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A NOTE ON THE DECIMAL EXPANSION OF RECIPROCALS OF MERSENNE PRIMES

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Abstract

Mersenne primes M_p are prime numbers of the form $2^p - 1$. The decimal expansion of $1/M_p$ is purely recurring, with period length L that divides $M_p - 1$. If p divides L, we partition the periodic part into p blocks, each of length ℓ. We show that there exists a permutation $B'_1, \ldots, B'_p$ of the blocks such that (i) $B'_1 = 2B'_p - 10^\ell$, $B'_{k+1} = 2B'_k$ for $1 \le k \le p-2$, $B'_p = 2B'_{p-1} + 1$, and (ii) $B'_1 + \cdots + B'_p = 10^\ell - 1$.

1. Introduction

The decimal expansion of $1/n$ has been studied for centuries, and was popularized by several authors; for instance, see [1, 5, 6, 10, 11]. When $n = p$ is a prime, a result of Midy [9] gives an intriguing "property of nines". An extension of this result by Ginsberg [3] more than a century and a half later led to a resurgence of interest in problems related to Midy's theorem by several authors; for instance, see [2, 4, 7, 8].

Mersenne primes M_p are prime numbers of the form $2^p - 1$. It is easy to see that M_p is prime implies p is prime. However, the converse is not true; for instance, $2^{11} - 1 = 23 \times 89$. We know that every prime divisor of M_p is of the form $2kp + 1$. There are only 51 such primes known to date; see www.mersenne.org for a complete and updated list.

For prime $q \neq 2, 5$, the decimal expansion of $1/q$ is purely recurring:

$$\frac{1}{q} = 0.\overline{c_1 \, \ldots \, c_L}.$$

Let $B(1, q)$ denote the smallest repeating block of digits in the decimal expansion of $1/q$:

$$B(1, q) = c_1 \, c_2 \, c_3 \, \ldots \, c_L.$$

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The number of digits L in $B(1,q)$ is the *period length* of $1/q$. Since $\gcd(q,10)=1$, $10^{q-1} \equiv 1 \pmod{q}$. Therefore, there exists a least positive integer h satisfying $10^h \equiv 1 \pmod{q}$, which we denote by $\operatorname{ord}_q 10$. It is easy to see that $\operatorname{ord}_q 10 \mid (q-1)$.

Consider the case $q = M_p$. Then both p and L divide $2^p - 2$. Throughout this note, we only consider those Mersenne primes M_p for which $p \mid L$. Since $1/M_p$ is the sum of an infinite geometric progression with first term $B(1,M_p)/10^L$ and common ratio $1/10^L$, we have

$$M_p \cdot B(1,M_p) = 10^L - 1. \tag{1}$$

Suppose $B(1,M_p)$ is divided into p blocks, each of length ℓ; thus $L = p\ell$. Let the blocks be given by

$$\begin{aligned}
B_1 &= c_1\, c_2\, c_3\, \ldots\, c_{\ell-1}\, c_\ell, \\
B_2 &= c_{\ell+1}\, c_{\ell+2}\, c_{\ell+3}\, \ldots\, c_{2\ell-1}\, c_{2\ell}, \\
B_3 &= c_{2\ell+1}\, c_{2\ell+2}\, c_{2\ell+3}\, \ldots\, c_{3\ell-1}\, c_{3\ell}, \\
&\ \vdots \\
B_p &= c_{(p-1)\ell+1}\, c_{(p-1)\ell+2}\, c_{(p-1)\ell+3}\, \ldots\, c_{p\ell-1}\, c_{p\ell}.
\end{aligned}$$

We have

$$\begin{aligned}
B_1 = \left\lfloor \frac{10^\ell}{2^p-1} \right\rfloor,\ B_1B_2 = \left\lfloor \frac{10^{2\ell}}{2^p-1} \right\rfloor,\ B_1B_2B_3 = \left\lfloor \frac{10^{3\ell}}{2^p-1} \right\rfloor,\ \ldots\ , \\
B_1B_2B_3\ldots B_p = \left\lfloor \frac{10^{p\ell}}{2^p-1} \right\rfloor.
\end{aligned} \tag{2}$$

The purpose of this note is to show that:

(i) $B_1' = 2B_p' - 10^\ell$, $B_{k+1}' = 2B_k'$ for $1 \le k \le p-2$ and $B_p' = 2B_{p-1}' + 1$ for some permutation $B_1', \ldots, B_p'$ of the blocks $B_1, \ldots, B_p$.

(ii) $B_1 + B_2 + B_3 + \cdots + B_p = 10^\ell - 1$.

Proposition 1. *Let $2^p - 1$ be prime such that p divides the periodic length of the decimal expansion of $1/(2^p-1)$. If the repeating block of the decimal expansion is partitioned into p blocks, each of length ℓ, then $10^\ell \equiv 2^r \pmod{2^p-1}$ for some $r \in \{1,\ldots,p-1\}$.*

Proof. The congruence $x^p \equiv 1 \pmod{2^p-1}$ has *at most* p (incongruent) solutions modulo 2^p-1 since 2^p-1 is prime. Since each of the p integers $1, 2, 2^2, \ldots, 2^{p-1}$ is a solution to the congruence, these must be all the solutions. Since $(2^p-1) \mid \left((10^\ell)^p - 1\right)$ by Equation (1), $x = 10^\ell$ is a solution to the above congruence, and so we have $10^\ell \equiv 2^r \pmod{2^p-1}$ for some $r \in \{0,1,2,\ldots,p-1\}$. Moreover, from $\operatorname{ord}_{2^p-1} 10 = L = p\ell$, we have $\operatorname{ord}_{2^p-1} 10^\ell = p$, so that $r > 0$. □

The main result of this note is the content of Theorem 1.

Theorem 1. *Let 2^p-1 be prime such that p divides the periodic length of the decimal expansion of $1/(2^p-1)$. Suppose the repeating block of the decimal expansion is partitioned into p blocks, each of length ℓ. Let r be such that $10^\ell \equiv 2^r \pmod{2^p-1}$, with $1 \le r \le p-1$, and let $s \equiv r^{-1} \pmod p$. Then*

(i)
$$B_{ks+1} = \begin{cases} 2B_{(k-1)s+1} & \text{if } 1 \le k \le p-1, k \ne p-r; \\ 2B_{(k-1)s+1} + 1 & \text{if } k = p-r; \\ 2B_{(k-1)s+1} - 10^\ell & \text{if } k = p; \end{cases}$$

where the block subscripts are taken modulo p and from the set $\{1, \dots, p\}$.

In particular,
$$B_{k+1} = \begin{cases} 2B_k & \text{if } 1 \le k \le p-2; \\ 2B_k + 1 & \text{if } k = p-1; \\ 2B_k - 10^\ell & \text{if } k = p; \end{cases}$$

when $r = 1$.

(ii) $B_1 + B_2 + B_3 + \cdots + B_p = 10^\ell - 1$.

Proof. Throughout this proof, we take the block subscripts modulo p and from the set $\{1, \dots, p\}$. By Proposition 1, $10^\ell \equiv 2^r \pmod{2^p-1}$ for some $r \in \{1, \dots, p-1\}$, and so $10^{k\ell} \equiv 2^{kr} \equiv 2^{kr \pmod p} \pmod{2^p-1}$ for each $k \in \{1, \dots, p\}$.

Fix $k \in \{1, \dots, p\}$. We have

$$B_1B_2B_3 \dots B_k = \left\lfloor \frac{10^{k\ell}}{2^p-1} \right\rfloor = \frac{10^{k\ell} - 2^{kr \pmod p}}{2^p-1}$$

by Equation (2). Hence

$$\begin{aligned} B_k &= B_1B_2B_3 \dots B_k - \left(B_1B_2B_3 \dots B_{k-1} \times 10^\ell\right) \\ &= \frac{10^{k\ell} - 2^{kr \pmod p}}{2^p-1} - \frac{10^\ell \left(10^{(k-1)\ell} - 2^{(k-1)r \pmod p}\right)}{2^p-1} \\ &= \frac{10^\ell \cdot 2^{(k-1)r \pmod p} - 2^{kr \pmod p}}{2^p-1}. \end{aligned} \tag{3}$$

Let $s \equiv r^{-1} \pmod p$. Then

$$\begin{aligned} B_{k+s} &= \frac{10^\ell \cdot 2^{(k-1)r+sr \pmod p} - 2^{(kr+sr) \pmod p}}{2^p-1} \\ &= \frac{10^\ell \cdot 2^{(k-1)r+1 \pmod p} - 2^{(kr+1) \pmod p}}{2^p-1}. \end{aligned} \tag{4}$$

From Equation (3) and Equation (4), we have

$$B_{k+s} = 2B_k,$$

except when $(k-1)r \equiv -1 \pmod{p}$ or when $kr \equiv -1 \pmod{p}$.

Write $k = (\lambda - 1)s + 1$, $1 \le \lambda \le p$. If $(k-1)r \equiv -1 \pmod{p}$, then $\lambda = p$, and

$$\begin{aligned} 2B_k - 10^\ell &= \frac{10^\ell \cdot 2 \cdot 2^{p-1} - 2 \cdot 2^{r-1}}{2^p - 1} - 10^\ell \\ &= \frac{10^\ell - 2^r}{2^p - 1} \\ &= B_{k+s} \end{aligned}$$

by Equation (3). If $kr \equiv -1 \pmod{p}$, then $\lambda = p - r$, and

$$\begin{aligned} 2B_k + 1 &= \frac{10^\ell \cdot 2 \cdot 2^{p-r-1} - 2 \cdot 2^{p-1}}{2^p - 1} + 1 \\ &= \frac{10^\ell \cdot 2^{p-r} - 1}{2^p - 1} \\ &= B_{k+s} \end{aligned}$$

by Equation (3). Thus,

$$B_{k+s} = \begin{cases} 2B_k & \text{if } k = (\lambda-1)s+1, \lambda \ne p-r, p; \\ 2B_k + 1 & \text{if } k = (p-r-1)s+1; \\ 2B_k - 10^\ell & \text{if } k = (p-1)s+1; \end{cases}$$

where the block subscripts are taken modulo p and from the set $\{1, \ldots, p\}$. Replacing k by $(\lambda - 1)s + 1$ with $1 \le \lambda \le p$, we may rewrite this as

$$B_{\lambda s+1} = \begin{cases} 2B_{(\lambda-1)s+1} & \text{if } 1 \le \lambda \le p-1, \lambda \ne p-r; \\ 2B_{(\lambda-1)s+1} + 1 & \text{if } \lambda = p-r; \\ 2B_{(\lambda-1)s+1} - 10^\ell & \text{if } \lambda = p; \end{cases}$$

where the block subscripts are taken modulo p and from the set $\{1, \ldots, p\}$. This completes the proof of part (i).

From Equation (3), we have

$$\begin{aligned} \sum_{k=1}^{p} B_k &= \frac{10^\ell \sum_{k=1}^{p} 2^{(k-1)r \pmod{p}} - \sum_{k=1}^{p} 2^{kr \pmod{p}}}{2^p - 1} \\ &= \frac{10^\ell \sum_{t=0}^{p-1} 2^t - \sum_{t=0}^{p-1} 2^t}{2^p - 1} \\ &= 10^\ell - 1. \end{aligned}$$

This completes the proof of part (ii). □

We close this note with some data concerning the decimal expansion of reciprocals of Mersenne primes. Table 1 lists the first thirteen Mersenne primes $2^p - 1$ together with the period length L of the decimal expansion of their reciprocals. The only primes p among these which do not divide L are $p = 2$ and $p = 17$.

p	$2^p - 1$	L	$p \mid L$
2	3	1	N
3	7	6	Y
5	31	15	Y
7	127	42	Y
13	8191	1365	Y
17	131071	3855	N
19	524287	74898	Y
31	2147483647	195225786	Y
61	2305843009213693951	1152921504606846975	Y
89	618970019642690137449562111	103161669940448356241593685	Y
107	162259276829213363391578010288127	162259276829213363391578010288126	Y
127	170141183460469231731687303715884105727	23307011432940990648176342974778644 62	Y
521	686479766013060971498190079908139321726 943530014330540939446345918554318339765 605212255964066145455497729631139148085 803712198799971664381257402829111505715 1	343239883006530485749095039954069660863 471765007165270469723172959277159169882 802606127982033072727748864815569574042 901856099399985832190628701414555752857 5	Y

Table 1: Table for lengths of decimal expansion of $1/(2^p - 1)$ when $2^p - 1$ is prime; (Y=Yes, N=No)

We look at the decimal expansion of $1/(2^p - 1)$ for $p = 3, 5, 7, 13$; the last has 1365 digits, so we list only the key digits here.

$$
\begin{aligned}
\frac{1}{7} &= 0.\overline{14 \quad 28 \quad 57} \\
\frac{1}{31} &= 0.\overline{032258 \quad 064516 \quad 129032 \quad 258064 \quad 516129} \\
\frac{1}{127} &= 0.\overline{007874 \quad 015748 \quad 031496 \quad 062992 \quad 125984 \quad 251968 \quad 503937} \\
\frac{1}{8191} &= 0.\overline{0001\ldots7568 \quad 0625\ldots4848 \quad 0039\ldots2178} \\
&\overline{0002\ldots5136 \quad 1250\ldots9696 \quad 0078\ldots4356} \\
&\overline{0004\ldots0272 \quad 2500\ldots9392 \quad 0156\ldots8712} \\
&\overline{0009\ldots0544 \quad 5000\ldots8784 \quad 0312\ldots7424} \\
&\overline{0019\ldots1089}.
\end{aligned}
$$

We note that p divides L in each case. Using the notation in Proposition 1, $r = 1$

in the first three examples and $r = 9$ in the last example. Notice that the blocks essentially double as move left to right in the first three examples, but essentially double after every three blocks in the last example. The explanation for the last mentioned fact is that 3 is the inverse of 9 modulo 13.

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