ON THE a-POINTS OF THE k^{th} DERIVATIVE OF AN L-FUNCTION IN THE SELBERG CLASS

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Abstract

Let $F(s)$ be a function from the Selberg class, $F^{(k)}(s)$ be the k^{th} derivative of $F(s)$, and a be a complex number. The solutions of $F^{(k)}(s)=a$ are called a-points of $F^{(k)}(s)$. In this paper, we study the distribution of the a-points of $F^{(k)}(s)$ and estimate the number of these a-points. Furthermore, we give an asymptotic formula for

$$\sum_{\rho_a^{(k)}:1<\gamma_a^{(k)}<T} x^{\rho_a^{(k)}} \quad as \qquad T\to\infty,$$

where x is a positive real number such that $x>1$ and $\rho_a^{(k)}=\beta_a^{(k)}+i\gamma_a^{(k)}$ denotes an a-point of the k^{th} derivative of $F(s)$.

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1. Introduction

An L-function in number theory is often defined by a Dirichlet series, or an Eulerian product on prime numbers, and satisfies certain analytical properties. The typical example of an L-function is the Riemann zeta function; it plays an important role in mathematical research, constitutes a first link between arithmetic and analysis and was used by Euler, Dirichlet, Tchebychev and Riemann to study the distribution of prime numbers. The distribution of the zeros of the Riemann zeta function $\zeta(s)$, as well as of other zeta and L-functions, is related to important questions in number theory. The famous (yet unproved) Riemann hypothesis, which claims that all nontrivial (nonreal) zeros of $\zeta(s)$ lie on the critical line $s = \frac{1}{2} + it$ with $t \in \mathbb{R}$, is of special interest. The Riemann hypothesis is the subject of several studies and research papers, has a variety of equivalent statements, and may be extended to a large class of L-functions (see for instance [2] and [10]). Dixit et al. [3] studied the vertical shifts s to $s + i\tau$. In [8], Li and Radziwill established results on the distribution of values of $\zeta\left(\frac{1}{2} + i(al + b)\right)$, where a and b are real numbers with $a > 0$, and l ranges over the integers in some dyadic interval $[T, 2T]$. Gonek [4] developed a uniform version of an explicit formula known as *Landau's formula* (it provides a connection between zeros of the zeta function and primes) which can be used in applications. The study of the derivatives of zeta functions was inspired by Speiser's paper [16]. Moreover, he proved that the Riemann hypothesis is equivalent to the absence of zeros of the derivative of the Riemann zeta function to the left of the critical line. Berndt [1] studied the number of zeros of the higher-order derivatives of the Riemann zeta function, and Spira [17] investigated the zero-free regions of the higher-order derivatives of the Riemann zeta function. Levinson and Montgomery [7] proved that, under certain conditions, the Riemann zeta function and its derivative have approximately the same number of zeros to the left of the critical line. They also studied the zeros of the higher-order derivatives of the Riemann zeta function, and assuming the Riemann hypothesis, they proved that there are a finite number of such zeros in the critical strip. Yildirim [20] considered the number of zeros of the first and higher-order derivatives of the Dirichlet L-functions. He obtained zero-free regions and an estimate of the number of zeros for these functions.

Let $F(s)$ be a function from the Selberg class, $F^{(k)}(s)$ be the k^{th} derivative of $F(s)$, and a be a complex number. The solutions of $F^{(k)}(s) = a$ are called *a-points* of $F^{(k)}(s)$. In this paper, we study the distribution of the a-points of $F^{(k)}(s)$ and we estimate the number of these a-points.

Theorem 1. *Let $F(s)$ be a function belonging to the Selberg class, k be a positive integer and a be a nonzero complex number. Then, for sufficiently large T we have*

$$N_k(a; T, F) = \frac{d_F T}{2\pi} \log T + \frac{T}{2\pi} \log\left(\lambda Q^2\right) - \frac{d_F T}{2\pi} + O\left(\log T\right),$$

where $N_k(a;T,F)$ denotes the number of a-points of $F^{(k)}(s)$ in the region $0<t<T$ and $E_2<\sigma<E_1$, where E_1 and E_2 are from Lemmas 1 and 3, respectively, as given below.

Remark 1. The case $a=0$ was already proved by R. Šimenas in [14]. In fact, he calculated the number of zeros of $F^{(k)}(s)$ in the region $|t|<T$ and $E_2<\sigma<E_1$; he obtained the following result:

$$\begin{aligned} N_k(0;-T,T,F) &= \sum_{\substack{\rho_0^{(k)}:\ -T<\gamma_0^{(k)}<T \\ E_2<\beta_0^{(k)}<E_1}} 1 \\ &= \frac{d_F T}{\pi}\log T + \frac{T}{\pi}\log\left(\lambda Q^2\right) - \frac{d_F T}{\pi} - \frac{T}{\pi}\log\ell + O\left(\log T\right), \end{aligned}$$

where $\ell \in \mathbb{N}$ is the least number such that the Dirichlet series coefficient a_ℓ for $F^{(k)}(s)$ does not vanish. In a similar way to Šimenas's work, we can easily prove the following result:

$$N_k(0;T,F) = \frac{d_F T}{2\pi}\log T + \frac{T}{2\pi}\log\left(\lambda Q^2\right) - \frac{d_F T}{2\pi} - \frac{T}{2\pi}\log\ell + O\left(\log T\right).$$

Remark 2. For the case $k=0$, we refer to Steuding's book [18, Chapter 7].

Theorem 2. *Let $F(s)$ be a function belonging to the Selberg class, k be a positive integer and a be a complex number. Then, for $x>1$ and sufficiently large T we have*

$$\begin{aligned} \sum_{\substack{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T \\ E_2<\beta_a^{(k)}<E_1}} x^{\rho_a^{(k)}} &= \frac{T}{2\pi} \sum_{\substack{l\geq 0 \\ n_0,n_1,\ldots,n_l\geq \ell \\ x=n_0n_1\ldots n_l}} \frac{(-1)^{k(l+1)}}{a^{l+1}}(\log n_0)^{k+1} \\ &\quad \times(\log n_1 \log n_2 \ldots \log n_l)^k a_{n_0}a_{n_1}\ldots a_{n_l} + O(\log T), \ \text{if } (a\neq 0) \end{aligned}$$

and

$$\begin{aligned} \sum_{\substack{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T \\ E_2<\beta_a^{(k)}<E_1}} x^{\rho_a^{(k)}} &= \frac{T}{2\pi} \sum_{\substack{l\geq 0 \\ n_0\geq \ell \\ n_1,\ldots,n_l\geq \ell+1 \\ x=n_0n_1\ldots n_l/\ell^{l+1}}} \left(\frac{-1}{a_\ell \log^k \ell}\right)^{l+1}(\log n_0)^{k+1} \\ &\quad \times(\log n_1 \log n_2 \ldots \log n_l)^k a_{n_0}a_{n_1}\ldots a_{n_l} + O(\log T), \ \text{if } (a=0), \end{aligned}$$

where $\ell \in \mathbb{N}$ is the least number such that the Dirichlet series coefficient a_ℓ for $F^{(k)}(s)$ does not vanish, $\rho_a^{(k)}=\beta_a^{(k)}+i\gamma_a^{(k)}$ denotes an a-point of the k^{th} derivative $F^{(k)}(s)$, while E_1 and E_2 are from Lemmas 1 and 3, respectively, as given below. If $a\neq 0$, the sum on the right-hand side is zero for $x\notin\mathbb{Z}$, and if $a=0$ with $\ell^n x\notin\mathbb{Z}$ for any $n\in\mathbb{N}$, then the sum on the right-hand side is zero.

2. Preliminary Lemmas and Equations

The Selberg class $\mathcal{S}$ was introduced by Selberg [13]. It consists of the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \qquad (\mathrm{Re}\,(s) > 1),$$

which satisfy the following conditions:

- *Ramanujan hypothesis*: $a(n) = O(n^{\epsilon})$.
- *Euler product*: For s with sufficiently large real part,
$$F(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right),$$
with suitable coefficients $b(p^k)$ satisfying $b(p^k) = O(p^{k\theta})$ for some $\theta < \frac{1}{2}$.
- *Analytic continuation*: There exists a non-negative integer m such that $(s-1)^m F(s)$ is an entire function of finite order (in the sequel m_F denotes the smallest integer m with this property).
- *Functional equation*: For $1 \leq j \leq r$, there exist positive real numbers Q_F, λ_j, and complex numbers μ_j, ω with $Re(\mu_j) \geq 0$ and $|\omega| = 1$ such that $\phi_F(s) = \omega\overline{\phi_F(1-\overline{s})}$, where
$$\phi_F(s) = F(s) Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j). \tag{2.1}$$

It follows from the Euler product that $F \in \mathcal{S}$ has no zeros in the half-plane $\sigma > 1$. Also, F admits zeros in the left-half plane, which are generated by the poles of the Γ-factor appearing in the functional equation. These zeros are called trivial and they are located at the points $s = -\frac{n+\mu_j}{\lambda_j}$, where $n \in \mathbb{N}$ and $1 \leq j \leq r$.

All other zeros are located inside the critical strip and they are called nontrivial zeros of F. It is expected that for every function in the Selberg class, the analogue of the Riemann hypothesis holds, i.e., all nontrivial zeros lie on the critical line $\mathrm{Re}\,(s) = \frac{1}{2}$. The *degree* of $F \in \mathcal{S}$ is defined by $d_F = 2\sum_{j=1}^{r} \lambda_j$. The logarithmic derivative of $F(s)$ has a Dirichlet series expansion, as given below:

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \Lambda_F(n) n^{-s}, \qquad (\mathrm{Re}\, s > 1)$$

where $\Lambda_F(n) = b(n)\log n$ is the generalized von Mangoldt function (supported on the prime powers). The Euler product of every $F \in \mathcal{S}$ can be written in the standard form $F(s) = \prod_p F_p(s)$, where $F_p(s) = 1 + \sum_{m=1}^{\infty} \frac{a_F(p^m)}{p^{ms}}$. It has been conjectured in [6] that the p-factors F_p are of polynomial type, hence we have

$$F_p(s) = \prod_{i=1}^{\nu_F} \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1}, \qquad |\alpha_i(p)| \leq 1.$$

In view of our investigations, the functional equation is of special interest. We rewrite the functional equation as

$$F(s) = \Delta_F(s)\overline{F(1-\overline{s})}, \tag{2.2}$$

where

$$\Delta_F(s) = \omega Q^{1-2s} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1-s) + \overline{\mu_j})}{\Gamma(\lambda_j s + \mu_j)}. \tag{2.3}$$

By Stirling's formula, we have

$$\Delta_F(\sigma + it) = \left(\lambda Q^2 t^{d_F}\right)^{\frac{1}{2} - \sigma - it} \exp\left(itd_F + \frac{i\pi(\mu - d_F)}{4}\right)\{\omega + O(1/t)\}$$

and

$$-\frac{\Delta_F'}{\Delta_F}(\sigma + it) = \log\left(\lambda Q^2 t^{d_F}\right) + O\left(\frac{1}{t}\right),$$

where $\mu = 2\sum_{j=1}^{r}(1 - 2\mu_j)$ and $\lambda = \prod_{j=1}^{r} \lambda_j^{2\lambda_j}$. Further, we have

$$\mu_F(\sigma) = \limsup_{t \to \pm\infty} \frac{\log|F(\sigma + it)|}{\log|t|} = \begin{cases} 0 & \text{for} \quad \sigma > 1, \\ (\frac{1}{2} - \sigma)d_F & \text{for} \quad \sigma < 0, \end{cases}$$

and

$$\mu_F(\sigma) \leq \frac{1}{2} d_F(1 - \sigma) \quad \text{for } 0 \leq \sigma \leq 1.$$

Also

$$F(\sigma + it) \ll_\epsilon t^{\mu_F(\sigma) + \epsilon}.$$

For more details, kindly see Lemma 2.1 in [9]. Moreover, by Cauchy's integral formula we have

$$F^{(k)}(s) = \frac{k!}{2\pi i} \int_{\mathbf{C}} \frac{F(w)}{(w-s)^{k+1}} ds, \tag{2.4}$$

where $\mathbf{C}$ is any arbitrary small circle centered at s. By using the last bound of $F(s)$ of (2.2), it follows that

$$F^{(k)}(s) \ll_\epsilon t^{\mu_F(\sigma) + \epsilon}. \tag{2.5}$$

Lemma 1. *Let $F(s)$ be a function belonging to the Selberg class, k be a positive integer and $a \in \mathbb{C}$. Then, there exists a real number $E_1 = E_1(k, a, F) \geq 1$ such that there is no a-point of $F^{(k)}(s)$ in the region $\{s \in \mathbb{C} : \sigma \geq E_1\}$.*

Proof. The case $a = 0$ was treated by Šimenas in [14]. Hence, we consider only the case $a \neq 0$. Since $F^{(k)}(s) = (-1)^k \sum_{n\geq 2} \frac{a_n(\log n)^k}{n^s}$, we have

$$|F^{(k)}(s)| \leq \sum_{n\geq 2} \frac{|a_n|(\log n)^k}{n^\sigma} \longrightarrow 0 \quad \text{as} \quad \sigma \longrightarrow \infty.$$

Hence, there exists $E_1 = E_1(k, a, F) \geq 1$ such that $|F^{(k)}(s)| < |a|$ for $\sigma \geq E_1$. Thus $F^{(k)}(s) \neq a$ in this half-plane. □

Lemma 2. *Let $F(s)$ be a function belonging to the Selberg class and k be a positive integer. Then, for $c > 1$ the following equation holds in the region $\{s \in \mathbb{C} : \sigma > c, |t| \geq \tau_k\}$ for sufficiently large τ_k:*

$$\begin{aligned} \overline{F(1-\overline{s})}^{(k)} &= \frac{(d_F)^k}{\pi^r \omega} Q^{2s-1} \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)\Gamma(\lambda_j s + 1 - \lambda_j - \overline{\mu_j}) \\ &\quad \times \left(\prod_{j=1}^{r} \sin \pi(-\lambda_j s + \lambda_j + \overline{\mu_j})\right) \log^k(s) \\ &\quad \times F(s)\left(1 + O\left(\frac{1}{|\log(s)|}\right)\right). \end{aligned}$$

Proof. Using Equations (2.2), (2.3) and applying the Euler reflection formula

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin \pi s}$$

with $s \neq 0, -1, -2, \cdots$, we obtain

$$\begin{aligned} \overline{F(1-\overline{s})} &= \frac{1}{\pi^r \omega} Q^{2s-1} F(s) \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)\Gamma(\lambda_j s + 1 - \lambda_j - \overline{\mu_j}) \\ &\quad \times \prod_{j=1}^{r} \sin \pi(-\lambda_j s + \lambda_j + \overline{\mu_j}). \end{aligned}$$

Further, by differentiating the previous equation k times we get

$$\begin{aligned} \overline{F(1-\overline{s})}^{(k)} &= \frac{1}{\pi^r \omega} Q^{2s-1} \left(\prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)\Gamma(\lambda_j s + 1 - \lambda_j - \overline{\mu_j})\right)^{(k)} \\ &\quad \times \prod_{j=1}^{r} \sin \pi(-\lambda_j s + \lambda_j + \overline{\mu_j}) F(s) \\ &\quad + \frac{1}{\pi^r \omega} Q^{2s-1} \sum_{n=0}^{k-1} \left(\prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)\Gamma(\lambda_j s + 1 - \lambda_j - \overline{\mu_j})\right)^{(n)} R_{n,k}(s), \end{aligned}$$

where

$$R_{n,k}(s) = \sum_{m=0}^{k-n}\sum_{l=0}^{m}\binom{k}{n}\binom{k-n}{m}\binom{m}{l}(2\log Q)^{k-n-m} \times \left(\prod_{j=1}^{r}\sin\pi(-\lambda_j s+\lambda_j+\overline{\mu_j})\right)^{(m-l)} F^{(l)}(s).$$

Using [20, Equation (13)], derivatives of the Gamma function can be estimated as follows:

$$\Gamma^{(j)}(s) = \Gamma(s)\,(\log s)^j\left(1+O\left(\frac{1}{s\log s}\right)\right)$$

in the region $\{s\in\mathbb{C}, \sigma\geq 1+\delta, |t|\geq 1\}$. Furthermore, with the use of the last estimated values of $\Gamma^{(j)}(s)$ and the equality $\log(as+b)=\log s+\log\left(a+\frac{b}{s}\right)$, it follows that for sufficiently large $|t|\geq\tau_k$ and $\sigma\geq c>1$ one has

$$\begin{aligned}
&\left(\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j)\Gamma(\lambda_j s+1-\lambda_j-\overline{\mu_j})\right)^{(k)}\\
&= \sum_{n_1+n_2+\ldots+n_{2r}=k}\binom{k}{n_1,n_2,\ldots,n_{2r}}\left(\Gamma(\lambda_1 s+\mu_1)\right)^{(n_1)}\\
&\quad\times(\Gamma(\lambda_1 s+1-\lambda_1-\overline{\mu_1}))^{(n_2)}\ldots(\Gamma(\lambda_{2r-1}s+\mu_{2r-1}))^{(n_{2r-1})}\\
&\quad\times\Gamma(\lambda_{2r}s+1-\lambda_{2r}-\overline{\mu_{2r}})^{(n_{2r})}\\
&= \log^k(s)\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j)\Gamma(\lambda_j s+1-\lambda_j-\overline{\mu_j})\left(1+O\left(\frac{1}{s\log(s)}\right)\right)\\
&\quad\times\sum_{n_1+n_2+\ldots+n_{2r}=k}\binom{k}{n_1,n_2,\ldots,n_{2r}}\lambda_1^{n_1+n_2}\lambda_2^{n_3+n_4}\ldots\lambda_r^{n_{2r-1}+n_{2r}}\\
&= (d_F)^k\log^k(s)\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j)\Gamma(\lambda_j s+1-\lambda_j-\overline{\mu_j})\left(1+O\left(\frac{1}{s\log(s)}\right)\right).
\end{aligned}$$

Moreover, using the same argument as above in the same region $F(s)\asymp 1$ and $F^{(l)}(s)=\sum_{n\geq 2}\frac{a_n(-\log n)^l}{n^s}\ll 1$ we get

$$\begin{aligned}
&\left|\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j)\Gamma(\lambda_j s+1-\lambda_j-\overline{\mu_j})\prod_{j=1}^{r}\sin\pi(-\lambda_j s+\lambda_j+\overline{\mu_j})F(s)\right|\\
&\asymp \left|\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j)\Gamma(\lambda_j s+1-\lambda_j-\overline{\mu_j})\log^k(s)e^{\pi(\lambda_1+\lambda_2+\ldots+\lambda_r)|t|}\right|
\end{aligned}$$

and

$$\left|\sum_{n=0}^{k-1}\left(\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j)\Gamma(\lambda_j s+1-\lambda_j-\overline{\mu_j})\right)^{(n)} R_{n,k}(s)\right|$$

$$\ll \left|\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j)\Gamma(\lambda_j s+1-\lambda_j-\overline{\mu_j})\log^{k-1}(s)e^{\pi(\lambda_1+\lambda_2+\ldots+\lambda_r)|t|}\right|.$$

As a consequence, we get

$$\begin{aligned}\overline{F(1-\overline{s})}^{(k)} &= \frac{(d_F)^k}{\pi^r\omega}Q^{2s-1}\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j)\Gamma(\lambda_j s+1-\lambda_j-\overline{\mu_j})\\ &\quad\times\left(\prod_{j=1}^{r}\sin\pi(-\lambda_j s+\lambda_j+\overline{\mu_j})\right)\log^k(s)F(s)\left(1+O\left(\frac{1}{|\log(s)|}\right)\right)\end{aligned}$$

in the region $\{s\in\mathbb{C}:\sigma>c,|t|\geq\tau_k\}$. □

Lemma 3. *Let $F(s)$ be a function belonging to the Selberg class, k be a positive integer and $a\in\mathbb{C}$. Then, there exists a real number $E_2=E_2(k,a,F)\leq 0$ such that there is no a-point of $F^{(k)}(s)$ in the region $\{s\in\mathbb{C}:\sigma\leq E_2,|t|\geq\tau_k\}$ for sufficiently large τ_k.*

Proof. It follows from Lemma 2 that $F^{(k)}(1-s)\to\infty$ as $\sigma\to\infty$ and there exists $E_2=E_2(k,a,F)\leq 0$ such that $|F^{(k)}(s)|>|a|$ for $\sigma\leq E_2$ and $|t|\geq\tau_k$. Thus $F^{(k)}(s)\neq a$ in this region. □

Remark 3. It can also be seen by Rouché's theorem [19, Section 3.42] and Lemma 2 that there is $N_k=N_k(a,F)<0$ such that $F^{(k)}(s)$ has an a-point around any trivial zero $s=-\frac{n+\mu_j}{\lambda_j}, n\in\mathbb{N}, 1\leq j\leq r$ of $F(s)$ in the region $\{s\in\mathbb{C}:\sigma\leq N_k,|t|\leq\tau_k\}$. Those a-points can be regarded as trivial a-points of $F^{(k)}(s)$.

Now, we consider the number $N_k(a;T,F)$ of a-points of $F^{(k)}(s), k\geq 1$ in the region $0<t<T$ and $E_2<\sigma<E_1$, where E_1 and E_2 are from Lemmas 1 and 3, respectively. The following lemma is a direct result of a combinations of Equation (2.5) and Jensen's formula.

Lemma 4. *Let $F(s)$ be a function belonging to the Selberg class, k be a positive integer and $a\in\mathbb{C}$. Then, for sufficiently large T, we have*

$$N_k(a;T+1,F)-N_k(a;T,F)\ll\log(T).$$

Lemma 5. *Let $F(s)$ be a function belonging to the Selberg class, k be a positive integer and $a \in \mathbb{C}$. Then, for any constants σ_1, σ_2 and $s \in \mathbb{C}$ with $\sigma_1 < \sigma < \sigma_2$ and large t, we have*

$$\frac{F^{(k+1)}(s)}{F^{(k)}(s)-a} = \sum_{|\gamma_a^{(k)}-t|\leq 1} \frac{1}{s-\rho_a^{(k)}} + O\left(\log t\right),$$

where $\rho_a^{(k)}$ denotes an a-point of the k^{th} derivative $F^{(k)}(s)$ and $\gamma_a^{(k)} = Im(\rho_a^{(k)})$.

Proof. It is clearly reflected from Equations (2.2), (2.3) and Stirling's formula that the function $(s-1)^{m_F}F(s)$ is of order one (see Lemma 3.3 in [15] for more details). Further, using Equation (2.4) the same result can easily be shown for the functions $(s-1)^{m_F+k}(F^{(k)}(s)-a),\ k \geq 1$. Hence, by the Hadamard factorization theorem we get

$$(s-1)^{m_F+k}(F^{(k)}(s)-a)$$
$$= \exp\left(A_{k,a}+B_{k,a}s\right) s^{m_{k,a}} \prod_{\rho_a^{(k)}\neq 0}\left(1-\frac{s}{\rho_a^{(k)}}\right)\exp\left(\frac{s}{\rho_a^{(k)}}\right),$$

where $A_{k,a}$ and $B_{k,a}$ are certain complex constants, $m_{k,a}$ is a positive integer and the product is taken over all a-points $\rho_a^{(k)}$ of $F^{(k)}(s)$. Hence, applying the property of the logarithmic derivative we obtain

$$\begin{aligned}\frac{F^{(k+1)}(s)}{F^{(k)}(s)-a} &= -\frac{A_{k,a}}{s-1}+B_{k,a}+\frac{m_{k,a}}{s}+\sum_{\rho_a^{(k)}\neq 0}\left(\frac{1}{s-\rho_a^{(k)}}+\frac{1}{\rho_a^{(k)}}\right)\\ &= \sum_{\rho_a^{(k)}\neq 0}\left(\frac{1}{s-\rho_a^{(k)}}+\frac{1}{\rho_a^{(k)}}\right)+O(1).\end{aligned}$$

Then, by the same argument due to Onozuka [12, Lemma 2.6] and using Lemma 4 we complete the proof. □

Lemma 6. *Let $F(s)$ be a function belonging to the Selberg class, k be a positive integer and $a \in \mathbb{C}$. Then, for $s \in \mathbb{C}$ with sufficiently large $\sigma \geq E_1$ we have*

$$\begin{aligned}\frac{F^{(k+1)}(s)}{F^{(k)}(s)-a} &= \sum_{\substack{l\,\geq\,0\\ n_0,\,n_1,\,\ldots,\,n_l\,\geq\,\ell}} \frac{(-1)^{k(l+1)}}{a^{l+1}}(\log n_0)^{k+1}(\log n_1 \log n_2 \ldots \log n_l)^k\\ &\qquad\times\frac{a_{n_0}a_{n_1}\ldots a_{n_l}}{n_0^s n_1^s \ldots n_l^s}, \qquad \text{if } \ a\neq 0\end{aligned}$$

and

$$\begin{aligned}\frac{F^{(k+1)}(s)}{F^{(k)}(s)-a} &= \sum_{\substack{l\,\geq\,0\\ n_0\,\geq\,\ell\\ n_1,\,\ldots,\,n_l\,\geq\,\ell+1}} \left(\frac{-1}{a_\ell \log^k \ell}\right)^{l+1}(\log n_0)^{k+1}(\log n_1 \log n_2 \ldots \log n_l)^k\\ &\qquad\times\frac{\ell^{(l+1)s}a_{n_0}a_{n_1}\ldots a_{n_l}}{n_0^s n_1^s \ldots n_l^s}, \qquad \text{if } \ a=0.\end{aligned}$$

Proof. When $a \neq 0$, we have

$$\begin{aligned}
\frac{F^{(k+1)}(s)}{F^{(k)}(s)-a} &= \frac{F^{(k+1)}(s)}{-a\left(1-\frac{F^{(k)}(s)}{a}\right)} = \frac{F^{(k+1)}(s)}{-a}\sum_{l\geq 0}\left(\frac{F^{(k)}(s)}{a}\right)^{l} \\
&= \frac{(-1)^{k+1}}{-a}\sum_{n_0\geq \ell}\frac{a_{n_0}(\log n_0)^{k+1}}{n_0^s} \\
&\quad \times \sum_{l\geq 0}\left(\frac{(-1)^k}{a}\sum_{n_1\geq \ell}\frac{a_{n_1}(\log n_1)^k}{n_1^s}\right)^{l} \\
&= \sum_{\substack{l\geq 0\\ n_0,n_1,..,n_l\geq \ell}}\frac{(-1)^{k(l+1)}}{a^{l+1}}(\log n_0)^{k+1} \\
&\quad \times(\log n_1\log n_2..\log n_l)^k\,\frac{a_{n_0}a_{n_1}..a_{n_l}}{n_0^s n_1^s..n_l^s}.
\end{aligned}$$

When $a = 0$, one has

$$\begin{aligned}
\frac{F^{(k+1)}(s)}{F^{(k)}(s)} &= -\frac{\sum_{n_0\geq \ell}\frac{a_{n_0}(\log n_0)^{k+1}}{n_0^s}}{\frac{a_\ell(\log \ell)^k}{\ell^s}\left(1+\frac{\ell^s}{a_\ell(\log\ell)^k}\sum_{n_1\geq \ell+1}\frac{a_{n_1}(\log n_1)^k}{n_1^s}\right)} \\
&= -\frac{\ell^s}{a_\ell(\log \ell)^k}\sum_{n_0\geq \ell}\frac{a_{n_0}(\log n_0)^{k+1}}{n_0^s} \\
&\quad \times\sum_{l\geq 0}\left(\frac{-\ell^s}{a_\ell(\log\ell)^k}\sum_{n_1\geq \ell+1}\frac{a_{n_1}(\log n_1)^k}{n_1^s}\right)^{l} \\
&= \sum_{\substack{l\geq 0\\ n_0\geq \ell\\ n_1,\dots,n_l\geq \ell+1}}\left(\frac{-1}{a_\ell\log^k\ell}\right)^{l+1}(\log n_0)^{k+1}(\log n_1\log n_2...\log n_l)^k \\
&\quad \times\frac{\ell^{(l+1)s}a_{n_0}a_{n_1}...a_{n_l}}{n_0^s n_1^s...n_l^s}.
\end{aligned}$$

□

3. Proof of Theorem 1

We now present a proof of our first main theorem.

Proof of Theorem 1. Let a be a complex number. We write $s = \sigma + it$ and $\rho_a^{(k)} = \beta_a^{(k)} + i\gamma_a^{(k)}$ with real numbers $\sigma, t, \beta_a^{(k)}$ and $\gamma_a^{(k)}$. The case $a = 0$ was already proved by Šimenas in [14] so here we assume $a \neq 0$. By the residue theorem we have

$$N_k(a;T,F) = \frac{1}{2\pi i}\oint_{\mathbf{R}}\frac{F^{(k+1)}(s)}{F^{(k)}(s)-a}ds + O(1),$$

where the integration is taken over a rectangular contour in the counterclockwise direction, denoted by $\mathbf{R}$ with vertices $u+ib, u+iT, v+iT, v+ib$ with some constants $b>0$, $u \geq E_1$, $v \leq E_2$ such that $F^{(k)}(s)$ has no a-point on the lines $t=T$ and $t=b$. Hence, we have

$$\begin{aligned} N_k(a;T,F) &= \frac{1}{2\pi i}\int_{\mathbf{R}} \frac{F^{(k+1)}(s)}{F^{(k)}(s)-a}ds + O(1) \\ &= \frac{1}{2\pi i}\left\{\int_{v+ib}^{u+ib} + \int_{u+ib}^{u+iT} + \int_{u+iT}^{v+iT} + \int_{v+iT}^{v+ib}\right\} \frac{F^{(k+1)}(s)}{F^{(k)}(s)-a}ds + O(1) \\ &= I_1 + I_2 + I_3 + I_4 + O(1). \end{aligned}$$

The first integral I_1 is independent of T so $\mathbf{I_1} = O(1)$. Next we consider I_2. Using Lemma 6 it easy to see that $\mathbf{I_2} = O(1)$. From Lemma 5, we get for I_3:

$$\begin{aligned} I_3 &= \frac{1}{2\pi i}\sum_{|\gamma_a^{(k)}-T|<1}\int_{u+iT}^{v+iT}\frac{1}{s-\rho_a^{(k)}}ds + O\left(\int_{u+iT}^{v+iT}(\log t)ds\right) \\ &= \frac{1}{2\pi i}\sum_{|\gamma_a^{(k)}-T|<1}\int_{u+iT}^{v+iT}\frac{1}{s-\rho_a^{(k)}}ds + O\left(\log T\right). \end{aligned}$$

Now we change the path of integration. If $\gamma_a^{(k)} < T$, we change the path to the upper semi circle with center $\rho_a^{(k)}$ and radius 1 (see Figure 1). If $\gamma_a^{(k)} > T$, we change the path to the lower semicircle with center $\rho_a^{(k)}$ and radius 1 (see Figure 2).

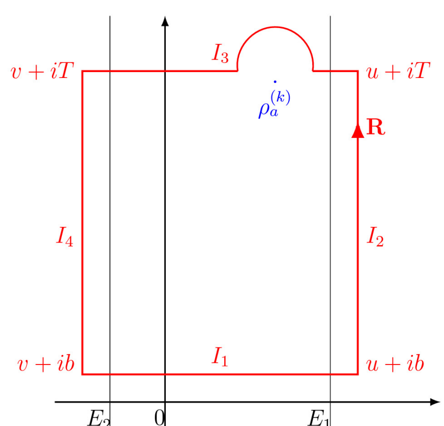

Figure 1: The case $\gamma_a^{(k)} < T$.

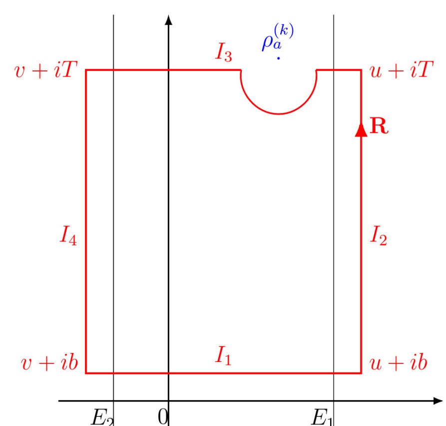

Figure 2: The case $\gamma_a^{(k)} > T$.

Therefore, $\frac{1}{s-\rho_a^{(k)}} \ll 1$ on the new path, which yields

$$I_3 = O\left(\sum_{|\gamma_a^{(k)}-T|<1} 1\right) + O\left(\log T\right).$$

Hence, by Lemma 4 we obtain $\mathbf{I_3} = O\left(\log T\right)$.

Finally, we have to estimate the value for I_4. Since $|F^{(k)}(s)| > |a|$ for $\sigma \leq E_2$ and $|t| \geq \tau_k$, one has

$$\begin{aligned}
I_4 &= \frac{1}{2\pi i}\int_{v+iT}^{v+i\tau_k} \frac{F^{(k+1)}(s)}{F^{(k)}(s)-a}ds + O\left(1\right) \\
&= \frac{1}{2\pi i}\int_{v+iT}^{v+i\tau_k} \frac{F^{(k+1)}(s)}{F^{(k)}(s)} \times \frac{1}{1-\frac{a}{F^{(k)}(s)}}ds + O\left(1\right) \\
&= \frac{1}{2\pi i}\int_{v+iT}^{v+i\tau_k} \frac{F^{(k+1)}(s)}{F^{(k)}(s)} \left(1 + \sum_{n\geq 1}\left(\frac{a}{F^{(k)}(s)}\right)^n\right) ds + O\left(1\right) \\
&= \frac{1}{2\pi i}\int_{v+iT}^{v+i\tau_k} \frac{F^{(k+1)}(s)}{F^{(k)}(s)}ds + \frac{1}{2\pi i}\int_{v+iT}^{v+i\tau_k} \frac{F^{(k+1)}(s)}{F^{(k)}(s)} \sum_{n\geq 1}\left(\frac{a}{F^{(k)}(s)}\right)^n ds \\
&\quad + O\left(1\right).
\end{aligned}$$

Combining Lemma 2 and Stirling's formula we get

$$F^{(k)}(v+it) \gg |t|^{\frac{r}{2}-d_F v+\frac{r}{2}d_F} e^{-\pi(\lambda_1+\lambda_2+\ldots+\lambda_r)|t|}(\log|t|)^k e^{\pi(\lambda_1+\lambda_2+\ldots+\lambda_r)|t|}.$$

Here we choose v such that $\frac{r}{2} - d_F v + \frac{r}{2}d_F > \frac{3}{2}$. Then, we obtain $F^{(k)}(v+it) \gg |t|^{\frac{3}{2}}$. Therefore

$$\sum_{n\geq 1}\left(\frac{a}{F^{(k)}(v+it)}\right)^n \ll |t|^{-\frac{3}{2}}.$$

Furthermore, from Lemma 2 we have $\frac{F^{(k+1)}(v+it)}{F^{(k)}(v+it)} \ll \log|t|$. Finally we get

$$\begin{aligned}
I_4 &= \frac{1}{2\pi i}\int_{v+iT}^{v+i\tau_k} \frac{F^{(k+1)}(s)}{F^{(k)}(s)}ds + O\left(\int_{v+iT}^{v+i\tau_k} \log(|t|)|t|^{-\frac{3}{2}}dt\right) + O\left(1\right) \\
&= \frac{1}{2\pi i}\int_{v+iT}^{v+i\tau_k} \frac{F^{(k+1)}(s)}{F^{(k)}(s)}ds + O\left(1\right).
\end{aligned} \tag{3.1}$$

The estimate of the first term of Equation (3.1) goes with the same method as per the estimation of I_3 in [14]. We have

$$I_4 = \frac{d_F T}{2\pi}\log T + \frac{T}{2\pi}\log\left(\lambda Q^2\right) - \frac{d_F T}{2\pi} + O\left(\log T\right).$$

Theorem 1 follows from the estimation of I_1, I_2, I_3 and I_4. □

4. Proof of Theorem 2

In this section, we present a proof of our second main theorem.

Proof of Theorem 2. Let a be a complex number. Since there are finitely many a-points in the region $0 < t < \tau_k$ and $E_2 < \sigma < E_1$, by the residue theorem we have

$$\sum_{\substack{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T \\ E_2<\beta_a^{(k)}<E_1}} x^{\rho_a^{(k)}} = \frac{1}{2\pi i}\oint_{\mathbf{R}} x^s \frac{F^{(k+1)}(s)}{F^{(k)}(s)-a}ds + O(1),$$

where τ_k is the same as in Lemma 2. The integration is taken over a rectangular contour in the counterclockwise direction, denoted by $\mathbf{R}$ with vertices $u+i\tau_k, u+iT, -v+iT, -v+i\tau_k$ with some constants $u \geq E_1$, $v \leq E_2$. When $F^{(k)}(s)$ has no a-point on the lines $t = T$ and $t = \tau_k$, then

$$\begin{aligned}\sum_{\substack{\rho_a^{(k)}:\ 0<\gamma_a^{(k)}<T \\ E_2<\beta_a^{(k)}<E_1}} x^{\rho_a^{(k)}} &= \frac{1}{2\pi i}\oint_{\mathbf{R}} x^s \frac{F^{(k+1)}(s)}{F^{(k)}(s)-a}ds + O(1) \\ &= \frac{1}{2\pi i}\left\{\int_{v+i\tau_k}^{u+i\tau_k} + \int_{u+i\tau_k}^{u+iT} + \int_{u+iT}^{v+iT} + \int_{v+iT}^{v+i\tau_k}\right\} \\ &\quad \times x^s \frac{F^{(k+1)}(s)}{F^{(k)}(s)-a}ds + O(1) \\ &= J_1 + J_2 + J_3 + J_4 + O(1).\end{aligned}$$

The first integral I_1 is independent of T, so $J_1 = O(1)$. Similar to the estimate of I_3, by Lemmas 4 and 5 the integral J_3 can be estimated as $J_3 = O(\log T)$. Next we consider J_4. Using Lemma 2 we obtain

$$\frac{F^{(k+1)}(v+it)}{F^{(k)}(v+it)-a} \ll \log t, \qquad t \geq \tau_k.$$

Hence

$$J_4 \ll \int_{\tau_k}^{T} x^v \log t dt \ll x^v T \log T \longrightarrow 0, \quad v \longrightarrow -\infty.$$

Now we proceed in order to estimate the value of J_2. In Lemma 6 the right-hand side is complicated; let us write it as $\sum_d \alpha(d) d^{-s}$. Hence, we have

$$J_2 = \frac{1}{2\pi i}\int_{u+i\tau_k}^{u+iT} x^s \sum_d \alpha(d) d^{-s} ds = \frac{1}{2\pi i}\sum_d \alpha(d)\int_{u+i\tau_k}^{u+iT}\left(\frac{x}{d}\right)^s ds.$$

The integral factor can be calculated as

$$\int_{u+i\tau_k}^{u+iT}\left(\frac{x}{d}\right)^s ds = \begin{cases} iT + O(1) & \text{for } x = d, \\ O(1) & \text{for } x \neq d. \end{cases}$$

which yields

$$J_2 = \begin{cases} \frac{T}{2\pi} \sum_{d,x=d} \alpha(d) + O(1) & \text{for} \quad x = d, \\ O\left(1\right) & \text{for} \quad x \neq d. \end{cases}$$

Combining the estimated values of J_1, J_2, J_3 and J_4, we finally obtain Theorem 2. □

5. Concluding Remarks

In this section, we present some problems that will be considered in a sequel to this article.

1. Prove an asymptotic for the sum of $F^{(k)}(\rho_a)X^{\rho_a}$, where $F^{(k)}(s)$ denotes the nth derivative of F, X is a positive real number, and ρ_a denotes a nontrivial a-point of $F(s)$. The sum is over the zeros with imaginary parts up to a height T, as $T \to \infty$. We can find a new form of the asymptotic formula, when X is a positive integer and that highlights the differences in the asymptotic expansions as X changes its arithmetic nature (see [11]).

2. Following [5], we can investigate the distribution of the fractional parts of the sequence $(\alpha\gamma_a^k)$, where α is any fixed non-zero real number and γ_a^k runs over the imaginary parts of the a-points of $F^{(k)}(s)$.

3. Study a-points of combinations of functions ϕ_F defined in (2.1) (see [3] for $a = 0$ and $F = \zeta$).

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References

[1] B. C. Berndt, The number of zeros for $\zeta^{(k)}(s)$, *J. Lond. Math. Soc.* **2** (4) (1970), 577-580.

[2] M. P. Chaudhary, K. Mazhouda, and M. Mekkaoui, Discrete measures and the Riemann hypothesis of some L-functions in the Selberg class, *Integers* **23** (2023), #A6, 16 pp.

[3] A. Dixit, N. Robles, A. Roy, and A. Zaharescu, Zeros of combinations of the Riemann ξ-function on bounded vertical shifts, *J. Number Theory* **149** (2015), 404-434.

[4] S. Gonek, An explicit formula of Landau and its applications to the theory of the Riemann zeta function, *Contemp. Math.* **143** (1993), 395-413.

[5] M-T. Jakhlouti, K. Mazhouda, and J. Steuding, On the distribution of the a-points of a Selberg class L-function modulo one, *Arch. Math. (Basel)* **104** (2015), 419-429.

[6] J. Kaczorowski and A. Perelli, A note on the degree conjecture for the Selberg class, *Rend. Circ. Mat. Palermo* **57** (2008), 443-448.

[7] N. Levinson and H. L. Montgomery, Zeros of the derivatives of the Riemann zeta-function, *Acta Math.* **133** (1974), 49-65.

[8] X. Li and M. Radziwill, The Riemann zeta-function on vertical arithmetic progression, *Int. Math. Res. Notices* **2** (2015), 325-354.

[9] K. Mazhouda and S. Omar, Mean-square of L-functions in the Selberg class, in *New Directions in Value-Distribution Theory of zeta and L-functions*, Ber. Math., Shaker Verlag, Aachen, 2009.

[10] K. Mazhouda and B. Sodaïgui, The Li-Sekatskii coefficients for the Selberg class, *Int. J. Math.* **33** (12) (2022), Paper No. 2250075, 23 pp.

[11] M. Mekkaoui, A. Derbal, and K. Mazhouda, On some sums at the a-points of the k-th derivatives of the Dirichlet L-functions, *Turkish J. Math.* **44** (2020), 1544-1560.

[12] T. Onozuka, On the a-points of the derivatives of the Riemann zeta function, *Eur. J. Math.* **3** (1) (2017), 53-76.

[13] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, in *Proceedings of the Amalfi Conference on Analytic Number Theory*, E. Bombieri et al. (eds.), Università di Salerno, 1992.

[14] R. Šimenas, Zeros of the higher-order derivatives of the functions belonging to the extended Selberg class, *Lith. Math. J.* **61** (1) (2021), 87-95.

[15] L. Smajlović, On Li's criterion for the Riemann hypothesis for the Selberg class, *J. Number Theory* **130** (2010), 828-851.

[16] A. Speiser, Geometrisches zur Riemannschen zetafunktion, *Math. Ann.* **110** (1934), 514-521.

[17] R. Spira, Zero-free regions of $\zeta^{(k)}(s)$, *J. Lond. Math. Soc.* **1** (1) (1965), 677-682.

[18] J. Steuding, Value-distribution of L-functions, in *Lecture Notes in Math 1877*, Springer, 2007.

[19] E. C. Titchmarsh, *The Theory of Functions (2nd ed.)*, Oxford University Press, 1939.

[20] C. Y. Yildirim, Zeros of derivatives of Dirichlet L-functions, *Turkish J. Math.* **20** (1996), 521-534.