PIECEWISE SYNDETIC SETS IN $\mathbb{N}^t$ AND $\mathbb{N}^X$

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Abstract

We give simple proofs that the basic facts concerning piecewise syndetic subsets of $\mathbb{N}$ apply equally well to piecewise syndetic subsets of $\mathbb{N}^t, t \geq 1$. Some of these facts apply also to piecewise syndetic subsets of $\mathbb{N}^X$ for any infinite set X.

1. Introduction

We write $\mathbb{N}$ for the set of positive integers $\{1, 2, 3, \dots\}$ and $[0, d]$ for the interval $\{0, 1, 2, \dots, d\}$. Let S be an infinite subset of $\mathbb{N}$. If there exists $d \in \mathbb{N}$ such that $S + [0, d]$ contains an infinite interval, then S is *syndetic*. If there exists $d \in \mathbb{N}$ such that $S + [0, d]$ contains arbitrarily large finite intervals of $\mathbb{N}$, then S is *piecewise syndetic*. (As usual, $S + [0, d]$ denotes the set $\{s + x : s \in S, x \in [0, d]\}$.) It is an elementary fact (see for example [21]) that every piecewise syndetic set S contains arbitrarily long arithmetic progressions.

We show that Fact 1 and Fact 2 below have natural extensions to $\mathbb{N}^t, t \geq 1$, and that (after extending the definition of "piecewise syndetic") Fact 1 extends also to $\mathbb{N}^X$ for any infinite set X.

Fact 1. The property of being piecewise syndetic is "partition regular;" that is, if $A_1 \cup A_2 \cup \dots \cup A_r \subseteq \mathbb{N}$ is piecewise syndetic then some A_i is piecewise syndetic.

In particular, if $\mathbb{N}$ is finitely colored, then some color class is piecewise syndetic. This fact appears to have been first stated in [5], first proved explicitly in [6], then discovered and proved independently by Hindman in [14], and proved again by Furstenberg in [12]. (It is also mentioned in [13], [16], [20], and [21].) An example showing that a set of positive upper density need not be piecewise syndetic is given in [3], and is mentioned in [21]. A "canonical version" of Fact 1 (where $\mathbb{N}$

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is colored with either finitely many colors or infinitely many colors) is given in [9]. An elementary proof (using van der Waerden's theorem) that if S is a piecewise syndetic subset of $\mathbb{N}$ then for each $k \geq 1$ the set

$$\{(a, d) : a, a + d, a + 2d, \ldots, a + kd \in S\}$$

is piecewise syndetic in $\mathbb{N}^2$, is given in [2]. In [13] (p. 56) is a proof of the interesting fact that if A, B are subsets of $\mathbb{N}$ with positive Banach density, then $A + B$ is piecewise syndetic. Other related work can be found in [8], [10], [11], and [22]. There are proofs of Facts 1 and 2 in [19].

To state Fact 2, we need the following definition.

Definition 1. For a finite subset $S = \{a_1 < a_2 < \cdots < a_n\}$ of $\mathbb{N}$, the *gap size* of S is

$$gs(S) = \max\{a_{j+1} - a_j : 1 \leq j \leq n - 1\}.$$

(If $|S| = 1$, we set $gs(S) = 1$.)

Fact 2. For all $r \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, there exists a (smallest) $B(f; r) \in \mathbb{N}$ such that if $[1, B(f; r)]$ is r-colored, there exists a monochromatic set S with $|S| > f(gs(S))$.

(There is certainly no analogous result for monochromatic arithmetic progressions. Jóseph Beck showed [1] in 1980 the existence of a 2-coloring of $\mathbb{N}$ for which any monochromatic arithmetic progression S with gap size d has $|S| \leq (1+\epsilon)\log_2 d$, for sufficiently large d. Similar results were proved in [7] and in [17].)

2. Extending Fact 1 to $\mathbb{N}^t$.

Definition 2. Let $t \in \mathbb{N}$. Any subset of $\mathbb{N}^t$ of the form $\mathbf{a} + [0, m - 1]^t$, where

$$\mathbf{a} \in \mathbb{N}^t, m \geq 1, [0, m - 1]^t = \{(x_1, x_2, \ldots, x_t) : 0 \leq x_i \leq m - 1, 1 \leq i \leq t\},$$

is called a *subcube* of $\mathbb{N}^t$ of *size* m^t.

Note that the word "*size*" here refers to cardinality. Thus, the subset $[1, 1]+[0, 4]^2$ of $\mathbb{N}^2$ has size 5^2. The interval $[10, 15]$ in $\mathbb{N}$ has size 6. A one-element subset of $\mathbb{N}^t$ is a subcube of size 1.

Definition 3. For any $t \geq 1$, if S is an infinite subset of $\mathbb{N}^t$, we say that S is *piecewise syndetic* if and only if for some $d \in \mathbb{N}, S + [0, d]^t$ contains arbitrarily large (finite) subcubes of $\mathbb{N}^t$.

Note that for $S \subseteq \mathbb{N}$, S is piecewise syndetic if and only if there exist $d \in \mathbb{N}$ and arbitrarily long intervals M of $\mathbb{N}$ such that if $[a, a+d] \subset M$, then $[a, a+d] \cap S \neq \emptyset$. (If d has this property then $S + [0, d]$ contains arbitrarily long intervals. For example, assume that S contains all multiples of 4. Then for any finite interval M, if $[a, a+3] \subset M$ then $[a, a+3] \cap S \neq \emptyset$, and in fact $S + [0, 3]$ will contain all of M except possibly for the first 3 and last 3 elements of M.)

If $S \subseteq \mathbb{N}$ is *not* piecewise syndetic then the negation of the previous property holds, i. e., for every $d \in \mathbb{N}$ there exists (a sufficiently large) $d_1 \in \mathbb{N}$ such that every interval $[a, a+d_1]$ in $\mathbb{N}$ contains a subinterval $E = [b, b+d]$ such that $E \cap S = \emptyset$.

For $\mathbb{N}^t, t \geq 2$, the same statements hold: If $S \subset \mathbb{N}^t$, then S is piecewise syndetic if and only if there exist $d \in \mathbb{N}$ and arbitrarily large subcubes M of $\mathbb{N}^t$ such that every subcube of M of size d^t meets S, and if S is *not* piecewise syndetic, then for every $d \in \mathbb{N}$ there exists $d_1 \in \mathbb{N}$ such that every subcube of $\mathbb{N}^t$ of size $(d_1)^t$ contains a subcube E of size d^t such that $E \cap S = \emptyset$.

The first statement (for $t \geq 2$) is less obvious than in the case $t = 1$, so let us illustrate it with an example for the case $t = 2$.

(In the next paragraph we write "*subsquare*" instead of "*subcube*" and "$d \times d$ subsquare" for any set of the form $(a, b) + [0, d-1]^2$.)

Suppose $S \subseteq \mathbb{N}^2$ and S has the property that for some fixed $d \in \mathbb{N}$ there are arbitrarily large subquares M of $\mathbb{N}^2$ such that every $d \times d$ subsquare of M meets S. Now fix a large subsquare M of size say m^2 such that every $d \times d$ subsquare of M meets S. Let us assume, for convenience of visualizing, that $m > 1000d$ and that the subsquare M has its sides parallel to the x and y axes of a coordinate plane. Consider the union, over all points $(x, y) \in M \cap S$, of all the $2d \times 2d$ subsquares $(x, y) + [0, 2d-1]^2$. These subsquares evidently cover all of M except perhaps for a vertical strip along the left side of M of width d, and a horizontal strip along the bottom of M, of height d. This shows that $S + [0, 2d-1]^2$ contains arbitrarily large subsquares, and hence S is piecewise syndetic.

The extension of Fact 1 to $\mathbb{N}^t$ has a simple proof.

Theorem 1. *If $r, t \in \mathbb{N}, A \subseteq \mathbb{N}^t, A$ is piecewise syndetic, and*

$$A = A_1 \cup A_2 \cup \cdots \cup A_r,$$

then some A_i is piecewise syndetic.

Proof. (We follow here the wonderful proof of Theorem 1 (in the case $t = 1$) which consists of the one-sentence statement in Chapter 14 of [15]: "One can verify combinatorially that the union of two sets which are not piecewise syndetic is not piecewise syndetic.")

Since A is piecewise syndetic, using induction on r it certainly suffices to show that if $B, C \subseteq \mathbb{N}^t$ and each of B, C is not piecewise syndetic then $B \cup C$ is not piecewise syndetic. Let $d \in \mathbb{N}$ be arbitrary. Since B is not piecewise syndetic, there exists d_1 (which depends on d) such that every subcube of $\mathbb{N}^t$ of size $(d_1)^t$ contains a subcube of size d^t which misses B. Since C is not piecewise syndetic, there exists d_2 (which depends on d_1) such that every subcube of $\mathbb{N}^t$ of size $(d_2)^t$ contains a subcube of size $(d_1)^t$ which misses C. Hence every subcube of $\mathbb{N}^t$ of size $(d_2)^t$ contains a subcube of $\mathbb{N}^t$ of size d^t which misses $B \cup C$. Since d_2 exists for every $d \in \mathbb{N}, B \cup C$ is not piecewise syndetic. □

3. Extending Fact 2 to $\mathbb{N}^t$.

A version of Fact 2 for $\mathbb{N}^t$ requires a definition of the *gap size* of a finite subset S of $\mathbb{N}^t$. This is Definition 5 below, which requires Definition 4.

Definition 4. Let $t \geq 1$ and let M be a subcube of $\mathbb{N}^t$. Let $d \in \mathbb{N}$ and let $S \subseteq \mathbb{N}^t$. We say that S *is d-dense in* M if every subcube of M of size d^t meets S.

The following definition is consistent with Definition 1 in the Introduction.

Definition 5. Let $t \geq 1$, let S be a fixed finite subset of $\mathbb{N}^t$, and let M be any subcube of $\mathbb{N}^t$ which contains S. Let $d(M)$ denote the smallest $d \in \mathbb{N}$ such that S is d-dense in M. The minimum $d(M)$, over all subcubes M of $\mathbb{N}^t$ which contain S, is denoted by $gs(S)$, and is called the *gap size* of S.

Theorem 2. *Let $r, t \in \mathbb{N}$ and let f be any nondecreasing function from $\mathbb{N}$ to $\mathbb{N}$. Then there exists a (smallest) $B_t(f;r) \in \mathbb{N}$ such that if $[1, B_t(f;r)]^t$ is r-colored, there exists a monochromatic set S such that $|S| > f(gs(S))$.*

Let us first give a short proof that the negation of Theorem 2 implies the negation of Theorem 1. After that we give a more constructive proof.

Proof 1 of Theorem 2. Assume now that Theorem 2 is false. That is, there are fixed $r, t \in \mathbb{N}$ and a nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$ there is an r - coloring χ_n of $[1, n]^t$ such that whenever $S \subseteq [1, n]^t$ is monochromatic with respect to χ_n, then $|S| \leq f(gs(S))$. We now define a chain of subsets $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ of $\{\chi_1, \chi_2, \chi_3, \dots\}$ in the following way. This is essentially an application of Kőnig's infinity lemma [18].

Let $\mathbb{N}^t = \{x_1, x_2, x_3, \dots\}$ be a fixed enumeration of the points of $\mathbb{N}^t$.

Let L_1 be an infinite subset of the set of colorings $\{\chi_1, \chi_2, \chi_3, \dots\}$ such that all of the colorings in L_1 agree at x_1. Let L_2 be an infinite subset of L_1 such that all of the colorings in L_2 agree on $\{x_1, x_2\}$. Continuing in this way, we obtain for each $u \geq 1$ an infinite set L_u of colorings (L_u is a subset of $\{\chi_1, \chi_2, \chi_3, \dots\}$), all of which agree on the set $\{x_1, x_2, \dots, x_u\}$.

Now we define a coloring χ of $\mathbb{N}^t$ which will contradict Theorem 1. For each $u \geq 1$, choose $h_u \in L_u$, and define the coloring χ by

$$\chi(x_u) = h_u(x_u), \quad u \geq 1.$$

By Theorem 1, some color class C of χ is piecewise syndetic, which means that for some fixed $d \in \mathbb{N}, C + [0,d]^t$ contains arbitrarily large (finite) subcubes M of $\mathbb{N}^t$. Then $S = M \cap C$ is monochromatic with respect to χ and S is d-dense in M, hence $gs(S) \leq d$, so $f(gs(S)) \leq f(d)$. As $|M|$ becomes large, so does $|S| = |M \cap C|$. Let M be contained in $\{x_1, x_2, \ldots, x_u\}$. Then χ restricted to $\{x_1, x_2, \ldots, x_u\}$ is identical to h_u restricted to $\{x_1, x_2, \ldots, x_u\}$, so by the assumption on h_u, $|S| \leq f(gs(S)) \leq f(d)$, contradicting Theorem 1, since d is fixed and S can be arbitrarily large. □

The following lemmas are needed for our second proof of Theorem 2.

Lemma 1. *Let $t \geq 1$ be given and fixed throughout this Lemma and proof. Let $1 = m_0 < m_1 < m_2 < \cdots$ be any strictly increasing sequence of positive integers. Then for each $r \geq 1$, the following statement S_r holds:*

S_r : Let M be any subcube of $\mathbb{N}^t$ of size $(m_r)^t$, and let an r-coloring of M be given, with color classes $C_1, C_2, \ldots, C_r$. Then there exist $i \in \mathbb{N}$, with $1 \leq i \leq r$, and a subcube M' of M of size $(m_i)^t$, such that C_i is m_{i-1} - dense in M'. (That is, every subcube of M' of size $(m_{i-1})^t$ meets the color class C_i.)

Proof. For $r = 1$, let M be any subcube of $\mathbb{N}^t$ of size $(m_1)^t$, and let M be 1-colored, with color class C_1. Then every subcube of M of size $(m_0)^t$ meets C_1.

Now let $r \geq 2$ and assume that S_{r-1} holds. Let M be any subcube of $\mathbb{N}^t$ of size $(m_r)^t$, and let an r-coloring of M be given, with color classes $C_1, C_2, \ldots, C_r$.

`Case 1`. If every subcube of M of size $(m_{r-1})^t$ meets C_r, then S_r holds by setting $i = r$ and $M' = M$, since then every size $(m_{r-1})^t$ subcube of M' meets C_r.

`Case 2`. There is a size $(m_{r-1})^t$ subcube M' of M which does not meet C_r. Now we have an $(r-1)$-coloring of a size $(m_{r-1})^t$ subcube of $\mathbb{N}^t$, and we are done by the induction hypothesis. □

Lemma 2. *Given $r, t \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, there exists (a smallest) $A_t(f; r) \in \mathbb{N}$ such that if $[1, A_t(f; r)]^t$ is r-colored, there exist $d, m \in \mathbb{N}$, a color class C, and a size m^t subcube M of $[1, A_t(f; r)]^t$, such that C is d-dense in M and $m > f(d)$.*

Proof. Let $r, t \in \mathbb{N}$ and let $f : \mathbb{N} \to \mathbb{N}$ be given. We apply Lemma 1 by defining

$$m_0 = 1, m_1 = 1 + f(m_0), m_2 = 1 + f(m_1), \ldots, m_r = 1 + f(m_{r-1}).$$

We can now show that $A_t(f; r) \leq m_r$. Let $[1, m_r]^t$ be r-colored, with color classes $C_1, \ldots, C_r$. By Lemma 1 there exists $i, 1 \leq i \leq r$, and a subcube M of $[1, m_r]^t$ of size

$(m_i)^t$ such that C_i is m_{i-1} - dense in M. Here M has size $(m_i)^t$, and $m_i > f(m_{i-1})$, so we can set $m = m_i,\ d = m_{i-1}, C = C_i$. □

Proof 2 of Theorem 2. We apply Lemma 2 with the given r, t and the function $g : \mathbb{N} \to \mathbb{N}$ given by $g(x) = xf(x) + x$, and we show that

$$B_t(f; r) \leq A_t(g; r).$$

Let $[1, A_t(g; r)]^t$ be r-colored. By Lemma 2 there exist $d, m \in \mathbb{N}$, a color class C, and a size m^t subcube M of $[1, A_t(g; r)]^t$ such that C is d-dense in M and $m > g(d)$.

Let $S = M \cap C$. Since S is monochromatic, there remains only to show that $|S| > f(gs(S))$. We have that S is d-dense in M (since C is d-dense in M), so $d \geq gs(S)$ by Definition 5. Since f is non-decreasing, $f(d) \geq f(gs(S))$.

The subcube M of size m^t contains $[\frac{m}{d}]^t$ pairwise disjoint subcubes of size d^t, and each of these meets S. Finally, since $m > g(d) = df(d) + d$, we have

$$|S| \geq ([\frac{m}{d}])^t > ([\frac{m}{d}] - 1)^t \geq ([\frac{df(d) + d}{d}] - 1)^t = (f(d))^t \geq f(d) \geq f(gs(S)).$$

□

4. Replacing t by an Infinite Set X.

Perhaps it is of interest to note that some of these results extend to finite colorings of $\mathbb{N}^X$, for any infinite set X. We outline this below, including some necessary modifications of the definitions.

Definition $\tilde{2}$. Let X be an infinite set. Any subset M of $\mathbb{N}^X$ of the form $M = \mathbf{a} + [0, m-1]^X$, where $\mathbf{a} \in \mathbb{N}^X, m \in \mathbb{N}$, is called a *subcube* of $\mathbb{N}^X$.

Of course, by Y^X we mean $\{f : X \to Y\}$.

Definition $\tilde{3}$. Let X be an infinite set. A subset S of $\mathbb{N}^X$ is called *piecewise syndetic* if for some $d \in \mathbb{N},\ S + [0, d]^X$ contains subcubes of $\mathbb{N}^X$ of the form $\mathbf{a} + [0, m-1]^X$ for arbitrarily large $m \in \mathbb{N}$.

Theorem $\tilde{1}$. If $r \in \mathbb{N}$, X is an infinite set, $A \subseteq \mathbb{N}^X$, A is piecewise syndetic, and

$$A = A_1 \cup A_2 \cup \cdots \cup A_r,$$

then some A_i is piecewise syndetic.

Proof. The proof is the same as the proof of Theorem 1. □

Definition $\tilde{4}$. Let X be an infinite set and let M be a subcube of $\mathbb{N}^X$. Let $d \in \mathbb{N}$ and let $S \subseteq \mathbb{N}^X$. We say that S is *d-dense* in M if every subcube of M of the form $\mathbf{b} + [0, d-1]^X$ meets S.

Definition $\tilde{5}$. Let X be an infinite set, let S be a fixed finite subset of $\mathbb{N}^X$, and let M be any subcube of $\mathbb{N}^X$ which contains S. Let $d(M)$ denote the smallest $d \in \mathbb{N}$ such that S is d-dense in M. The minimum of $d(M)$, over all subcubes M of $\mathbb{N}^X$ which contain S, is denoted by $gs(S)$ and is called the *gap size of S*.

It seems that Theorem 2 has no analogue in the context of $\mathbb{N}^X$. However, Lemmas 1 and 2 make perfect sense, with the same proofs, if they are re-stated in the following way.

Lemma $\tilde{1}$. *Let X be a fixed infinite set. Let $1 = m_0 < m_1 < m_2 < \cdots$ be any strictly increasing sequence of positive integers. Then for each $r \geq 1$, the following statement S_r holds:*

S_r : Let M be any subcube of $\mathbb{N}^X$ of the form $\boldsymbol{a} + [0, m_r - 1]^X$, and let an r-coloring of M be given, with color classes $C_1, C_2, \ldots, C_r$. Then there exist $i \in \mathbb{N}$, with $1 \leq i \leq r$, and a subcube M' of M of the form $\mathbf{b} + [0, m_i - 1]^X$, such that C_i is m_{i-1} - dense in M'. (That is, every subcube of M' of the form $\mathbf{c} + [0, m_{i-1} - 1]^X$ meets the color class C_i.)

Lemma $\tilde{2}$. *Given $r \in \mathbb{N}$ and an infinite set X, and given $f : \mathbb{N} \to \mathbb{N}$, there exists (a smallest) $A_X(f;r) \in \mathbb{N}$ such that if $[1, A_X(f;r)]^X$ is r-colored, there exist $d, m \in \mathbb{N}$, a color class C, and a subcube M of $[1, A_X(f;r)]^X$ of the form $M = \boldsymbol{a} + [0, m-1]^X$ such that C is d-dense in M and $m > f(d)$.*

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