COMBINATORIAL PROOFS OF MERCA'S IDENTITIES INVOLVING THE SUM OF DIFFERENT PARTS CONGRUENT TO r MODULO m IN ALL PARTITIONS OF n

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Abstract

We give combinatorial proofs of several recent results due to Merca on the sum of different parts congruent to r modulo m in all partitions of n. The proofs make use of some well-known involutions from the literature and some new involutions introduced here.

1. Introduction

A *partition* λ of n is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of positive integers that add up to n. We refer to the integers λ_i as the *parts* of λ. As usual, we denote by $p(n)$ the number of partitions of n. Note that $p(x) = 0$ if x is not a non-negative integer, and since the empty partition $\emptyset$ is the only partition of 0, we have that $p(0) = 1$.

Let m, n, and r be nonnegative integers such that $0 \leq r < m$. Denote by $a_{r,m}(n)$ the sum of all different parts congruent to r modulo m in all partitions of n. Thus, for a partition λ of n, a part $mj + r$ of λ contributes $mj + r$ to $a_{r,m}(n)$ regardless of its multiplicity. We denote by $s_m(n)$ the sum of different parts that appear at least m times in partitions of n. Thus, for a partition λ of n, a part j of λ that occurs at least m times contributes j to $s_m(n)$ regardless of its multiplicity.

Recently Merca [15] proved several results relating $a_{m,r}(n)$, $s_m(n)$, and numbers of restricted partitions. In this article, we give combinatorial proofs of several results in [15]. Combinatorial proofs of [15, Theorem 1.3 and Corollary 1.4] are given in [14]. In [15] the author gives a combinatorial proof of Theorem 1.6. We prove combinatorially Theorem 1.3 and Corollaries 4.2, 4.4, 4.6, 4.7(i), 4.9, 4.10, 5.2, 5.3, 6.3, 6.3, 7.2, 7.3 of [15]. The corollaries are limiting cases of inequalities obtained in [15] by truncating theta series.

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2. Combinatorial Proofs of Theorems of Merca

We first introduce some notation. We denote by $\mathcal{P}(n)$ the set of partitions of n and by $\mathcal{P}$ the set of all partitions. The following convention is used for all other sets of partitions: if $\mathcal{A}(n)$ denotes a set of partitions of n, then

$$\mathcal{A} = \bigcup_{n\geq 0} \mathcal{A}(n).$$

We write $\lambda \vdash n$ to mean that λ is a partition of n. We also write $|\lambda| = n$ to mean that the parts of λ add up to n. The number of parts of λ is called the *length* of λ and is denoted by $\ell(\lambda)$. If $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)}$ are partitions, we write $(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)}) \vdash n$ to mean $|\lambda^{(1)}|+|\lambda^{(2)}|+\cdots+|\lambda^{(k)}| = n$. We define $p_{e-o}(n) := p_e(n)-p_o(n)$, where $p_e(n)$ (respectively $p_o(n)$) is the number of partitions of n with an even (respectively odd) number of parts. An *overpartition* of n is a partition of n in which the first occurrence of a part may be overlined. We denote by $\overline{\mathcal{P}}(n)$ the set of overpartitions of n. As in the case of partitions, we define $\overline{p}_{e-o}(n) := \overline{p}_e(n) - \overline{p}_o(n)$, where $\overline{p}_e(n)$ (respectively $\overline{p}_o(n)$) is the number of overpartitions of n with an even (respectively odd) number of parts. Given a set $\mathcal{A}(n)$ of partitions or overpartitions, we denote by $\mathcal{A}_e(n)$ (respectively $\mathcal{A}_o(n)$) the subset of $\lambda \in \mathcal{A}(n)$ with $\ell(\lambda)$ even (respectively odd). We denote by $\mathcal{Q}(n)$ the set of partitions of n into distinct parts and write $q(n)$ for $|\mathcal{Q}(n)|$. Similarly, we denote by $q_{odd}(n)$ (respectively $q_{even}(n)$) the number of partitions of n into distinct parts all odd (respectively even).

Theorem 1 ([15, Theorem 1.3]). *Let $m, n,$ and r be nonnegative integers such that $0 \leq r < m$. We have*

(i) $a_{r,m}(n) = \sum_{j=0}^{\infty}(mj+r)p(n-mj-r);$

(ii) $s_m(n) = \sum_{j=0}^{\infty} jp(n-mj).$

Proof. We note that the combinatorial proof of this theorem is implicit in the combinatorial proof of [15, Theorem 1.6]. We write it here in clearer form since it is used in subsequent proofs.

(i) Let $\mathcal{A}_{r,j,m}(n)$ be the set of overpartitions of n with exactly one part overlined and only a part equal to $mj + r$ maybe overlined. Clearly

$$a_{m,r}(n) = \sum_{j=0}^{\infty}(mj+r)|\mathcal{A}_{r,j,m}(n)|.$$

We denote by $f_{r,j,m} : \mathcal{A}_{r,j,m}(n) \to \mathcal{P}(n-mj-r)$ the transformation defined by $f_{r,j,m}(\lambda) = \mu$, where μ is the partition obtained from λ by removing its unique overlined part. Since $f_{r,j,m}$ is a bijection, this completes the proof of (i).

(ii) Let $\mathcal{S}_{j,m}(n)$ be the set of partitions of n in which part j occurs at least m times. Clearly

$$s_m(n) = \sum_{j=0}^{\infty} j|\mathcal{S}_{j,m}(n)|.$$

We denote by $g_{j,m} : \mathcal{S}_{a,m}(n) \to \mathcal{P}(n-mj)$ he transformation defined by $g_{j,m}(\nu) = \eta$, where η is the partitions obtained from ν by removing m parts equal to j. Since $g_{j,m}$ is a bijection, this completes the proof of (ii). □

Example 1. Let $n = 47, m = 3, j = 4$, and $r = 2$.

(i) Given $\lambda = (15, \overline{14}, 14, 2, 2) \in \mathcal{A}_{2,4,3}(47)$, we have

$$f_{2,4,3}(\lambda) = f_{2,4,3}(15, \overline{14}, 14, 2, 2) = (15, 14, 2, 2) \in \mathcal{P}(47 - 14) = \mathcal{P}(33).$$

Conversely, for $\mu = (15, 14, 2, 2) \in \mathcal{P}(33)$, we have

$$f^{-1}_{2,4,3}(\mu) = f^{-1}_{2,4,3}(15, 14, 2, 2) = (15, \overline{14}, 14, 2, 2) \in \mathcal{A}_{2,4,3}(33 + 14) = \mathcal{A}_{2,4,3}(47).$$

(ii) Given $\nu = (13, 10, 10, 4, 4, 4, 4, 1, 1) \in \mathcal{S}_{4,3}(47)$, we have

$$g_{4,3}(\nu) = g_{4,3}(13, 10, 10, 4, 4, 4, 4, 1, 1) = (13, 10, 10, 4, 1, 1) \in \mathcal{P}(47 - 12) = \mathcal{P}(35).$$

Conversely, for $\eta = (13, 10, 10, 4, 1, 1) \in \mathcal{P}(35)$, we have

$$\begin{aligned} g^{-1}_{4,3}(\eta) &= g^{-1}_{4,3}(13, 10, 10, 4, 1, 1) \\ &= (13, 10, 10, 4, 4, 4, 4, 1, 1) \in \mathcal{S}_{4,3}(35 + 12) = \mathcal{S}_{4,3}(47). \end{aligned}$$

Theorem 2 ([15, Corollary 4.2]). *Let m, n, and r be nonnegative integers such that $0 \leq r < m$. Then*

$$a_{m,r}(n) + 2\sum_{k=1}^{\infty}(-1)^k a_{m,r}(n - k^2) = \sum_{j=0}^{\infty}(mj + r)p_{e-o}(n - mj - r). \tag{1}$$

Proof. By Theorem 1, the left-hand side of Identity (1) equals

$$\sum_{j=0}^{\infty}(mj + r)\left(p(n - mj - r) + 2\sum_{k=1}^{\infty}(-1)^k p(n - k^2 - mj - r)\right).$$

Thus, to prove the theorem, it suffices to show combinatorially that for all $n \geq 0$,

$$p_{e-o}(n) = p(n) + 2\sum_{k=1}^{\infty}(-1)^k p(n - k^2). \tag{2}$$

For $n \geq 1$, let $\overline{\mathcal{P}}^*(n)$ be the set of overpartitions of n that are not of the form

$$(\underbrace{a, a, \ldots, a}_{a \text{ times}}) \text{ or } (\overline{a}, \underbrace{a, \ldots, a}_{a-1 \text{ times}}).$$

In [2], Andrews defined an involution φ_{A_1} on $\overline{\mathcal{P}}^*(n)$ that reverses the parity of the length of overpartitions and thus proved combinatorially that

$$1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} = \sum_{n=0}^{\infty} \overline{p}_{e-o}(n) q^n. \tag{3}$$

The restriction $\varphi_{A_1} : \overline{\mathcal{P}}^*_e(n) \to \overline{\mathcal{P}}^*_o(n)$ is a bijection.

We define

$$\mathcal{P}\overline{\mathcal{P}}(n) := \{(\alpha, \beta) \vdash n \mid \alpha \in \mathcal{P}, \beta \in \overline{\mathcal{P}}\}$$
$$\mathcal{P}\overline{\mathcal{P}}^*(n) := \{(\alpha, \beta) \vdash n \mid \alpha \in \mathcal{P}, \beta \in \overline{\mathcal{P}}^*\}.$$

Moreover, set $\mathcal{P}\overline{\mathcal{P}}^{**}(n) := \mathcal{P}\overline{\mathcal{P}}(n) \setminus \mathcal{P}\overline{\mathcal{P}}^*(n)$.

The transformation $(\alpha, \beta) \mapsto (\alpha, \varphi_{A_1}(\beta))$ is an involution on $\mathcal{P}\overline{\mathcal{P}}^*(n)$ that reverses the parity of $\ell(\beta)$. Hence

$$\begin{aligned} |\{(\alpha, \beta) \in& \mathcal{P}\overline{\mathcal{P}}(n) \mid \beta \in \overline{\mathcal{P}}_e\}| - |\{(\alpha, \beta) \in \mathcal{P}\overline{\mathcal{P}}(n) \mid \beta \in \overline{\mathcal{P}}_o\}| \\ &= |\{(\alpha, \beta) \in \mathcal{P}\overline{\mathcal{P}}(n) \mid \beta \in \overline{\mathcal{P}}^{**}_e\}| - |\{(\alpha, \beta) \in \mathcal{P}\overline{\mathcal{P}}(n \mid \beta \in \overline{\mathcal{P}}^{**}_o\}|. \end{aligned} \tag{4}$$

Since $\overline{p}_{e-o}(0) = 1$ and, if $k > 0$, $\overline{\mathcal{P}}^{**}(k)$ is either empty or consists of exactly two "square" overpartitions of length congruent to k modulo 2, mapping $(\alpha, \beta) \mapsto \alpha$, we see that the right-hand side of Equation (4) equals

$$p(n) + 2\sum_{k=1}^{\infty}(-1)^k p(n - k^2).$$

Next, we define the set

$$\mathcal{P}\mathcal{Q}(n) := \{(\alpha, \eta) \vdash n \mid \alpha \in \mathcal{P}, \eta \in \mathcal{Q}\}.$$

If $n \geq 1$, we define an involution ψ on $\mathcal{P}\mathcal{Q}(n)$ by

$$\psi(\alpha, \eta) := \begin{cases} (\alpha \setminus (\alpha_1), \eta \cup (\alpha_1) & \text{if } \alpha_1 > \eta_1, \\ (\alpha \cup (\eta_1), \eta \setminus (\eta_1) & \text{if } \alpha_1 \leq \eta_1. \end{cases}$$

Clearly, $\psi(\alpha, \eta)$ reverses the parity of $\ell(\eta)$. Thus, if $n \geq 1$,

$$|\{(\alpha, \eta) \in \mathcal{P}\mathcal{Q}(n) \mid \eta \in \mathcal{Q}_e\}| - |\{(\alpha, \eta) \in \mathcal{P}\mathcal{Q}(n) \mid \eta \in \mathcal{Q}_o\}| = 0,$$

and if $n = 0$ the difference is 1 since $\{(\alpha, \eta) \in \mathcal{PQ}(0)\} = \{(\emptyset, \emptyset)\}$.

To prove combinatorially that Identity (2) holds, write $(\alpha, \beta) \in \mathcal{P}\overline{\mathcal{P}}(n)$ as $(\alpha, \overline{\beta}, \widetilde{\beta})$, where $\overline{\beta}$ is the partition consisting of the overlined parts of β and $\widetilde{\beta}$ is the partition consisting of the nonoverlined parts of β. Fix a partition $\widetilde{\beta}$ and define

$$\mathcal{P}\overline{\mathcal{P}}_{\widetilde{\beta}}(n) := \{(\alpha, \beta) = (\alpha, \overline{\beta}, \widetilde{\beta}) \in \mathcal{P}\overline{\mathcal{P}}(n)\},$$

the set of pairs (α, β) in $\mathcal{P}\overline{\mathcal{P}}(n)$ such that the nonoverlined parts in the overpartition β are precisely the parts of $\widetilde{\beta}$. Let $(\gamma, \zeta) := \psi(\alpha, \overline{\beta})$ and define ξ to be the overpartition with overlined parts precisely the parts of ζ and nonoverlined parts precisely the parts of $\widetilde{\beta}$. Then, $(\gamma, \xi) \in \mathcal{P}\overline{\mathcal{P}}_{\widetilde{\beta}}(n)$ and $\ell(\xi)$ and $\ell(\overline{\beta})$ have different parity. Thus,

$$\begin{aligned} |\{(\alpha, \overline{\beta}, \widetilde{\beta}) \in \mathcal{P}\overline{\mathcal{P}}_{\widetilde{\beta}}(n) \mid \ell(\overline{\beta}) \text{ even}\}| - |\{(\alpha, \overline{\beta}, \widetilde{\beta}) \in \mathcal{P}\overline{\mathcal{P}}_{\widetilde{\beta}}(n) \mid \ell(\overline{\beta}) \text{ odd}\}| \\ = |\{(\emptyset, \emptyset, \widetilde{\beta}) \in \mathcal{P}\overline{\mathcal{P}}(n)\}| = 1. \end{aligned}$$

Summing over all $\widetilde{\beta}$, we obtain

$$\begin{aligned} &|\{(\alpha, \beta) \in \mathcal{P}\overline{\mathcal{P}}(n) \mid \ell(\beta) \text{ even}\}| - |\{(\alpha, \beta) \in \mathcal{P}\overline{\mathcal{P}}(n) \mid \ell(\beta) \text{ odd}\}| \\ &\quad = |\{(\emptyset, \emptyset, \widetilde{\beta}) \in \mathcal{P}\overline{\mathcal{P}}(n) \mid \ell(\widetilde{\beta}) \text{ even}\}| - |\{(\emptyset, \emptyset, \widetilde{\beta}) \in \mathcal{P}\overline{\mathcal{P}}(n) \mid \ell(\widetilde{\beta}) \text{ odd}\}| \\ &\quad = p_{e-o}(n). \end{aligned}$$

□

Example 2. For an example using Andrews' involution φ_{A_1}, we refer the reader to [2]. Here we illustrate the transformation ψ. Consider the pair $(\alpha, \beta) \in \mathcal{P}\overline{\mathcal{P}}(94)$ with $\alpha = (9, 9, 3, 3, 3, 1)$ and $\beta = (\overline{9}, 9, 9, 7, 7, \overline{6}, 6, 6, \overline{2}, 2, \overline{1}, 1, 1)$. Hence $\ell(\beta) = 13$, $\overline{\beta} = (9, 6, 2, 1)$ and $\widetilde{\beta} = (9, 9, 7, 7, 6, 6, 2, 1, 1)$. Since $\alpha_1 = 9 \leq \overline{\beta}_1 = 9$,

$$\psi(\alpha, \overline{\beta}) = ((9, 9, 9, 3, 3, 3, 1), (6, 2, 1)).$$

Then $\xi = (9, 9, 7, 7, \overline{6}, 6, 6, \overline{2}, 2, \overline{1}, 1, 1)$ and $\ell(\xi) = 12$. Thus, (α, β) "cancels" with $((9, 9, 9, 3, 3, 3, 1), \xi)$.

Now consider the pair $(\alpha, \beta) \in \mathcal{P}\overline{\mathcal{P}}(106)$ with

$$\alpha = (12, 9, 9, 3, 3, 3, 1) \qquad \text{and} \qquad \beta = (\overline{9}, 9, 9, 7, 7, \overline{6}, 6, 6, \overline{2}, 2, \overline{1}, 1, 1).$$

Then, $\ell(\beta) = 13$, $\overline{\beta} = (9, 6, 2, 1)$ and $\widetilde{\beta} = (9, 9, 7, 7, 6, 6, 2, 1, 1)$. Since $\alpha_1 = 12 > \overline{\beta}_1 = 9$, we have

$$\psi(\alpha, \overline{\beta}) = ((9, 3, 3, 3, 1), (12, 9, 6, 2, 1)).$$

Then $\xi = (\overline{12}, \overline{9}, 9, 9, 7, 7, \overline{6}, 6, 6, \overline{2}, 2, \overline{1}, 1, 1)$ and $\ell(\xi) = 14$. Thus, (α, β) "cancels" with $((9, 3, 3, 3, 1), \xi)$.

Theorem 3 ([15, Corollary 4.4 (i)]). *Let m and n be nonnegative integers. Then*

$$s_m(n) + 2\sum_{j=1}^{\infty}(-1)^j s_m(n-j^2) = \sum_{j=0}^{\infty} j p_{e-o}(n-mj). \tag{5}$$

Proof. By Theorem 1, the left-hand side of Identity (5) equals

$$\sum_{j=0}^{\infty} j \left(p(n-mj) + 2\sum_{j=1}^{\infty}(-1)^j p(n-j^2-mj) \right).$$

Then, the combinatorial proof of Identity (2) provided in the proof of Theorem 2 completes the argument. □

Theorem 4 ([15, Corollary 4.6]). *Let $m, n,$ and r be nonnegative integers such that $0 \le r < m$. Then*

$$a_{m,r}(n) + 2\sum_{k=1}^{\infty}(-1)^k a_{m,r}(n-2k^2) = \sum_{j=0}^{\infty}(mj+r) q_{odd}(n-mj-r). \tag{6}$$

Proof. By Theorem 1, the left-hand side of Identity (6) equals

$$\sum_{j=0}^{\infty}(mj+r) \left(p(n-mj-r) + 2\sum_{k=1}^{\infty}(-1)^k p(n-2k^2-mj-r) \right).$$

Thus, to prove the theorem, it suffices to show combinatorially that for all $n \ge 0$,

$$q_{odd}(n) = p(n) + 2\sum_{k=1}^{\infty}(-1)^k p(n-2k^2). \tag{7}$$

Doubling all parts in Andrews' proof of Identity (3), shows combinatorially that the right-hand side of Identity (7) equals

$$|\{(\alpha,\beta) \in \mathcal{P}\overline{\mathcal{P}}(n) \mid \beta \text{ has even parts, } \beta \in \overline{\mathcal{P}}_e\}| \\ - |\{(\alpha,\beta) \in \mathcal{P}\overline{\mathcal{P}}(n) \mid \beta \text{ has even parts, } \beta \in \overline{\mathcal{P}}_o\}|.$$

In [10], Gupta constructed an involution φ_G on the set of partitions of n with at least one even part or at least one repeated part. By construction, the parity of the number of even parts in λ is different than the parity of the number of even parts in $\varphi_G(\lambda)$. This gives a combinatorial proof for

$$q_{odd}(n) = p_e(n,2) - p_o(n,2),$$

where $p_e(n,2)$ (respectively $p_o(n,2)$) is the number of partitions of n with and even (respectively odd) number of even parts.

If $n \geq 1$, the involution ψ of Theorem 2 is well defined when restricted to pairs of partitions in $\mathcal{PQ}(n)$ in which both partitions have only even parts. Thus, if $n \geq 1$,

$$|\{(\alpha,\eta) \in \mathcal{PQ}(n) \mid \alpha,\eta \text{ have even parts, } \ell(\eta) \text{ even}\}| \\ - |\{(\alpha,\eta) \in \mathcal{PQ}(n) \mid \alpha,\eta \text{ have even parts, } \ell(\eta) \text{ odd}\}| = 0,$$

and if $n = 0$ the difference is 1 since $\{(\alpha,\eta) \in \mathcal{PQ}(0) \mid \alpha,\eta \text{ have even parts}\} = \{(\emptyset,\emptyset)\}$.

Next, we write $(\alpha,\beta) \in \mathcal{P}\overline{\mathcal{P}}(n)$ where β has only even parts as $(\alpha^o,\alpha^e,\overline{\beta},\widetilde{\beta})$, where α^e (respectively α^o) is the partition consisting of the even (respectively odd) parts of α and $\overline{\beta},\widetilde{\beta}$ are as in the proof of Theorem 2. We fix a partition $\widetilde{\beta}$ with even parts and a partition α^o with odd parts, and define

$$\mathcal{P}\overline{\mathcal{P}}_{\widetilde{\beta},\alpha^o}(n) := \{(\alpha,\beta) = (\alpha^o,\alpha^e,\overline{\beta},\widetilde{\beta}) \in \mathcal{P}\overline{\mathcal{P}}(n)\}.$$

Using the involution ψ on $(\alpha^e,\overline{\beta})$ and proceeding as in the proof of Theorem 2, we obtain

$$|\{(\alpha^o,\alpha^e,\overline{\beta},\widetilde{\beta}) \in \mathcal{P}\overline{\mathcal{P}}_{\widetilde{\beta},\alpha^o}(n) \mid \ell(\overline{\beta}) \text{ even}\}| - |\{(\alpha^o,\alpha^e,\overline{\beta},\widetilde{\beta}) \in \mathcal{P}\overline{\mathcal{P}}_{\widetilde{\beta},\alpha^o}(n) \mid \ell(\overline{\beta}) \text{ odd}\}| \\ = |\{(\alpha^o,\emptyset,\emptyset,\widetilde{\beta}) \in \mathcal{P}\overline{\mathcal{P}}(n)\}| = 1.$$

Summing over all α^o and $\widetilde{\beta}$, we obtain

$$\begin{aligned} &|\{(\alpha,\beta) \in \mathcal{P}\overline{\mathcal{P}}(n) \mid \beta \text{ has even parts, } \ell(\beta) \text{ even}\}| \\ &\quad - |\{(\alpha,\beta) \in \mathcal{P}\overline{\mathcal{P}}(n) \mid \beta \text{ has even parts, } \ell(\beta) \text{ odd}\}| \\ &= |\{(\alpha^o,\widetilde{\beta}) \vdash n \mid \alpha^o \text{ has odd parts, } \widetilde{\beta} \text{ has even parts, } \ell(\widetilde{\beta}) \text{ even}\}| \\ &\quad - |\{(\alpha^o,\widetilde{\beta}) \vdash n \mid \alpha^o \text{ has odd parts, } \widetilde{\beta} \text{ has even parts, } \ell(\widetilde{\beta}) \text{ odd}\}| \\ &= q_{odd}(n). \end{aligned}$$

The last equality above follows from Gupta's involution. □

For an example of the use of Gupta's transformation, see [10].

Theorem 5 ([15, Corollary 4.7(i)]). *Let m and n be nonnegative integers. Then*

$$s_m(n) + 2\sum_{k=1}^{\infty}(-1)^k s_m(n-2k^2) = \sum_{j=0}^{\infty} j q_{odd}(n-mj).$$

Proof. By Theorem 1, the left-hand side of Identity (2) equals

$$\sum_{j=0}^{\infty} j \left(p(n-mj) + 2\sum_{k=1}^{\infty}(-1)^k p(n-2k^2-mj) \right).$$

Then, the combinatorial proof of Identity (7) provided in the proof of Theorem 4 completes the argument. □

Theorem 6 ([15, Corollary 4.9]). *Let $m, n,$ and r be nonnegative integers such that $0 \le r < m$. Then*

$$\sum_{k=0}^{\infty}(-1)^{k(k+1)/2}a_{m,r}(n-k(k+1)/2)=\sum_{j=0}^{\infty}(mj+r)q\left(\frac{n-mj-r}{2}\right). \tag{8}$$

Proof. By Theorem 1, the left-hand side of Identity (8) equals

$$\sum_{j=0}^{\infty}(mj+r)\sum_{k=0}^{\infty}(-1)^{k(k+1)/2}p(n-k(k+1)/2-mj-r).$$

Then, to prove the theorem, it suffices to prove combinatorially that for all $n \ge 0$,

$$\sum_{k=0}^{\infty}(-1)^{k(k+1)/2}p(n-k(k+1)/2)=q\left(\frac{n}{2}\right). \tag{9}$$

Denote by $\mathcal{PED}(n)$ the set of partitions of n with even parts distinct and odd parts unrestricted. Let $\mathcal{PED}^*(n)$ be the subset of partitions in $\mathcal{PED}(n)$ not of the form

$$(\underbrace{2a+1,2a+1,\ldots,2a+1}_{a \text{ times}}) \text{ or } (\underbrace{2a-1,2a-1,\ldots,2a-1}_{a \text{ times}}).$$

In [2], Andrews defined an involution φ_{A_2} on $\mathcal{PED}^*(n)$ that reverses the parity of the length of partitions and thus proved combinatorially that

$$\sum_{n=0}^{\infty}(-1)^{j(j+1)/2}q^{j(j+1)/2}=\sum_{n=0}^{\infty}ped_{e-o}(n)q^n,$$

where $ped_{e-o}(n) := |\mathcal{PED}_e(n)| - |\mathcal{PED}_o(n)|$. We define

$$\mathcal{PPED}(n) := \{(\lambda,\mu)\vdash n \mid \lambda\in\mathcal{P},\mu\in\mathcal{PED}\}$$
$$\mathcal{PPED}^*(n) := \{(\lambda,\mu)\vdash n \mid \lambda\in\mathcal{P},\mu\in\mathcal{PED}^*\}.$$

Moreover, set $\mathcal{PPED}^{**}(n) = \mathcal{PPED}(n)\setminus\mathcal{PPED}^*(n)$.

The mapping $(\alpha,\mu)\mapsto(\alpha,\varphi_{A_2}(\mu))$ is an involution of $\mathcal{PPED}^*(n)$ that reverses the parity of $\ell(\mu)$. Then,

$$\begin{aligned}&|\{(\lambda,\mu)\in\mathcal{PPED}(n)\mid\mu\in\mathcal{PPED}_e\}|-|\{(\lambda,\mu)\in\mathcal{PPED}(n)\mid\mu\in\mathcal{PPED}_o\}| \quad (10)\\ &\quad=|\{(\lambda,\mu)\in\mathcal{PPED}(n)\mid\mu\in\mathcal{PPED}_e^{**}\}|-|\{(\lambda,\mu)\in\mathcal{PPED}(n)\mid\mu\in\mathcal{PPED}_e^{**}\}|.\end{aligned}$$

We have that $ped_{e-o}(0) = 1$ and, if $k > 0$, $\mathcal{PPED}(k) \setminus \mathcal{PPED}^*(k)$ is either empty of consists of exactly one partition with a parts, all equal to $2a+1$ or all equal to $2a-1$. In each case, a is congruent to k modulo 2. Hence, the right-hand side of Equation (10) equals

$$\sum_{k=0}^{\infty}(-1)^{k(k+1)/2}p(n-k(k+1)/2).$$

We write $(\lambda,\mu) \in \mathcal{PPED}(n)$ as $(\lambda^e,\lambda^o,\mu^e,\mu^o)$, where λ^e (respectively λ^o) is the partition consisting of the even (respectively odd) parts of λ; and μ^e, μ^o are defined similarly. Note that μ^e is a partition with distinct even parts.

Fix λ^o and μ^o two partitions with odd parts. Define

$$\mathcal{PPED}_{\lambda^o,\mu^o}(n) := \{(\lambda,\mu) = (\lambda^e,\lambda^o,\mu^e,\mu^o) \in \mathcal{PPED}(n)\}.$$

Using the involution ψ of the proof of Theorem 2 on (λ^e,μ^e), we obtain

$$\begin{aligned}|\{(\lambda^e,\lambda^o,\mu^e,\mu^o) \in \mathcal{PPED}_{\lambda^o,\mu^o}(n) \mid \ell(\mu^e) \text{ even}\}|& \\ - |\{(\lambda^e,\lambda^o,\mu^e,\mu^o) \in \mathcal{PPED}_{\lambda^o,\mu^o}(n) \mid \ell(\mu^e) \text{ odd}\}|& \\ = |\{(\emptyset,\lambda^o,\emptyset,\mu^o) \in \mathcal{PPED}_{\lambda^o,\mu^o}(n)\}| = 1.&\end{aligned}$$

Summing over all λ^o and μ^o, we obtain

$$\begin{aligned}|\{(\lambda,\mu) \in \mathcal{PPED}(n) \mid \ell(\mu) \text{ even}\}|& \\ - |\{(\lambda,\mu) \in \mathcal{PPED}(n) \mid \ell(\mu) \text{ odd}\}|& \\ = |\{(\lambda^o,\mu^o) \vdash n \mid \lambda^o,\mu^o \text{ have odd parts, } \ell(\mu^o) \text{ even}\}|& \\ - |\{(\lambda^o,\mu^o) \vdash n \mid \lambda^o,\mu^o \text{ have odd parts, } \ell(\mu^o) \text{ odd}\}|.&\end{aligned}$$

In [5, Proposition 4], Ballantine and Welch proved that

$$\begin{aligned}|\{(\alpha,\beta) \vdash n \mid \alpha,\beta \text{ have odd parts, } \ell(\beta) \text{ even}\}|& \\ - |\{(\alpha,\beta) \vdash n \mid \alpha,\beta \text{ have odd parts, } \ell(\beta) \text{ odd}\}|& \\ = q_{even}(n),&\end{aligned}$$

where $q_{even}(n)$ is the number of distinct partitions with even parts. Hence, the left-hand side of Equation (10) equals $q_{even}(n)$.

Finally, the transformation that maps a partition $\lambda \vdash n$ with distinct even parts to the partition $\mu \vdash n/2$ with $\mu_i = \lambda_i/2$ for all $1 \leq i \leq \ell(\lambda)$ is a bijection and shows that $q_{even}(n) = q(n/2)$. This completes the proof. □

Remark 1. The involution of [5, Proposition 4] is by no means simple and it is illustrated with several examples in [5].

Theorem 7 ([15, Corollary 4.10]). *Let m and n be nonnegative integers. Then*

$$\sum_{k=0}^{\infty}(-1)^{k(k+1)/2}s_m(n-k(k+1)/2)=\sum_{j=0}^{\infty}jq\left(\frac{n-mj}{2}\right). \tag{11}$$

Proof. By Theorem 1, the left-hand side of Identity (11) equals

$$\sum_{j=0}^{\infty}j\sum_{k=1}^{\infty}(-1)^{k(k+1)/2}p(n-k(k+1)/2-mj).$$

Then, the combinatorial proof of Identity (9) provided in the proof of Theorem 6 completes the argument. □

We remark a slight error in [14, Corollary 2.13] whose statement is the same as Identity (11) if $m \mid n$. In [14, Corollary 2.13], the right-hand side of Identity (11) is set to 0 if $m \nmid n$. However, it is easily verified that, for example, for $n = 10$ and $m = 3$ the left-hand side of Identity (11) equals 2.

The *rank* $r(\lambda)$ of a partition λ is defined [8] as the largest part of λ minus the number of parts in λ. Thus, $r(\lambda) = \lambda_1 - \ell(\lambda)$. Let

$$\mathcal{N}(n) := \{\lambda \vdash n \mid r(\lambda) \geq 0\}\} \qquad \text{and} \qquad \mathcal{R}(n) := \{\lambda \vdash n \mid r(\lambda) > 0\}\},$$

and define $N(n) = |\mathcal{N}(n)|$ and $R(n) = |\mathcal{R}(n)|$.

Theorem 8 ([15, Corollary 5.2]). *Let $m, n,$ and r be nonnegative integers such that $0 \leq r < m$. Then*

(i) $\displaystyle\sum_{j=0}^{\infty}(-1)^j a_{m,r}(n-j(3j+1)/2)=\sum_{j=0}^{\infty}(mj+r)N(n-mj-r)$

(ii) $\displaystyle\sum_{j=1}^{\infty}(-1)^{j+1} a_{m,r}(n-j(3j+1)/2)=\sum_{j=0}^{\infty}(mj+r)R(n-mj-r).$

Proof. As was the case in Theorems 2 - 7, Theorem 1 implies that it is enough to prove combinatorially that for all $n \geq 0$ we have

(i*) $\displaystyle\sum_{j=0}^{\infty}(-1)^j p(n-j(3j+1)/2)=N(n)$

(ii*) $\displaystyle\sum_{j=1}^{\infty}(-1)^{j+1} p(n-j(3j+1)/2)=R(n).$

Since $p(n) - N(n) = |\{\lambda \vdash n \mid r(\lambda) < 0\}|$ and, by conjugation, the number of partitions of n with negative rank equals the number of partitions of n with positive rank, statements (i*) and (ii*) are equivalent.

For $j \in \mathbb{Z}$, set $a(j) := j(3j+1)/2$. In [7], Bressoud and Zeilberger constructed an involution

$$\varphi_{BZ} : \bigcup_{j \in 2\mathbb{Z}} \mathcal{P}\left(n - a(j)\right) \to \bigcup_{j \in 2\mathbb{Z}+1} \mathcal{P}\left(n - a(j)\right)$$

as follows. Let $\lambda \in \mathcal{P}(n - a(j))$ and define $\varphi_{BZ}(\lambda)$ to be

$$(\ell(\lambda) + 3j - 1, \lambda_1 - 1, \ldots, \lambda_{\ell(\lambda)} - 1) \in \mathcal{P}(n - a(j-1)) \text{ if } \ell(\lambda) + 3j \geq \lambda_1,$$

$$(\lambda_2 + 1, \ldots, \lambda_{\ell(\lambda)} + 1, 1^{\lambda_1 - 3j - \ell(\lambda) - 1}) \in \mathcal{P}(n - a(j+1)) \text{ if } \ell(\lambda) + 3j < \lambda_1,$$

where 1^i means that there are i parts equal to 1 in the partition. Since $\mathcal{R}(n) = \{\lambda \in \mathcal{P}(n - a(0)) \mid \ell(\lambda) < \lambda_1\}$, restricting φ_{BZ} we obtain an involution

$$\varphi_{BZ} : \mathcal{R}(n) \cup \bigcup_{j \geq 2 \text{ even}} \mathcal{P}\left(n - a(j)\right) \to \bigcup_{j \geq 1 \text{ odd}} \mathcal{P}\left(n - a(j)\right).$$

This completes the combinatorial proof of the theorem. □

Example 3. Let $n = 20$ and $j = 0$. Then $a(0) = 0$ and $\lambda = (10, 8, 2) \in \mathcal{R}(20) = \mathcal{P}(20 - a(0))$ has $\ell(\lambda) = 3$ and $\lambda_1 = 10$. Thus,

$$\varphi_{BZ}(10, 8, 2) = (9, 3, 1^6) \in \mathcal{P}(20 - a(1)) = \mathcal{P}(20 - 2) = \mathcal{P}(18).$$

Let $n = 20$ and $j = 2$. Then, $a(2) = 7$ and $\lambda = (4, 3, 3, 2, 1) \in \mathcal{P}(13) = \mathcal{P}(20 - a(2))$ has $\ell(\lambda) = 5$ and $\lambda_1 = 4$. Thus $\ell(\lambda) + 3j \geq \lambda_1$ and

$$\varphi_{BZ}(4, 3, 3, 2, 1) = (5 + 6 - 1, 3, 2, 2, 1) = (10, 3, 2, 2, 1) \in \mathcal{P}(18) = \mathcal{P}(20 - a(1)).$$

Theorem 9 ([15, Corollary 5.3]). *Let m and n be nonnegative integers. Then*

(i) $\displaystyle\sum_{j=0}^{\infty} (-1)^j s_m(n - j(3j+1)/2) = \sum_{j=0}^{\infty} jN(n - mj)$

(ii) $\displaystyle\sum_{j=1}^{\infty} (-1)^{j+1} s_m(n - j(3j+1)/2) = \sum_{j=0}^{\infty} jR(n - mj).$

Proof. Theorem 1 implies that it is enough to prove Identities (i*) and (ii*) given in the proof of Theorem 8. □

Garden of Eden partitions we introduced by Hopkins and Sellers in [11] in connection to the game Bulgarian solitaire. They are partitions λ with all parts less than $\ell(\lambda)-1$. Hence they are precisely the partitions with rank at most -2. Denote by $G(n)$ the number of Garden of Eden partitions of n.

Theorem 10 ([15, Corollaries 6.2 and 6.3]). *Let $m, n,$ and r be nonnegative integers such that $0 \le r < m$. Then*

(i) $\displaystyle\sum_{j=0}^{\infty}(-1)^{j+1}a_{m,r}(n-3j(j+1)/2) = \sum_{j=0}^{\infty}(mj+r)G(n-mj-r)$

(ii) $\displaystyle\sum_{j=1}^{\infty}(-1)^{j+1}s_m(n-3j(j+1)/2) = \sum_{j=0}^{\infty}jG(n-mj).$

Proof. Theorem 1 implies that to prove both identities is suffices to show that for all $n \ge 0$,

$$G(n) = \sum_{j\ge 1}(-1)^{j+1}p(n-3j(j+1)/2).$$

A combinatorial proof of this identity is given by Hopkins and Sellers in [11]. They give an involution similar to Bressoud and Zeilberger's involution φ_{BZ} described in the proof of Theorem 8. $\square$

Given a partitions λ, we denote by $m_\lambda(1)$ the number of parts equal to 1 in λ and by $w(\lambda)$ the number of parts greater than $m_\lambda(1)$ in λ. Then the *crank* $cr(\lambda)$ of λ is defined [3] as

$$cr(\lambda) := \begin{cases} \lambda_1 & \text{if } m_\lambda(1) = 0, \\ w(\lambda) - m_\lambda(1) & \text{if } m_\lambda(1) > 0. \end{cases}$$

Let

$$\mathcal{C}(n) := \{\lambda \vdash n \mid cr(\lambda) \ge 0\}\},$$
$$\mathcal{D}(n) := \{\lambda \vdash n \mid cr(\lambda) > 0\}\},$$

and define $C(n) = |\mathcal{C}(n)|$ and $D(n) = |\mathcal{D}(n)|$.

Theorem 11 ([15, Corollary 7.2]). *Let $m, n,$ and r be nonnegative integers such that $0 \le r < m$. Then*

(i) $\displaystyle\sum_{j=0}^{\infty}(-1)^{j}a_{m,r}(n-j(j+1)/2) = \sum_{j=0}^{\infty}(mj+r)C(n-mj-r)$

(ii) $\displaystyle\sum_{j=1}^{\infty}(-1)^{j+1}a_{m,r}(n-j(j+1)/2) = \sum_{j=0}^{\infty}(mj+r)D(n-mj-r).$

Proof. Theorem 1 implies that it is enough to prove combinatorially that for all $n \geq 0$ we have

(i**) $\displaystyle\sum_{j=0}^{\infty}(-1)^j p(n-j(j+1)/2) = C(n)$

(ii**) $\displaystyle\sum_{j=1}^{\infty}(-1)^{j+1} p(n-j(j+1)/2) = D(n).$

Berkovich and Gravan [6] proved combinatorially that $D(n)$ is also equal to the number of partitions of n with negative crank, i.e., $p(n) - C(n)$. Thus statements (i**) and (ii**) are equivalent.

Given a partition λ, the smallest positive integer that is not a part of λ is called the *minimal excludant* of λ and is denoted by $\text{mex}(\lambda)$ (see [9, 4]). For example,

$$\text{mex}(7,7,4,2,1,1) = 3.$$

If n, j are nonnegative integers with $0 < j(j+1)/2 \leq n$, and $\lambda \in \mathcal{P}(n-j(j+1)/2)$, the transformation that adds parts $1, 2, \ldots, j$ to λ is a bijection from $\mathcal{P}(n-j(j+1)/2)$ to the set of partitions $\lambda \in \mathcal{P}(n)$ with $\text{mex}(\lambda) > j$. This shows combinatorially that, for $n, j \geq 0$, we have

$$p\left(n - \frac{j(j+1)}{2}\right) - p\left(n - \frac{(j+1)(j+2)}{2}\right) = |\{\lambda \in \mathcal{P}(n) \mid \text{mex}(\lambda) = j+1\}|.$$

Therefore, we have a combinatorial proof that

$$\sum_{j \geq 0}(-1)^j p\left(n - \frac{j(j+1)}{2}\right) = |\{\lambda \in \mathcal{P}(n) \mid \text{mex}(\lambda) \text{ odd}\}|.$$

Hopkins, Sellers, and Yee [12], and also Konan [13], proved combinatorially that

$$C(n) = |\{\lambda \in \mathcal{P}(n) \mid \text{mex}(\lambda) \text{ odd}\}|.$$

This completes the combinatorial proof of Theorem 11. □

Theorem 12 ([15, Corollary 7.3]). *Let m and n be nonnegative integers. Then*

(i) $\displaystyle\sum_{j=0}^{\infty}(-1)^j s_m(n-j(j+1)/2) = \sum_{j=0}^{\infty} jC(n-mj)$

(ii) $\displaystyle\sum_{j=1}^{\infty}(-1)^{j+1} s_m(n-j(j+1)/2) = \sum_{j=0}^{\infty} jD(n-mj).$

Proof. Theorem 1 implies that it is enough to prove Identities (i**) and (ii**) given in the proof of Theorem 11. □

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