SIERPIŃSKI AND RIESEL NUMBERS IN NARAYANA'S COW SEQUENCE

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Abstract

In this paper, we show that there are infinitely many Sierpiński numbers and infinitely many Riesel numbers in Narayana's Cow Sequence.

1. Introduction

In 1956, Riesel [16] proved that there are infinitely many odd integers k such that $k \cdot 2^n - 1$ is composite for all positive integers n. Four years later, in 1960, Sierpiński [17] proved that there are infinitely many odd integers ℓ such that $\ell \cdot 2^n + 1$ is composite for all positive integers n. Today, such integers k and ℓ are called *Riesel numbers* and *Sierpiński numbers*, respectively. Since the original findings of Riesel and Sierpiński, many papers have been published that investigate the existence of Riesel numbers and/or Sierpiński numbers in various other integer sequences:

- binomial coefficients [1];
- polygonal numbers [3, 4];
- Lucas sequence [5].
- Carmichael numbers [6];
- perfect powers [10, 12, 14];
- Ruth-Aaron pairs [11];
- in the image of various polynomials [13];
- Fibonacci sequence [15];

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In this article, we show that there are infinitely many Riesel numbers and infinitely many Sierpiński numbers in Narayana's cow sequence. We call such numbers *NC-Riesel numbers* and *NC-Sierpiński numbers*, respectively.

Narayana's cow sequence is the ternary recurrent sequence satisfying $N_0 = 0$, $N_1 = N_2 = 1$ and $N_t = N_{t-1} + N_{t-3}$ for all $t \geq 3$. The first fifteen terms of the sequence are

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129.$$

Recent interest in Narayana's cow sequence has produced several papers that explore the intersection of this sequence with other sequences [7, 8, 9].

2. Constructing Riesel and Sierpiński Numbers

The method commonly used to produce Riesel or Sierpiński numbers involves using a covering system of the integers. A *covering system of the integers* is a collection of congruences such that every integer satisfies at least one congruence in the collection. For example, the collection

$$\begin{aligned} n &\equiv 1 \pmod{2} \\ n &\equiv 0 \pmod{3} \\ n &\equiv 0 \pmod{4} \\ n &\equiv 6 \pmod{8} \\ n &\equiv 2 \pmod{12} \\ n &\equiv 10 \pmod{24} \end{aligned} \tag{1}$$

is a covering system of the integers. To see this, one can check that every positive integer, up to the lease common multiple of the moduli, satisfies at least one of the congruences in the collection. For an integer $m \geq 2$, let p_m be a prime divisor of $2^m - 1$. When $n \equiv r \pmod{m}$, notice that

$$k \cdot 2^n - 1 \equiv k \cdot 2^r - 1 \pmod{p_m}.$$

Thus, if $n \equiv r \pmod{m}$ and $k \equiv 2^{-r} \pmod{p_m}$, then $k \cdot 2^n - 1$ is divisible by p_m. With the covering system in (1), choosing k to satisfy

$$\begin{aligned} k &\equiv 2 \pmod{3} \\ k &\equiv 1 \pmod{7} \\ k &\equiv 1 \pmod{5} \\ k &\equiv 4 \pmod{17} \\ k &\equiv 10 \pmod{13} \\ k &\equiv 237 \pmod{241} \end{aligned} \tag{2}$$

ensures that for any positive integer n, $k \cdot 2^n - 1$ is divisible by at least one of the primes in the set $\{3, 5, 7, 13, 17, 241\}$. Hence, using the Chinese remainder theorem, we can find infinitely many Riesel numbers by further choosing $k \equiv 1 \pmod 2$ with $2k - 1 > 241$. Consequently, any $k \equiv 8086751 \pmod{11184810}$ is a Riesel number. A similar method shows that any $\ell \equiv 3098059 \pmod{11184810}$ is a Sierpiński number.

We note that the method described above relies on the moduli of (2) being pairwise relatively prime. The following theorem of Bang [2] ensures that any covering system with distinct moduli will yield infinitely many Riesel numbers and infinitely many Sierpiński numbers, provided 6 is not a modulus in the covering system.

Theorem 1 (Bang)**.** *For any integer $m > 1$ and $m \neq 6$, there exists a prime p such that p divides $2^m - 1$ and p does not divide $2^{\tilde{m}} - 1$ for any $\tilde{m} < m$.*

3. NC-Riesel and NC-Sierpiński Numbers

From our discussion in Section 2, any N_t in Narayana's cow sequence satisfying $N_t \equiv 8086751 \pmod{1184810}$ will be a NC-Riesel number. For integers a and $m \geq 2$, let $\mathcal{I}(a, m) = \{j : N_j \equiv a \pmod m\}$. Then for any

$$\begin{aligned} t &\in \mathcal{I}(8086751, 1184810) \\ &= \mathcal{I}(1,2) \cap \mathcal{I}(2,3) \cap \mathcal{I}(1,7) \cap \mathcal{I}(1,5) \cap \mathcal{I}(4,17) \cap \mathcal{I}(10,13) \cap \mathcal{I}(237,241), \end{aligned}$$

N_t will be a NC-Riesel number.

It is useful to note that for $m \geq 2$, Narayana's cow sequence is periodic modulo m. For example,

$$\begin{aligned} N_0 &= 0 \\ N_1 &= 1 \\ N_2 &= 1 \\ N_3 &= 1 \\ N_4 &= 2 \equiv 0 \pmod 2 \\ N_5 &= 3 \equiv 1 \pmod 2 \\ N_6 &= 4 \equiv 0 \pmod 2 \\ N_7 &= 6 \equiv 0 \pmod 2 \\ N_8 &= 9 \equiv 1 \pmod 2 \\ N_9 &= 13 \equiv 1 \pmod 2, \end{aligned} \tag{3}$$

and we see that $N_t \equiv N_{n-7} \pmod 2$ for every $t \geq 7$. Thus, $t \in \mathcal{I}(1,2)$ if and only if $t \equiv x \pmod 7$ for some $x \in \{1, 2, 3, 5\}$.

Due to the periodic nature of Narayana's cow sequence modulo m, we denote the length of the period modulo m by $\mathcal{P}(m)$, and we modify our previous notation so that

$$\mathcal{I}(a,m) \pmod{\mathcal{P}(p)} := \{j \pmod{\mathcal{P}(p)} : N_j \equiv a \pmod{m} \text{ with } 0 \le j \le \mathcal{P}(p)-1\}.$$

Then

$$\begin{aligned}
\mathcal{I}(1,2) \pmod{7} &= \{1,2,3,5\}\\
\mathcal{I}(2,3) \pmod{8} &= \{4\}\\
\mathcal{I}(1,7) \pmod{57} &= \{1,2,3,23,27,44,51,52,55\}\\
\mathcal{I}(1,5) \pmod{31} &= \{1,2,3,7,12,18,25,26,29\}\\
\mathcal{I}(4,17) \pmod{288} &= \{6,36,38,79,103,114,127,146,189,247,257,\\
&\quad 260,269,273\}\\
\mathcal{I}(10,13) \pmod{168} &= \{14,19,41,62,73,77,108,127,157\}\\
\mathcal{I}(237,241) \pmod{9680} &= \{70,766,842,1038,1125,1185,1415,1461,1856,\\
&\quad 2030,2240,2280,2489,2984,2998,3657,4162,\\
&\quad 4209,4513,4605,4663,4827,4977,5206,5423,\\
&\quad 6227,6287,6539,6735,7344,7993,8012,8141,\\
&\quad 8654,8934,9092,9385,9607\}.
\end{aligned}$$

Considering the intersection of these sets yields the following theorem.

Theorem 2. *If* $t \equiv x \pmod{718391520}$ *for some*

$$\begin{aligned}
x \in \{&3629412,13386852,73151172,96325092,101203812,119499012,\\
&175604292,198778212,212194692,240247332,268299972,328064292,\\
&337821732,360995652,384169572,402464772,425638692,467107812,\\
&476865252,495160452,564682212,587856132,592734852,611030052,\\
&629325252,652499172,703725732\},
\end{aligned}$$

then N_t *is a NC-Riesel number.*

To obtain an analogous result for Sierpiński numbers, we look for a number in Narayana's cow sequence satisfying $N_t \equiv 3098059 \pmod{11184810}$. In other words, we want to find

$$\begin{aligned}
t &\in \mathcal{I}(8086751,1184810)\\
&= \mathcal{I}(1,2)\cap\mathcal{I}(1,3)\cap\mathcal{I}(6,7)\cap\mathcal{I}(4,5)\cap\mathcal{I}(13,17)\cap\mathcal{I}(3,13)\cap\mathcal{I}(4,241).
\end{aligned}$$

There are 180 values $0 \le x < 718391520$ in the intersection of the sets

$$\begin{aligned}
\mathcal{I}(1,2) \pmod{7} &= \{1,2,3,5\}\\
\mathcal{I}(1,3) \pmod{8} &= \{1,2,3,6,9,10\}\\
\mathcal{I}(6,7) \pmod{57} &= \{7,9,12,29,43,53\}\\
\mathcal{I}(4,5) \pmod{31} &= \{6,8,10,15,16,27\}\\
\mathcal{I}(13,17) \pmod{288} &= \{9,28,31,48,84,86,89,124,130,149,165,168,175,\\
&\quad 176,200,212,246\}\\
\mathcal{I}(3,13) \pmod{168} &= \{5,18,31,40,64,82,84,95,99,106,107,110,113,114,\\
&\quad 115,132,134,159\}\\
\mathcal{I}(4,241) \pmod{9680} &= \{6,224,424,555,852,905,1313,1513,1762,1989,2283,\\
&\quad 2528,2660,2755,2929,2963,3114,3242,3373,3769,3859,\\
&\quad 3928,4026,4057,4293,4577,4741,4840,4861,5031,5158,\\
&\quad 5259,5401,5966,6484,6572,6700,6786,6991,8558,8684,\\
&\quad 9041,9360,9549,9634,9665\}.
\end{aligned}$$

The smallest is $x = 1966553$. Hence, for any $t \equiv 1966553 \pmod{718391520}$, N_t is a NC-Sierpiński number. This leads us to the following theorem.

Theorem 3. *There are infinitely many NC-Sierpiński numbers.*

We noted at the end of Section 2 that any covering system with distinct moduli will yield infinitely many Riesel numbers and infinitely many Sierpiński numbers, provided 6 is not a modulus in the covering system. However, these covering systems are not guaranteed to produce NC-Riesel or NC-Sierpiński numbers. For example, any $k \equiv 4442323 \pmod{11184810}$ can be shown to be a Riesel number by using the method described in Section 2 and the covering system

$$\begin{aligned}
n &\equiv 0 \pmod{2}\\
n &\equiv 1 \pmod{3}\\
n &\equiv 1 \pmod{4}\\
n &\equiv 7 \pmod{8}\\
n &\equiv 11 \pmod{12}\\
n &\equiv 3 \pmod{24}.
\end{aligned}$$

However, $\mathcal{I}(4442323, 11184810)$ is an empty set.

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