mathbf{x}'$ in $\text{NIM}_{\mathcal{H}}$ such that $g(\mathbf{x}') = z$.

Let us recall the SG theorem stating that the SG function of the disjunctive compound of impartial games is a function of the SG function values of components (namely, the so-called NIM-sum of the SG values [10, 13, 20, 21]). Furthermore, in disjunctive compounds, each move to a lower SG value can be realized by moving to a lower SG value in one of the components.

We will give two examples demonstrating that conjunctive and selective compounds do not have such properties. Our first example shows that the SG values of conjunctive and selective compounds are not uniquely defined by the SG values of the components.

Example 1. Given $V_1 = \{1,2,3,4\}$ and $V_2 = \{5,6,7,8\}$, define $\mathcal{H}_1 \subseteq 2^{V_1}$ and $\mathcal{H}_2 \subseteq 2^{V_2}$ as follows:

$$\begin{aligned}\mathcal{H}_1 &= \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\} \text{ and}\\ \mathcal{H}_2 &= \{\{5,6\},\{6,7\},\{7,8\},\{8,5\}\}.\end{aligned}$$

Then consider positions $\mathbf{a}^1 = (0,4,4,0)$ and $\mathbf{a}^2 = (0,3,3,0)$ in $\text{N}\textsc{im}_{\mathcal{H}_1}$ and $\text{N}\textsc{im}_{\mathcal{H}_2}$, respectively. For these positions, we have

$$m(\mathbf{a}^1) = m(\mathbf{a}^2) = 0,\ y_{\mathcal{H}_1}(\mathbf{a}^1) = 4,\text{ and } y_{\mathcal{H}_2}(\mathbf{a}^2) = 3.$$

Since both positions are long, we have

$$\mathcal{U}(\mathbf{a}^1) = h_{\mathcal{H}_1}(\mathbf{a}^1) = 4 \text{ and } \mathcal{U}(\mathbf{a}^2) = h_{\mathcal{H}_2}(\mathbf{a}^2) = 3.$$

For the pair of positions $\mathbf{b}^1 = (0,4,4,0)$ and $\mathbf{b}^2 = (4,6,6,4)$, we have

$$m(\mathbf{b}^1) = 0, m(\mathbf{b}^2) = 4,\ y_{\mathcal{H}_1}(\mathbf{b}^1) = 4,\text{ and } y_{\mathcal{H}_2}(\mathbf{b}^2) = 2.$$

Since $\mathbf{b}^1$ is long and $\mathbf{b}^2$ is short, we get

$$\mathcal{U}(\mathbf{b}^1) = h_{\mathcal{H}_1}(\mathbf{b}^1) = 4 \text{ and } \mathcal{U}(\mathbf{b}^2) = v(m(\mathbf{b}^2), y_{\mathcal{H}_2}(\mathbf{b}^2)) = 3.$$

By Theorem 1, both hypergraphs are JM+, since they are conjunctive compounds of two copies of $\binom{[2]}{1}$. Consequently, the SG values of $\mathbf{a}^i$ and $\mathbf{b}^i$ are the same for both $i = 1,2$.

Let us first consider the conjunctive compound $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. For position $\mathbf{a} = (\mathbf{a}^1, \mathbf{a}^2)$, we have

$$\begin{aligned}m(\mathbf{a}) &= \min\{m(\mathbf{a}^1), m(\mathbf{a}^2)\} = 0 \text{ and}\\ y_{\mathcal{H}}(\mathbf{a}) &= \min\{y_{\mathcal{H}_1}(\mathbf{a}^1), y_{\mathcal{H}_2}(\mathbf{a}^2)\} = 3.\end{aligned}$$

Since $\mathbf{a}$ is long, $\mathcal{U}(\mathbf{a}) = h_{\mathcal{H}}(\mathbf{a}) = 3$. For position $\mathbf{b} = (\mathbf{b}^1, \mathbf{b}^2)$, we have

$$\begin{aligned}m(\mathbf{b}) &= \min\{m(\mathbf{b}^1), m(\mathbf{b}^2)\} = 0 \text{ and}\\ y_{\mathcal{H}}(\mathbf{b}) &= \min\{y_{\mathcal{H}_1}(\mathbf{b}^1), y_{\mathcal{H}_2}(\mathbf{b}^2)\} = 2.\end{aligned}$$

Since $\mathbf{b}$ is also long, $\mathcal{U}(\mathbf{b}) = h_{\mathcal{H}}(\mathbf{b}) = 4$. By Theorem 1 $\mathcal{H}$ is JM+, and thus, $\mathcal{U}(\mathbf{a}) \neq \mathcal{U}(\mathbf{b})$ implies that the SG values of $\mathbf{a}$ and $\mathbf{b}$ are different.

Let us next consider the selective compound $\mathcal{H} = \mathcal{H}_1 \boxplus \mathcal{H}_2$. Then, by applying Theorem 2 to $\mathcal{H}$, we compute SG values of $\mathbf{a} = (\mathbf{a}^1, \mathbf{a}^2)$ and $\mathbf{b} = (\mathbf{b}^1, \mathbf{b}^2)$ as follows.

For position $\mathbf{a}$, we have $M(\mathbf{a}) = m(\mathbf{a}^1) + m(\mathbf{a}^2) = 0$ and $Y_{\mathcal{H}}(\mathbf{a}) = y_{\mathcal{H}_1}(\mathbf{a}^1) + y_{\mathcal{H}_2}(\mathbf{a}^2) = 7$. Hence, the SG value $\mathcal{G}(\mathbf{a})$ is given by $\mathcal{U}(\mathbf{a}) = h_{\mathcal{H}}(\mathbf{a}) = 7$. For position $\mathbf{b}$ we have $M(\mathbf{b}) = m(\mathbf{b}^1) + m(\mathbf{b}^2) = 4$ and $Y(\mathbf{b}) = y_{\mathcal{H}_1}(\mathbf{b}^1) + y_{\mathcal{H}_2}(\mathbf{b}^2) = 6$. Hence, $\mathcal{G}(\mathbf{b}) = \mathcal{U}(\mathbf{b}) = h_{\mathcal{H}}(\mathbf{b}) = 14$. Thus, $\mathcal{G}(\mathbf{a}) \neq \mathcal{G}(\mathbf{b})$. $\square$

The next example shows that to move to a position with smaller SG value in a selective compound, it may be necessary to increase the SG value in some of the component games.

Example 2. Let us consider two copies of the hypergraph on 3 vertices consisting of all proper subsets (Moore's game on 3 vertices), the positions $\mathbf{a}^1 = \mathbf{a}^2 = (4, 4, 5)$, and the position $\mathbf{a} = (\mathbf{a}^1, \mathbf{a}^2)$ in the selective compound. We have

$$m(\mathbf{a}^1) = m(\mathbf{a}^2) = 4,\ y_{\mathcal{H}_1}(\mathbf{a}^1) = y_{\mathcal{H}_2}(\mathbf{a}^2) = 1,$$

and hence, $\mathcal{U}(\mathbf{a}^1) = \mathcal{U}(\mathbf{a}^2) = v(4, 1) = 1$. Since both games are JM, we have $\mathcal{G}(\mathbf{a}^1) = \mathcal{G}(\mathbf{a}^2) = 1$.

In the selective compound, we have

$$M(\mathbf{a}) = 8, Y(\mathbf{a}) = 2, \text{ and } \mathcal{U}(\mathbf{a}) = v(8, 2) = 4.$$

By Theorem 2, $\mathcal{U}$ is the SG function of the compound game, and hence, we must have a move $\mathbf{a} \to \mathbf{b}$ such that $\mathcal{U}(\mathbf{b}) = 2$. It is easy to argue that for such a position $\mathbf{b}$ we must have $Y(\mathbf{b}) = 1$ and $M(\mathbf{b}) \equiv 1 \mod 2$. The only move (up to the symmetry between the two games) yielding these values is to $\mathbf{b}^1 = (3, 3, 4)$ in one of the games and to $\mathbf{b}^2 = (4, 4, 4)$ in the other one. However the SG value $\mathcal{G}(\mathbf{b}^1) = \mathcal{U}(\mathbf{b}^1) = 2$ is larger than $\mathcal{G}(\mathbf{a}^1) = 1$. $\square$

4. Technical Lemmas

In this section, we present several lemmas which will be used to show our main theorems.

For positions $\mathbf{x}, \mathbf{x}' \in \mathbb{Z}_+^V$ we define

$$\|\mathbf{x}' - \mathbf{x}\|_+ \ = \sum_{\substack{i \in V \\ x'_i > x_i}} x'_i - x_i;$$

in particular, we have $\|\mathbf{x}' - \mathbf{x}\|_+ \ = \ 0$ if $\mathbf{x}' \leq \mathbf{x}$, that is, $x'_i \leq x_i$ for all $i \in V$.

Lemma 1. *For positions $\mathbf{x}, \mathbf{x}' \in \mathbb{Z}_+^V$ we have*

$$h_{\mathcal{H}}(\mathbf{x}) \geq h_{\mathcal{H}}(\mathbf{x}') - \|\mathbf{x}' - \mathbf{x}\|_+.$$

In particular, function $h_{\mathcal{H}}$ is monotone with respect to the componentwise relation $\geq$.

Proof. By the definition of the height, if $\mathbf{x} \geq \mathbf{x}'$ then $h_{\mathcal{H}}(\mathbf{x}) \geq h_{\mathcal{H}}(\mathbf{x}')$. Also, if $\mathbf{e}_j$ is the jth unit vector for $j \in V$ then we have $h_{\mathcal{H}}(\mathbf{x} - \mathbf{e}_j) \geq h_{\mathcal{H}}(\mathbf{x}) - 1$. $\square$

For two integers $a, b \in \mathbb{Z}_+$ with $a \leq b$ we denote by $[a, b]$ the set of integers between a and b, that is, all i such that $a \leq i \leq b$. Similarly, for two positions $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^V$ with $\mathbf{a} \leq \mathbf{b}$ we denote by

$$[\mathbf{a}, \mathbf{b}] \ = \ \{x \in \mathbb{Z}_+^V \mid a_i \leq x_i \leq b_i, \ i \in V\}$$

the set of integer vectors between $\mathbf{a}$ and $\mathbf{b}$.

Given a hypergraph $\mathcal{H} \subseteq 2^V$, a move $\mathbf{x} \to \mathbf{x}'$ is called an *H-move* if $x_i' < x_i$ for $i \in H$ and $x_i' = x_i$ for $i \notin H$.

Lemma 2 (Contiguity Lemma)**.** *Given a position $\mathbf{x} \in \mathbb{Z}_+^V$, an edge $H \in \mathcal{H}$, and two H-moves $\mathbf{x} \to \mathbf{a}$, $\mathbf{x} \to \mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b}$, then each position $\mathbf{c} \in [\mathbf{a}, \mathbf{b}]$ can be reached by an H-move from $\mathbf{x}$, and we have*

$$\{h_{\mathcal{H}}(\mathbf{c}) \mid \mathbf{c} \in [\mathbf{a}, \mathbf{b}]\} \ = \ [h_{\mathcal{H}}(\mathbf{a}), h_{\mathcal{H}}(\mathbf{b})] \, .$$

Proof. Since both $\mathbf{x} \to \mathbf{a}$ and $\mathbf{x} \to \mathbf{b}$ are H-moves (with the same edge $H \in \mathcal{H}$), any position $\mathbf{c} \in [\mathbf{a}, \mathbf{b}]$ satisfies $\mathbf{c} < \mathbf{x}$ and $\{i \in V \mid \mathbf{c}_i < \mathbf{x}_i\} = H$, proving that $\mathbf{x} \to \mathbf{c}$ is an H-move. Moreover, by the monotonicity of $h_{\mathcal{H}}$ in Lemma 1, we have $\{h_{\mathcal{H}}(\mathbf{c}) \mid \mathbf{c} \in [\mathbf{a}, \mathbf{b}]\} \subseteq [h_{\mathcal{H}}(\mathbf{a}), h_{\mathcal{H}}(\mathbf{b})]$. To show the converse inclusion, let us define $p = \sum_{i \in V}(b_i - a_i)$, and consider a sequence of positions $\mathbf{x}^0$,$\mathbf{x}^1$, $\ldots$, $\mathbf{x}^p$, such that $\mathbf{x}^0 = \mathbf{b}$, $\mathbf{x}^p = \mathbf{a}$, and for all $j = 1, \ldots, p$ $\mathbf{x}^j$ is obtained from $\mathbf{x}^{j-1}$ by decreasing one of its components by one unit. Then, again by Lemma 1, we have $h_{\mathcal{H}}(\mathbf{x}^{j-1}) \geq h_{\mathcal{H}}(\mathbf{x}^j) \geq h_{\mathcal{H}}(\mathbf{x}^{j-1}) - 1$, which proves the converse inclusion. $\square$

Given a hypergraph $\mathcal{H} \subseteq 2^V$, a move $\mathbf{x} \to \mathbf{x}'$ is called *slow* if $x_i - x_i' \leq 1$ for all $i \in V$. We denote by $\mathbf{x}^{s(H)}$ the set of positions obtained from $\mathbf{x}$ by a slow H-move, that is,

$$x_i^{s(H)} = \begin{cases} x_i - 1 & \text{if } i \in H, \\ x_i & \text{otherwise.} \end{cases}$$

Lemma 3. *Consider a JM+ hypergraph $\mathcal{H}$ and a position $\mathbf{x} \in \mathbb{Z}_+^V$ with $h_{\mathcal{H}}(\mathbf{x}) > 0$. Then, there exists an integer t such that*

(L0) *$t = h_{\mathcal{H}}(\mathbf{x})$ if $m(\mathbf{x}) = 0$ and $m(\mathbf{x}) \leq t < h_{\mathcal{H}}(\mathbf{x})$ if $m(\mathbf{x}) > 0$;*

(L1) *for all $t < z < h_{\mathcal{H}}(\mathbf{x})$ there exists a move $\mathbf{x} \to \mathbf{x}'$ such that $0 \leq m(\mathbf{x}') < m(\mathbf{x})$, $y_{\mathcal{H}}(\mathbf{x}') \geq y_{\mathcal{H}}(\mathbf{x})$, and $h_{\mathcal{H}}(\mathbf{x}') = z$;*

(L2) *for* $z = t < h_{\mathcal{H}}(\mathbf{x})$ *there exists a move* $\mathbf{x} \to \mathbf{x}'$ *such that* $m(\mathbf{x}') = 0$*,* $y_{\mathcal{H}}(\mathbf{x}') \geq y_{\mathcal{H}}(\mathbf{x})$*, and* $h_{\mathcal{H}}(\mathbf{x}') = z$*;*

(L3) *for all* $m(\mathbf{x}) \leq z < t$ *there exists a move* $\mathbf{x} \to \mathbf{x}'$ *such that* $m(\mathbf{x}') = 0$ *and* $h_{\mathcal{H}}(\mathbf{x}') = z$*.*

Proof. Let us first consider a position $\mathbf{x} \in \mathbb{Z}_+^V$ with $m(\mathbf{x}) > 0$. By property (MD), there exists a $j \in H \in \mathcal{H}$ such that $h_{\mathcal{H}}(\mathbf{x}^{s(H)}) = h_{\mathcal{H}}(\mathbf{x}) - 1$ and $x_j = m(\mathbf{x})$. Set $\mathbf{a}^0 = \mathbf{x}^{s(H)}$ and define $\mathbf{c}^0$ by

$$c_i^0 = \begin{cases} 0 & \text{if } i = j \\ x_i - 1 & \text{if } i \in H \setminus \{j\} \\ x_i & \text{if } i \notin H. \end{cases}$$

We claim that $t = h_{\mathcal{H}}(\mathbf{c}^{(0)})$ has the desired properties. Clearly, this choice satisfies (L0).

For any $\mathbf{x}' \in [\mathbf{c}^0, \mathbf{a}^0]$, there exists a move $\mathbf{x} \to \mathbf{x}'$. Note that $m(\mathbf{x}') < m(\mathbf{x})$. By this, we have $\mathbf{x}' - m(\mathbf{x}')\mathbf{e} \geq \mathbf{x} - m(\mathbf{x})\mathbf{e}$, which implies that $y_{\mathcal{H}}(x') \geq y_{\mathcal{H}}(x)$. Thus, (L2) holds by taking $\mathbf{x}' = \mathbf{c}^0$, since $m(\mathbf{c}^0) = 0$. Moreover, since $[h_{\mathcal{H}}(\mathbf{c}^0), h_{\mathcal{H}}(\mathbf{a}^0)] = [t, h_{\mathcal{H}}(\mathbf{x}) - 1]$, Lemma 2 implies (L1).

Let us show (L3). By property (MTF), $\mathcal{H}_{V\setminus\{1\}}$ has a transversal edge H'. By property (MD), for any two edges H and H', there exists a chain $H_0\,(= H), H_1, \dots, H_r\,(= H')$. Let us then define positions $\mathbf{a}^k$ $(k = 1, \dots, r)$ by

$$a_i^k = \begin{cases} 0 & \text{if } i \in H_{k-1} \cap H_k \\ x_i - 1 & \text{if } i \in H_k \setminus H_{k-1} \\ x_i & \text{if } i \notin H_k, \end{cases}$$

and $\mathbf{b}^k$ $(k = 0, \dots, r)$ by

$$b_i^k = \begin{cases} 0 & \text{if } i \in H_k \\ x_i & \text{if } i \notin H_k. \end{cases}$$

Consider the set of positions $I = [\mathbf{b}^0, \mathbf{c}^0] \cup \bigcup_{k=1}^r [\mathbf{b}^k, \mathbf{a}^k]$. Note that from any position $\mathbf{x}'$ in I there is a move $\mathbf{x} \to \mathbf{x}'$ such that $m(\mathbf{x}') = 0$. Moreover, for any $i = 1, \dots, r$ we have $\|\mathbf{b}^{i-1} - \mathbf{a}^i\|_+ = 1$, implying that $h_{\mathcal{H}}(\mathbf{a}^i) \geq h_{\mathcal{H}}(\mathbf{b}^{i-1}) - 1$. By $h_{\mathcal{H}}(\mathbf{b}^r) = m$ and Lemma 2, we have

$$\{h_{\mathcal{H}}(\mathbf{x}') \mid \mathbf{x}' \in I\} = [m(\mathbf{x}), t],$$

which completes the case of $m(\mathbf{x}) > 0$.

Let us finally consider a position $\mathbf{x} \in \mathbb{Z}_+^V$ such that $m(\mathbf{x}) = 0$. We claim that $t = h_{\mathcal{H}}(\mathbf{x})$ satisfies the desired property, that is, (L3), while (L0), (L1), and (L2) are automatically satisfied for this t.

Consider $W = \{i \in V \mid x_i > 0\}$ and the corresponding induced subhypergraph $\mathcal{H}_W$. By the definition of the height, there exists an edge $H \in \mathcal{H}_W$ such that $h_{\mathcal{H}}(\mathbf{x}^{s(H)}) = h_{\mathcal{H}}(\mathbf{x}) - 1$. By property (MTF), there exists an edge $H' \in \mathcal{H}_W$ that intersects all other edges of $\mathcal{H}_W$. By property (C), we have again a chain $H = H_0, H_1, \ldots, H_r = H'$. Similar to the above construction, we have a series of H_k-moves $k = 0, \ldots, r$ such that the range of $h_{\mathcal{H}}$ values includes all integers z such that $0 \leq z < h_{\mathcal{H}}(x)$. □

Lemma 4. *Assume that $\mathcal{H} \subseteq 2^V$ satisfies properties* (MTF) *and* (C)*. Then, for every position $\mathbf{x} \in \mathbb{Z}_+^V$ with $m(\mathbf{x}) > 0$ and pair of integers $(\mu, \eta) \neq (m(\mathbf{x}), y_{\mathcal{H}}(\mathbf{x}))$ such that $0 \leq \mu \leq m(\mathbf{x})$ and $m(\mathbf{x}) - \mu \leq \eta \leq y_{\mathcal{H}}(\mathbf{x})$, there exists a move $\mathbf{x} \to \mathbf{x}'$ such that $m(\mathbf{x}') = \mu$ and $y_{\mathcal{H}}(\mathbf{x}') = \eta$.*

Proof. Let us consider first the case when $\mu = m(\mathbf{x})$. Then, (L3) of Lemma 3, applied to the truncated vector $\mathbf{x} - m(\mathbf{x})\mathbf{e}$, implies the claim.

Let us consider next the case when $\mu < m(\mathbf{x})$. Assume that $x_j = m(\mathbf{x})$. Let $H \in \mathcal{H}$ be an edge with $j \in H$, and $H' \in \mathcal{H}$ be a transversal edge of $\mathcal{H}_{V \setminus \{j\}}$. By property (C), we have a chain $H_0\,(= H), H_1, \ldots, H_r\,(= H')$. Define positions $\mathbf{a}^k$ and $\mathbf{b}^k$, $k = 0, \ldots, r$ by

$$a_i^k = \begin{cases} \mu & \text{if } i \in H_{k-1} \cap H_k \\ x_i - 1 & \text{if } i \in H_k \setminus H_{k-1} \\ x_i & \text{if } i \notin H_k \end{cases} \quad \text{and} \quad b_i^k = \begin{cases} \mu & \text{if } i \in H_k \\ x_i & \text{if } i \notin H_k, \end{cases}$$

assuming that $H_{-1} = \{j\}$. For $k = 0, \ldots, r$, set $I_k = [\mathbf{b}^k, \mathbf{a}^k]$ and $I = \bigcup_{k=0}^r I_k$. We claim that the set of positions I is a certificate of the lemma. Indeed, clearly for any position $\mathbf{x}' \in I$ we have $m(\mathbf{x}') = \mu$. Hence, it is enough to show that

$$\{y_{\mathcal{H}}(\mathbf{x}') \mid \mathbf{x}' \in I\} \supseteq [m(\mathbf{x}) - \mu, y_{\mathcal{H}}(\mathbf{x})]. \tag{7}$$

Note also that $y_{\mathcal{H}}(\mathbf{x}') = h_{\mathcal{H}}(\mathbf{x}' - \mu\mathbf{e})$ for any position $\mathbf{x}' \in I$. Hence, for $k = 0, \ldots, r$, Lemma 2 implies that

$$\{y_{\mathcal{H}}(\mathbf{x}') \mid \mathbf{x}' \in I_k\} = [y_{\mathcal{H}}(\mathbf{b}^k), y_{\mathcal{H}}(\mathbf{a}^k)].$$

Since $y_{\mathcal{H}}(\mathbf{a}^0) \geq y_{\mathcal{H}}(\mathbf{x})$, $y_{\mathcal{H}}(\mathbf{b}^r) = m(\mathbf{x}) - \mu$, and $y_{\mathcal{H}}(\mathbf{a}^k) \geq y_{\mathcal{H}}(\mathbf{b}^{k-1}) - 1$ for $k = 1, \ldots, r$, we obtain (7), which completes the proof. □

Lemma 5. *Assume that $\mathcal{H} = \mathcal{H}_1 \boxplus \cdots \boxplus \mathcal{H}_p \subseteq 2^V$ is a selective compound of transversal free hypergraphs $\mathcal{H}_i$, $i = 1, \ldots, p$. Then, $\mathcal{H}$ itself is transversal free, and for every position $\mathbf{a} \in \mathbb{Z}_+^V$ and move $\mathbf{a} \to \mathbf{b}$ the following relations hold:*

(i) $h_{\mathcal{H}}(\mathbf{a}) > h_{\mathcal{H}}(\mathbf{b}) \geq M(\mathbf{a}) \geq M(\mathbf{b})$*;*

(ii) $v(M(\mathbf{a}), Y(\mathbf{a})) < M(\mathbf{a})$ *if and only if* $M(\mathbf{a}) > \binom{Y(\mathbf{a})+1}{2}$*;*

(iii) $Y(\mathbf{b}) \geq M(\mathbf{a}) - M(\mathbf{b})$.

Proof. By the definition of the height, it strictly decreases with every move. Moreover, the $m(\mathbf{a}^i)$ values share this property. To complete the proof of (i), assume that $\mathbf{a} \to \mathbf{b}$ is an H-move for some $H \in \mathcal{H}$. By the definition of selective compound, we have $H \cap V_i \in \mathcal{H}_i \cup \{\emptyset\}$ for all $i = 1, \ldots, p$. Since all these hypergraphs are transversal free, by our assumption, there exist edges $H_i \in \mathcal{H}_i$ such that $H_i \cap H = \emptyset$ for all $i = 1, \ldots, p$. Hence, even after the $\mathbf{a} \to \mathbf{b}$ move, we still can make at least $m(\mathbf{a}^i)$ slow H_i-moves from $\mathbf{b}$. Since these moves for $i = 1, ..., p$ are all moves in $\textsc{Nim}_{\mathcal{H}}$, the inequality $h_{\mathcal{H}}(\mathbf{b}) \geq M(\mathbf{a})$ follows. The same argument shows also that we can make at least $m(\mathbf{a}^i) - m(\mathbf{b}^i)$ slow H_i-moves from $\mathbf{b}$, without decreasing $m(\mathbf{b}^i)$, for all $i = 1, ..., p$, thus, proving (iii). Finally, (ii) follows by the definition (1). $\square$

Lemma 6. *Assume that $\mathcal{H} = \mathcal{H}_1 \boxplus \cdots \boxplus \mathcal{H}_p \subseteq 2^V$ is a selective compound of transversal free hypergraphs $\mathcal{H}_i \subseteq 2^{V_i}$, $i = 1, \ldots, p$. Then, for all positions $\mathbf{x} \in \mathbb{Z}_+^V$ and moves $\mathbf{x} \to \mathbf{x}'$ of $\textsc{Nim}_{\mathcal{H}}$ we have*

$$\mathcal{U}(M(\mathbf{x}), Y(\mathbf{x}), h_{\mathcal{H}}(\mathbf{x})) \neq \mathcal{U}(M(\mathbf{x}'), Y(\mathbf{x}'), h_{\mathcal{H}}(\mathbf{x}')).$$

Proof. To prove this statement, we consider four cases, depending on the types of the positions $\mathbf{x}$ and $\mathbf{x}'$. For simplicity, we use $\mathcal{U}(\mathbf{x})$ for $\mathcal{U}(M(\mathbf{x}), Y(\mathbf{x}), h_{\mathcal{H}}(\mathbf{x}))$.

If $M(\mathbf{x}) \leq \binom{Y(\mathbf{x})+1}{2}$ and $M(\mathbf{x}') \leq \binom{Y(\mathbf{x}')+1}{2}$ then $\mathcal{U}(\mathbf{x}) = h_{\mathcal{H}}(\mathbf{x}) = h_{\mathcal{H}}(\mathbf{x}) > h_{\mathcal{H}}(\mathbf{x}') = h_{\mathcal{H}}(\mathbf{x}') = \mathcal{U}(\mathbf{x}')$, since every move strictly decreases the height, by its definition.

If $M(\mathbf{x}) \leq \binom{Y(\mathbf{x})+1}{2}$ and $M(\mathbf{x}') > \binom{Y(\mathbf{x}')+1}{2}$ then we have $\mathcal{U}(\mathbf{x}) = h_{\mathcal{H}}(\mathbf{x}) > M(\mathbf{x}) \geq M(\mathbf{x}') > v(M(\mathbf{x}'), Y(\mathbf{x}')) = \mathcal{U}(\mathbf{x}')$, proving the claim. Here the first two inequalities are implied by (i) of Lemma 5, while the last inequality follows by (ii) of this lemma.

If $M(\mathbf{x}) > \binom{Y(\mathbf{x})+1}{2}$ and $M(\mathbf{x}') \leq \binom{Y(\mathbf{x}')+1}{2}$ then we have $\mathcal{U}(\mathbf{x}') = h_{\mathcal{H}}(\mathbf{x}') \geq M(\mathbf{x})$, by (i) of Lemma 5, and $M(\mathbf{x}) > v(M(\mathbf{x}), Y(\mathbf{x})) = \mathcal{U}(\mathbf{x})$, by (ii) of this lemma.

Finally, if $M(\mathbf{x}) > \binom{Y(\mathbf{x})+1}{2}$ and $M(\mathbf{x}') > \binom{Y(\mathbf{x}')+1}{2}$ then, by (3b) and (i) of Lemma 5, we have either $M(\mathbf{x}) = M(\mathbf{x}')$ and $Y(\mathbf{x}) > Y(\mathbf{x}')$, or $M(\mathbf{x}) > M(\mathbf{x}')$.

If we have $Y(\mathbf{x}) > Y(\mathbf{x}')$, then $v(M(\mathbf{x}), Y(\mathbf{x})) \neq v(M(\mathbf{x}'), Y(\mathbf{x}'))$ follows by (1). If $Y(\mathbf{x}) = Y(\mathbf{x}')$ then by definition (1) we could have $v(M(\mathbf{x}), Y(\mathbf{x})) = v(M(\mathbf{x}'), Y(\mathbf{x}'))$ if and only if $M(\mathbf{x}') = M(\mathbf{x}) - \alpha(Y(\mathbf{x}) + 1)$ for some positive integer α, implying $M(\mathbf{x}') \leq M(\mathbf{x}) - Y(\mathbf{x}) - 1$. From this, by Lemma 5 (iii), it follows that $Y(\mathbf{x}') \geq Y(\mathbf{x}) + 1$ contradicting $Y(\mathbf{x}) = Y(\mathbf{x}')$. Hence, we must have $v(M(\mathbf{x}), Y(\mathbf{x})) \neq v(M(\mathbf{x}'), Y(\mathbf{x}'))$. $\square$

5. Proof of Theorem 1

In this section we prove that the family of JM+ games is closed under conjunctive compound. In fact, we show that the same holds for each of the three properties (MTF), (MD), and (C).

Lemma 7. *Assume that hypergraphs $\mathcal{H}^1 \subseteq 2^{V_1}$ and $\mathcal{H}^2 \subseteq 2^{V_2}$ satisfy property* (MTF) *and $V_1 \cap V_2 = \emptyset$. Then $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$ also satisfies* (MTF).

Proof. Let us note first that for any edge $H = H_1 \cup H_2$ of $\mathcal{H}$ with $H_i \in \mathcal{H}^i$, $i = 1, 2$, there exist edges $H_i' \in \mathcal{H}^i$, $i = 1, 2$, such that $H_i' \cap H_i = \emptyset$, since $\mathcal{H}^i$ are transversal free for $i = 1, 2$. Thus, $H' = H_1' \cup H_2' \in \mathcal{H}$ is disjoint from H implying that $\mathcal{H}$ is transversal free.

Let us consider next a proper subset $S \subsetneq V_1 \cup V_2$ such that $\mathcal{H}_S \neq \emptyset$, and define $S_i = S \cap V_i$ for $i = 1, 2$. Without loss of generality, we assume that $S_1 \neq V_1$. Then, by the minimal transversal freeness of $\mathcal{H}^1$, there exists an edge $H_1 \in \mathcal{H}^1_{S_1}$ that intersects all other edges of $\mathcal{H}^1_{S_1}$. Let us then consider an arbitrary edge $H_2 \in \mathcal{H}^2_{S_2}$. Such an H_2 exists since $\mathcal{H}_S \neq \emptyset$. Then the set $H = H_1 \cup H_2$ is a transversal edge of $\mathcal{H}_S$. $\square$

Lemma 8. *Assume that hypergraphs $\mathcal{H}^1 \subseteq 2^{V_1}$ and $\mathcal{H}^2 \subseteq 2^{V_2}$ satisfy property* (MD) *and $V_1 \cap V_2 = \emptyset$. Then $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$ also satisfies* (MD).

Proof. Let us consider a position $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2) \in \mathbb{Z}_+^V$, where $V = V_1 \cup V_2$, and note that

$$h_{\mathcal{H}}(\mathbf{x}) \;=\; \min\{h_{\mathcal{H}^1}(\mathbf{x}^1), h_{\mathcal{H}^2}(\mathbf{x}^2)\}.$$

By our assumptions, for both $i = 1, 2$ we have edges $H_i \in \mathcal{H}^i$ and H_i-moves $\mathbf{x}^i \to \mathbf{y}^i$ such that $h_{\mathcal{H}^i}(\mathbf{y}^i) = h_{\mathcal{H}^i}(\mathbf{x}^i) - 1$ and $m(\mathbf{y}^i) < m(\mathbf{x}^i)$. Then with the edge $H = H_1 \cup H_2 \in \mathcal{H}$ we can move from $\mathbf{x}$ to $\mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2)$ and have $h_{\mathcal{H}}(\mathbf{y}) = h_{\mathcal{H}}(\mathbf{x}) - 1$ and $m(\mathbf{y}) < m(\mathbf{x})$. $\square$

Lemma 9. *Assume that hypergraphs $\mathcal{H}^1 \subseteq 2^{V_1}$ and $\mathcal{H}^2 \subseteq 2^{V_2}$ satisfy property* (C) *and $V_1 \cap V_2 = \emptyset$. Then $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$ also satisfies* (C).

Proof. Let us consider two edges $H, H' \in \mathcal{H}$ and set $H_i = H \cap V_i$, $H_i' = H' \cap V_i$, and $S_i = H_i \cup H_i'$, for $i = 1, 2$. By our assumption, there are two chains $F_0, F_1, \dots, F_p$ in $\mathcal{H}^1_{S_1}$ and $G_0, G_1, \dots, G_r$ in $\mathcal{H}^2_{S_2}$ such that $F_0 = H_1$, $F_p = H_1'$, $G_0 = H_2$, and $G_r = H_2'$. Let us then define the following chain in $\mathcal{H}$:

$$F_0 \cup G_0,\; F_1 \cup G_0,\; \dots,\; F_p \cup G_0,\; F_p \cup G_1,\; \dots,\; F_p \cup G_r.$$

Then we have $H = F_0 \cup G_0$, $H' = F_p \cup G_r$, and all the above edges are contained by $H \cup H'$. $\square$

The following claim can easily be seen.

Lemma 10. *The conjunctive compound $\mathcal{F} = \mathcal{H} \otimes \binom{[k]}{1}$ for some positive integer k satisfies property* (C) *if $\mathcal{H}$ satisfies it or $\mathcal{H} = \binom{[\ell]}{1}$ for some positive integer ℓ.* □

Proof of Theorem 1. Since the operation $\oplus$ is associative and commutative, we can restrict ourselves to the case $p = 2$. If both games are JM+ then the claim follows by Lemmas 7, 8, and 9. It is also easy to see that the two pile $\text{NIM}_2 = \text{NIM}_{\binom{[2]}{1}}$ game satisfies both properties (MTF) and (MD). Thus, the statement follows by Lemma 10. □

6. Proof of Theorem 2

Let us note first that for $p = 1$ the statement of the theorem means that minimal transversal free minimum decreasing hypergraphs that have the chain property are JM. This was already proven in [8, Theorem 2]. Hence, we can assume in the sequel that $p \geq 2$, $\mathcal{H} = \mathcal{H}_1 \boxplus \cdots \boxplus \mathcal{H}_p \subseteq 2^V$, where $\mathcal{H}_i \subseteq 2^{V_i}$, $i = 1, \ldots, p$, and $V = V_1 \cup \cdots \cup V_p$.

To simplify notation we introduce $\mathcal{U}(\mathbf{x}) = \mathcal{U}(M(\mathbf{x}), Y(\mathbf{x}), h_{\mathcal{H}}(\mathbf{x}))$ for $\mathbf{x} \in \mathbb{Z}_+^V$. To prove the theorem we shall show that the function $\mathcal{U}(\mathbf{x})$ satisfies the sufficient conditions for a function to be an SG function, namely that

(D) for all $\mathbf{a} \in \mathbb{Z}_+^V$ and moves $\mathbf{a} \to \mathbf{b}$, we have $\mathcal{U}(\mathbf{a}) \neq \mathcal{U}(\mathbf{b})$;

(E) for all $\mathbf{a} \in \mathbb{Z}_+^V$ and values $0 \leq Z < \mathcal{U}(\mathbf{a})$ there exists a move $\mathbf{a} \to \mathbf{b}$ such that $\mathcal{U}(\mathbf{b}) = Z$.

Property (D) follows by Lemma 6, since JM+ hypergraphs are transversal free. To prove property (E), consider the next two cases:

(E1) for all $\mathbf{a} \in \mathbb{Z}_+^V$ with $M(\mathbf{a}) \leq \binom{Y(\mathbf{a})+1}{2}$ and integers Z with $M(\mathbf{a}) \leq Z < h_{\mathcal{H}}(\mathbf{a})$, there exists a move $\mathbf{a} \to \mathbf{b}$ such that $M(\mathbf{b}) \leq \binom{Y(\mathbf{b})+1}{2}$ and $h_{\mathcal{H}}(\mathbf{b}) = Z$;

(E2) for all $\mathbf{a} \in \mathbb{Z}_+^V$ and integers Z with $0 \leq Z < \min(M(\mathbf{a}), v(M(\mathbf{a}), Y(\mathbf{a}))))$, there exists a move $\mathbf{a} \to \mathbf{b}$ such that $M(\mathbf{b}) > \binom{Y(\mathbf{b})+1}{2}$ and $v(M(\mathbf{b}), Y(\mathbf{b})) = Z$.

It is easy to see that properties (E1) and (E2) imply property (E) by (2a) and (2b).

To prove (E1) let us consider a position $\mathbf{a} = (\mathbf{a}^1, \ldots, \mathbf{a}^p) \in \mathbb{Z}_+^V$ with $M(\mathbf{a}) \leq \binom{Y(\mathbf{a})+1}{2}$. By Lemma 3, there exist thresholds $m(\mathbf{a}^i) \leq t_i \leq h_{\mathcal{H}_i}(\mathbf{a}^i)$, $i = 1, ..., p$, satisfying the claims of the lemma. Let us set $T = t_1 + \cdots + t_p$.

For an integer Z with $T \leq Z < h_{\mathcal{H}}(\mathbf{a})$, let us choose integers z_i such that $t_i \leq z_i \leq h_{\mathcal{H}_i}(\mathbf{a}^i)$ and $Z = z_1 + \cdots + z_p$. Let us note that, by the above definitions, we must have $m(\mathbf{a}^i) = 0$ whenever $t_i = z_i = h_{\mathcal{H}_i}(\mathbf{a}^i)$. Let us define $Q = \{i \in [p] \mid z_i < h_{\mathcal{H}_i}(\mathbf{a}^i)\}$ and note that $Q \neq \emptyset$ and $m(\mathbf{a}^i) > 0$ for all $i \in Q$. Thus, by (L1) and (L2) of Lemma 3, for every $i \in Q$ there exists a move $\mathbf{a}^i \to \mathbf{b}^i$ such that $m(\mathbf{b}^i) \leq m(\mathbf{a}^i)$, $y_{\mathcal{H}_i}(\mathbf{b}^i) \geq y_{\mathcal{H}_i}(\mathbf{a}^i)$, and $h_{\mathcal{H}_i}(\mathbf{b}^i) = z_i$. We define $\mathbf{b}^i = \mathbf{a}^i$ for $i \in [p] \setminus Q$, and set $\mathbf{b} = (\mathbf{b}^1, \dots, \mathbf{b}^p)$. In this way we get that $\mathbf{a} \to \mathbf{b}$ is a move in $\text{N}\textsc{im}_{\mathcal{H}}$, satisfying $M(\mathbf{b}) \leq M(\mathbf{a})$, $Y(\mathbf{b}) \geq Y(\mathbf{a})$, and $h_{\mathcal{H}}(\mathbf{b}) = Z$. Thus, by our assumptions, it follows that $M(\mathbf{b}) \leq M(\mathbf{a}) \leq \binom{Y(\mathbf{a})+1}{2} \leq \binom{Y(\mathbf{b})+1}{2}$.

For an integer Z with $M(\mathbf{a}) \leq Z < T$, choose integers $m(\mathbf{a}^i) \leq z_i \leq t_i$ such that $Z = z_1 + \cdots + z_p$, and define $Q = \{i \in [p] \mid z_i < h_{\mathcal{H}_i}(\mathbf{a}^i)\}$ as above. We have $Q \neq \emptyset$, because $Z < T$. Furthermore, for $i \in [p] \setminus Q$ we have $t_i = h_{\mathcal{H}_i}(\mathbf{a}^i)$, implying $m(\mathbf{a}^i) = 0$ by (L0) of Lemma 3. By (L2) and (L3) of Lemma 3, for every $i \in Q$ there exists a move $\mathbf{a}^i \to \mathbf{b}^i$ such that $m(\mathbf{b}^i) = 0$ and $h_{\mathcal{H}_i}(\mathbf{b}^i) = z_i$. We define $\mathbf{b}^i = \mathbf{a}^i$ for $i \in [p] \setminus Q$, and set $\mathbf{b} = (\mathbf{b}^1, \dots, \mathbf{b}^p)$. Then $\mathbf{a} \to \mathbf{b}$ is a move in $\text{N}\textsc{im}_{\mathcal{H}}$ satisfying $M(\mathbf{b}) = 0$ and $h_{\mathcal{H}}(\mathbf{b}) = Z$. Thus, it trivially follows that $0 = M(\mathbf{b}) \leq \binom{Y(\mathbf{b})+1}{2}$. This completes the proof of property (E1).

To prove property (E2) we need a few more observations. Note that by (1), for a fixed integer $y \in \mathbb{Z}_+$ we have

$$U(y) = \{v(m,y) \mid m \in \mathbb{Z}_+\} \;=\; \left[\binom{y+1}{2}, \binom{y+1}{2} + y\right],$$

and also that the sets $U(y)$, $y \in \mathbb{Z}_+$ partition the set of nonnegative integers. Consequently, for every integer $z \in \mathbb{Z}_+$ there exists a unique integer $y \in \mathbb{Z}_+$ such that $z \in U(y)$. We denote this unique integer as $y = \eta(z)$. Note also that for every integer z we have

$$z \;=\; v(z+1, \eta(z)) \quad \text{and} \quad z+1 \;>\; \binom{\eta(z)+1}{2}. \tag{8}$$

Consider a position $\mathbf{a} \in \mathbb{Z}_+^V$ and a value $0 \leq Z < \min(M(\mathbf{a}), v(M(\mathbf{a}), Y(\mathbf{a})))$, as in (E2), and choose a largest integer $\alpha \geq 0$ such that $M = Z + 1 + \alpha(\eta(Z)+1) \leq M(\mathbf{a})$ and set $Y = \eta(Z)$. Note that we have

$$0 \leq M(\mathbf{a}) - M \leq \eta(Z). \tag{9}$$

We construct a position $\mathbf{b} \in \mathbb{Z}_+^V$ such that $\mathbf{a} \to \mathbf{b}$ is a move in $\text{N}\textsc{im}_{\mathcal{H}}$, $M(\mathbf{b}) = M$, and $Y(\mathbf{b}) = Y$. By (8) and the fact that $\alpha \geq 0$, this construction verifies property (E2), and completes our proof of the theorem.

To see the construction, note first that $M \leq M(\mathbf{a}) = \sum_{i=1}^{p} m(\mathbf{a}^i)$ by our choices above. Thus, there exist integer values μ_i, $i = 1, \dots, p$, such that

$$\sum_{i=1}^{p} \mu_i \;=\; M, \text{ and } 0 \leq \mu_i \leq m(\mathbf{a}^i) \text{ for all } i = 1, \dots p.$$

Then observe that we have

$$\sum_{i=1}^{p} (m(\mathbf{a}^i) - \mu_i) \;=\; M(\mathbf{a}) - M \;\leq\; Y \;\leq\; Y(\mathbf{a}) \;=\; \sum_{i=1}^{p} y_{\mathcal{H}_i}(\mathbf{a}^i),$$

by (9) and by our definitions (3a) and (3b). Thus, there exist integers η_i for $i = 1, \dots, p$ such that

$$m(\mathbf{a}^i) - \mu_i \;\leq\; \eta_i \;\leq\; y_{\mathcal{H}_i}(\mathbf{a}^i) \quad \text{for all } i = 1, \dots, p,$$

and $Y = \sum_{i=1}^{p} \eta_i$. Define $Q = \{i \in [p] \mid (\mu_i, \eta_i) \neq (m(\mathbf{a}^i), y_{\mathcal{H}_i}(\mathbf{a}^i))\}$. By (E2) we have $v(M, Y) = Z < v(m(\mathbf{a}), Y(\mathbf{a}))$ implying $(M, Y) \neq (m(\mathbf{a}), Y(\mathbf{a}))$, and thus, $Q \neq \emptyset$.

Now we can apply Lemma 4 for each of the JM+ games $\mathcal{H}_i$, $i \in Q$, and derive the existence of moves $\mathbf{a}^i \to \mathbf{b}^i$ in $\text{NIM}_{\mathcal{H}_i}$ such that $m(\mathbf{b}^i) = \mu_i$ and $y_{\mathcal{H}_i}(\mathbf{b}^i) = \eta_i$. Defining $\mathbf{b}^i = \mathbf{a}^i$ for $i \in [p] \setminus Q$ and setting $\mathbf{b} = (\mathbf{b}^1, \dots, \mathbf{b}^p)$ completes our construction and the proof of the theorem. $\square$

7. Combinatorial Properties of JM+ Hypergraphs

In this section we show that no two of the three properties (MTF), (MD), and (C) imply neither the third one, nor JM. For a hypergraph $\mathcal{H} \subseteq 2^V$ we denote by $\min \mathcal{H}$ the family of inclusionwise minimal edges of $\mathcal{H}$. For a subhypergraph $\mathcal{F} \subseteq \mathcal{H}$ we denote by $V(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} F$ the set of vertices that it covers. Along with property (MD) consider the following stronger combinatorial property (MD*).

(MD*) For every subhypergraph $\mathcal{F} \subseteq \min \mathcal{H}$ such that $V(\mathcal{F}) \neq V$ there exist edges $F \in \mathcal{F}$ and $H \in \mathcal{H}$ such that $H \cap V(\mathcal{F}) \subseteq F$ and $H \setminus F \neq \emptyset$.

It will be technically easier to verify (MD*), since (MD) involves " for all $\mathbf{x} \in \mathbb{Z}_+^V$".

A move $\mathbf{x} \to \mathbf{x}'$ is called a *height move* if $h_{\mathcal{H}}(\mathbf{x}') = h_{\mathcal{H}}(\mathbf{x}) - 1$.

Lemma 11. *Property* (MD*) *implies* (MD).

Proof. Consider a position $\mathbf{x} \in \mathbb{Z}_+^V$ with $m(\mathbf{x}) > 0$ and define $\mathcal{F}(\mathbf{x}) \subseteq \mathcal{H}$ to be the subhypergraph of those edges $H \in \mathcal{H}$ for which there exists a $\mathbf{x} \to \mathbf{x}'$ H-move such that $h_{\mathcal{H}}(\mathbf{x}') = h_{\mathcal{H}}(\mathbf{x}) - 1$. If $V(\mathcal{F}(\mathbf{x})) = V$ for all positions $\mathbf{x} \in \mathbb{Z}_+^V$, then property (MD) holds. Otherwise there exists a position $\mathbf{x}$ with $m(\mathbf{x}) > 0$ such that $V(\mathcal{F}(\mathbf{x})) \subsetneq V$. Without loss of generality, we can assume that $\mathcal{F} \subseteq \min \mathcal{H}$. Thus, by property (MD*), we have edges $F \in \mathcal{F}(\mathbf{x})$ and $H \in \mathcal{H}$ such that $H \cap V(\mathcal{F}(\mathbf{x})) \subseteq F$ and $H \setminus V(\mathcal{F}(\mathbf{x})) \neq \emptyset$. Clearly, we can assume that $H \in \min \mathcal{H}$. Then there exists a sequence of height moves that involves F by the definition of $\mathcal{F}(\mathbf{x})$. In this sequence replace one F-move by an H-move. This way we get another height sequence, and that contradicts the fact that $H \notin \mathcal{F}(\mathbf{x})$. This contradiction proves our claim. $\square$

Let us add that the inverse implication is not true. The following small example shows this. Consider $V = \{1, 2, 3, 4, 5\}$, $H_1 = \{1, 2\}$, $H_2 = \{2, 3\}$, $H_3 = \{3, 4\}$, $H_4 = \{1, 4, 5\}$, and $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$. For the subhypergraph $\mathcal{F} =$

$\{H_1, H_2, H_3\}$, property (MD*) fails to hold. It is however not difficult to see that $\mathcal{H}$ satisfies property (MD).

For a subset $S \subseteq [n]$ we denote by $\binom{[n]}{S}$ the hypergraph consisting of all edges $H \subseteq V$ such that $|H| \in S$. If $S = \{i\}$, then we simply write $\binom{[n]}{i}$. These hypergraphs are called *symmetric*. We say that $S \subseteq [n]$ has a gap, if there are integers $0 < i < j < k \leq n$ such that $i, k \in S$ and $j \notin S$.

Remark 1. It is easy to see that symmetric hypergraphs satisfy property (MD*). Furthermore, if S has a gap, then $\binom{[n]}{S}$ does not satisfy property (C). We also recall from [9] that symmetric JM hypergraphs have a simple arithmetic characterization.

Due to our results, properties (MTF), (MD), (C), and JM define 10 possible regions, one of which is related to the JM+ hypergraphs; see Figure 1. We will show that, among the remaining 9 regions, 7 are not empty; the status of the last two is open.

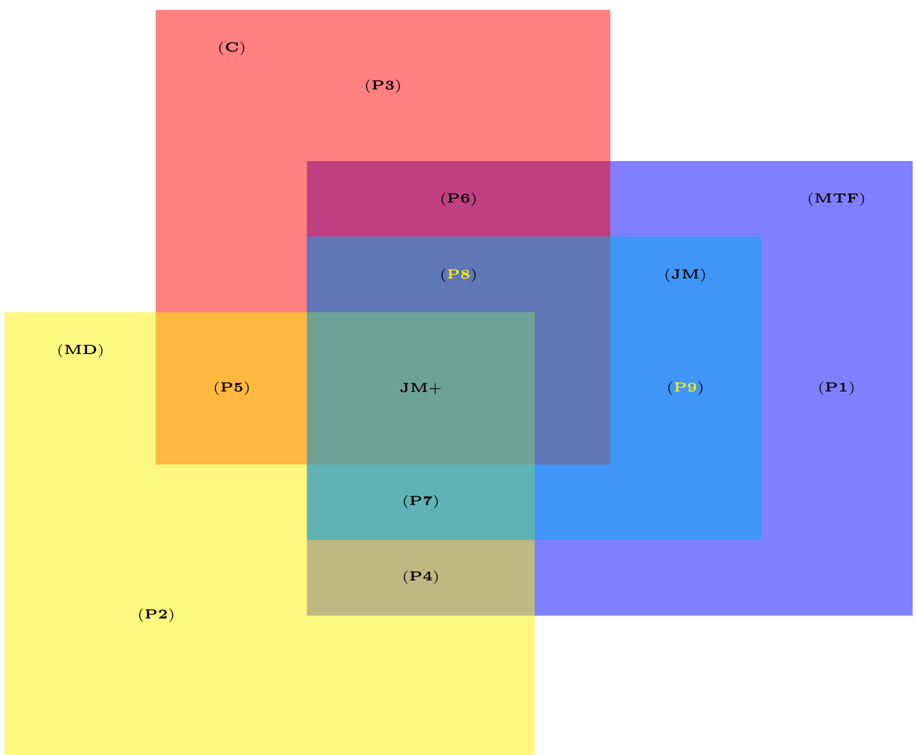

Figure 1: The 10 regions defined by properties (MTF), (MD), (C), and (JM).

Theorem 3. *The following seven (negative) statements hold:*

(P1) *Property* (MTF) *implies none of JM,* (MD)*, and* (C)*.*

(P2) *Property* (MD) *implies neither* (MTF) *nor* (C)*.*

(P3) *Property* (C) *implies neither* (MTF) *nor* (MD)*.*

(P4) *The conjunction of* (MTF) *and* (MD) *does not imply JM, and hence, it does not imply* (C)*.*

(P5) *The conjunction of* (MD) *and* (C) *does not imply JM, and hence, it does not imply* (MTF)*.*

(P6) *The conjunction of* (MTF) *and* (C) *does not imply JM, and hence, it does not imply* (MD)*.*

(P7) *The conjunction of JM and* (MD) *does not imply* (C)*.*

Proof. For (P1) we consider the "cube", that is, the hypergraph with 8 vertices and 6 edges corresponding to the 2 dimensional faces of a 3-dimensional cube. It was shown in [8, Section 7] that this hypergraph satisfies property (MTF), but none of the others.

For (P2) we consider, for example, $\binom{[4]}{\{1,3\}}$. As we noted above, it satisfies (MD*), and hence, (MD), but not (C). It does not satisfy (MTF) either, since if S is a subset of size 2, then the induced subhypergraph does not have a transversal edge.

For (P3) we consider the following hypergraph on 10 vertices: define

$T_1 = \{v_1, v_2, v_3\}$, $T_2 = \{v_4, v_5, v_6\}$, $T_3 = \{v_7, v_8, v_9\}$, $V = \{v_0, v_1, \ldots, v_9\}$, and

$$\mathcal{H} = \{T_1, T_2, T_3\} \cup \left\{ H \subseteq V \;\middle|\; \begin{array}{l} |H| = 4 \text{ and } v_0 \notin H, \text{ or} \\ |H| = 5 \text{ and } v_0 \in H \end{array} \right\}.$$

This hypergraph does not satisfy (MTF), since if we delete v_0, the reduced hypergraph still has no transversal edge. It does not satisfy property (MD) either, because for position $\mathbf{x} = (1, 2, 2, 2, 2, 2, 2, 2, 2, 2)$ the only height moves are with sets T_i, $i = 1, 2, 3$. Finally, it is easy to verify that it satisfies property (C).

For claims (P4), (P5), and (P7) it is enough to consider symmetric hypergraphs $\binom{[2]}{1}$, $\binom{[5]}{2}$, and $\binom{[6]}{\{2,4\}}$, respectively.

Finally, for (P6) we consider a 10-vertex hypergraph similar to one considered for (P3):

$$\mathcal{H} = \{T_1, T_2, T_3\} \;\cup\; \left\{ H \subseteq V \;\middle|\; \begin{array}{l} |H| = 4,\; v_0 \notin H, \text{ and } H \cap \{v_1, v_4, v_7\} \neq \emptyset, \text{ or} \\ |H| = 5 \end{array} \right\}.$$

To see property (MTF), note first that $\mathcal{H}$ has no transversal edge. Furthermore, if $S \subseteq V$ has $|S| \leq 5$, then any two edges of $\mathcal{H}_S$ intersect. If $|S| \geq 6$, then any edge $H \in \mathcal{H}_S$ such that $|H| = 5$ and $H \supseteq \{v_1, v_4, v_7\} \cap S$ is a transversal of $\mathcal{H}_S$.

For property (C), consider two edges $H, H' \in \mathcal{H}$. If $|H \cup H'| \leq 4$, then clearly, a chain from H to H' exists. On the other hand, if $|H \cup H'| \geq 5$, choose an edge $H'' \subseteq H \cup H'$ of size 5. Note that we can reach H' from H by a chain through H''. Thus, $\mathcal{H}$ satisfies (C).

To show that $\mathcal{H}$ is not JM, consider the position $\mathbf{x} = (6, 7, \ldots, 7)$. Note that $m(\mathbf{x}) = 6$, $y_{\mathcal{H}}(\mathbf{x}) = 3$, and $h_{\mathcal{H}}(\mathbf{x}) = 21$. Since $\binom{y_{\mathcal{H}}(\mathbf{x})+1}{2} = m(\mathbf{x})$, we have $\mathcal{U}(\mathbf{x}) = h_{\mathcal{H}}(\mathbf{x}) = 21$. We will show that there exists no move from $\mathbf{x} \to \mathbf{x}'$ such that $\mathcal{U}(\mathbf{x}') = 20$. Assume that such a move exists. Since $m(\mathbf{x}') \leq m(\mathbf{x}) = 6$, position $\mathbf{x}'$ is long with $h_{\mathcal{H}}(\mathbf{x}') = 20$. This implies that $\mathbf{x} \to \mathbf{x}'$ is a height move. For this position $\mathbf{x}$, the only height moves are with T_i, $i = 1, 2, 3$. Consequently, $y_{\mathcal{H}}(\mathbf{x}') < 3$ and $m(\mathbf{x}') = 6$, implying that $\mathbf{x}'$ is a short position. Thus, $\mathbf{x}'$ is both short and long, a contradiction. $\square$

8. Open Problems

For the next two negative statements we have no examples; in other words, we do not know if the corresponding two regions in Figure 1 are empty:

(P8) The conjunction of JM and (C) does not imply (MD);

(P9) Property JM does not imply the disjunction of (MD) and (C).

We also do not know if JM+ could be replaced by JM in Theorems 1 or 2.

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