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# ARITHMETICAL SELF-SIMILAR COMPACT SETS

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**Abstract**

A natural number $n$ is called stable if $\|3^k n\| = \|n\| + 3k$, where $\|n\|$ is the complexity of $n$. For each natural number $n$, the number $3^a n$ is stable for some $a \geq 0$. We define the stable complexity of $n$ as $\|n\|_{\mathrm{st}} := \|3^a n\| - 3a$. We show that the closure $K$ of the set of all fractions $n/3^{\lfloor \|n\|_{\mathrm{st}}/3 \rfloor}$ has remarkable properties: it is self-similar, $3K''' = K$ (where $K^{'''}$ is the third derived set of $K$), it is well-ordered, and has certain arithmetical noteworthy properties. We pose the question of the uniqueness of this compact set. This question raises other questions on the complexity of natural numbers, which we are also unable to answer.

## 1. Introduction

The *complexity* $\|n\|$ of a natural number $n$ is the least number of 1's needed to write an expression for $n$ using only the addition $+$, the product $*$, the unit 1, and parentheses $(,)$. So $\|9\| = 6$, since $9 = (1+1+1)*(1+1+1)$ and there is no other expression for 9 with less than 6 unit symbols. Richard Guy [10] popularized several problems on this concept. For example, does $\|2^n\| = 2n$ hold for all $n \geq 1$?

It is rather frequent that $\|3n\| = \|n\| + 3$. Therefore, it is natural to consider those fractions $n/3^{\lfloor \|n\|/3 \rfloor}$, for $n$ satisfying $\|3n\| = \|n\| + 3$, which remain invariant when $n$ is changed to $3n$. A number is called *stable* when $\|n3^k\| = \|n\| + 3k$ for all $k \geq 1$.

In [9] we posed some conjectures on these fractions, which were proved (some in a modified form) by Altman and Zelinsky [1], Altman [2], and Altman and Arias de Reyna [7]. In particular, it was shown, for each $m \geq 1$, that there is $a \geq 0$ such that $n = m3^a$ is stable [1, Theorem 13]. In this case, the *stable complexity* of $m$ is defined as $\|m\|_{\mathrm{st}} := \|m3^a\| - 3a$.

The main purpose of this paper is to show how the underlying structure of integer complexity (described by the above mentioned conjectures) is almost equivalent

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to the existence of a compact set $K$ of rational numbers with certain remarkable properties. These properties are,

1. The compact set $K$ is self-similar, as shown by the equation $3K''' = K$, where $K^{'''}$ is the third derived set of $K$.

2. The compact set $K$ is well-ordered, using the reverse of the usual order.

3. For each natural number $m$ with $3 \nmid m$ there is a non-negative integer $\kappa(m)$ such that
$$K = \{0\} \cup \Big\{\frac{m}{3^k} : 3 \nmid m, k \geq \kappa(m)\Big\}.$$

There are additional properties of $K$, which are described in detail in Theorem 1.

A compact set $K \subset [0, \infty)$ satisfying the above properties will be called an *arithmetical compact set*. The aim of this paper is to present such a compact set, based on integer complexity, and to pose the question of its uniqueness. It should be noted that it is possible to define another arithmetical compact set (based on addition chains) satisfying $2K' = K$.

In Section 2 we define arithmetical compact sets, Definition 4. This definition makes more transparent the role of the conjectures in [9].

Section 3 is devoted to proving the existence of the arithmetical compact set $K$ derived from integer complexity. This is done by translating into the language of [9] the results of [1], [2], and [7] where the conjectures are refined and proved.

The compact set $K$ was first glimpsed while working on [9]. The fact that $K$ contains all fractions $2^n/3^{\lfloor 2n/3 \rfloor}$, together with the fact that Theorem 1 below holds, seems to determine the remaining elements of $K$. This suggests the uniqueness of $K$. The conjectures formulated in [9] were aimed at using this idea to obtain precise rules for constructing $K$.

In Section 4 we discuss the uniqueness of $K$ when certain self-similarity parameters and its largest element are specified. Since the derivate sets $K^{(n)}$, for all $n \geq 1$, are also arithmetical compact sets, we can pose the question of whether the set $K$ is unique under the additional condition that its largest element is 2. The answer is probably no. The set candidate to show this fact is an arithmetical compact set $H$ constructed by Harry Altman [8]. However, verifying that $H \neq K$ turns out to be not trivial. In the Appendix we give some of the first elements of $K$, which are also the first elements of $H$. In this section we discuss possible candidates for elements $q \in K \smallsetminus H$. Proving that these elements exist is, however, beyond our computational capability.

## 2. Arithmetical Self-Similar Compact Sets

For any set $A \subset \mathbf{R}$, we denote by $A'$ the *derived set* of $A$, that is, the set of accumulation points of $A$. For any nonnegative integer $n$, $A^{(n)}$ is the $n$-th *derived set*, with $A^{(0)} := A$.

We consider the set of real numbers $\mathbf{R}$ ordered by the *reverse usual order*, that is, $x \preccurlyeq y$ if and only if $x \geq y$. A *well-ordered* set is a set that is isomorphic to an ordinal. Let $K$ be a well-ordered set with order type $\omega^\omega + 1$. For $0 \leq \alpha \leq \omega^\omega$, we denote by $K[\alpha]$ the corresponding element in $K$. Then, we have $K \subset [K[\omega^\omega], K[0]]$. This will be the case for our main example.

### 2.1. Self-Similar Compact Sets

**Definition 1.** Let $\rho > 1$ be a real number and $\mu$ a natural number. A non empty compact set $K \subset [0, +\infty)$ will be called a *self-similar compact set* of *module* $\mu$ and *ratio* $\rho$ if $K$ is well-ordered by the relation of order $\preccurlyeq$ and satisfies

$$\rho K^{(\mu)} = K. \tag{1}$$

**Proposition 1.** *Let $K$ be a self-similar compact set. Then $K[\omega^\omega] = 0$, $K[0] > 0$, and $K \subset [0, K[0]]$.*

*Proof.* Since $K$ is a compact set, there exists $a > 0$ such that $K \subset [-a, a]$. Since $K$ is self-similar, we have $K^{(\mu)} = K/\rho$, for some $\rho$ and $\mu$. Applying successively this relation we obtain $K^{(n\mu)} = K/\rho^n \subset [-a/\rho^n, a/\rho^n]$. Since $\rho > 1$, it follows that $K^{(\omega)} := \bigcap_{n=0}^{\infty} K^{(n)} = \{0\}$. Therefore, $K[\omega^\omega] = 0$. So, the order type of $K$ is $\omega^\omega + \alpha$ for some $\alpha < \omega^\omega$. Let's see that $\alpha = 1$. For some $n$ we have $K^{(n\mu)} \subset [0, K[0]]$, so $K$ does not contain negative elements and the order type of $K$ is $\omega^\omega + 1$. For any $y \in K$ we have $K[0] \preccurlyeq y \preccurlyeq K[\omega^\omega] = 0$. Therefore, $0 \leq y \leq K[0]$. Since $K[0] \neq K[\omega^\omega]$, we have $K[0] > 0$. □

**Proposition 2.** *Let $K$ be a self-similar compact set of module $\mu$ and ratio $\rho$. Then*

$$K = \{0\} \cup \bigcup_{n=0}^{\infty} (T_0 \cup T_1 \cup \cdots \cup T_{\mu-1})/\rho^n, \tag{2}$$

*where $T_u := K^{(u)} \setminus K^{(u+1)}$.*

*Proof.* For $n \geq 0$, we have $K^{(n)} \supset K^{(n+1)}$, and $\bigcap_{n=0}^{\infty} K^{(n)} = \{0\}$, as we have seen in Proposition 1. Define $T_u := K^{(u)} \setminus K^{(u+1)}$, for $u \in \mathbf{Z}_{\geq 0}$. The sets $T_u$ are disjoint and

$$K \setminus \{0\} = \bigcup_{u=0}^{\infty} T_u. \tag{3}$$

Applying the relation $K^{(\mu)} = K/\rho$ successively, we obtain $K^{(u+n\mu)} = K^{(u)}/\rho^n$. Therefore, $T_{u+n\mu} = T_u/\rho^n$. This fact, together with (3), implies (2). □

**Proposition 3.** *Let $K$ be a self-similar compact set of module $\mu$ and ratio $\rho$. Let $\alpha$ be an ordinal satisfying $0 \leq \alpha \leq \omega^\omega$. The following hold,*

$$K^{(n)}[\alpha] = K[\omega^n(1+\alpha)], \quad \text{for } n \geq 1.$$
$$T_0[\alpha] = \begin{cases} K[\alpha] & \text{if } \alpha < \omega, \\ K[\alpha+1] & \text{if } \alpha \geq \omega. \end{cases}$$
$$T_u[\alpha] = K[\omega^u(\alpha+1)], \quad \text{for } u \geq 1.$$

*Proof.* The set $K'$ is the set of limit points of $K$, which correspond to the limit ordinals in $\omega^\omega + 1$. Such ordinals can be written as $\omega(1+\alpha)$. Therefore, $K'[\alpha] = K[\omega(1+\alpha)]$. It follows that

$$K''[\alpha] = K'[\omega(1+\alpha)] = K[\omega(1+\omega(1+\alpha))] = K[\omega^2(1+\alpha)].$$

A similar argument allows us to deduce that $K^{(n)}[\alpha] = K[\omega^n(1+\alpha)]$.

The elements in $T_0$ correspond to the set of ordinals which consists of 0 and all successor ordinals less than $\omega^\omega$. The map that sends $\alpha$ to $\alpha$ or to $\alpha+1$, according to whether $\alpha < \omega$ or $\alpha \geq \omega$, is injective, increasing, and its image consists of all successors and 0. This implies the formula for $T_0[\alpha]$ claimed above. For $u > 0$, the elements in $T_u$ are the elements in $K^{(u)} = \{K[\omega^u(1+\alpha)] \colon 0 \leq \alpha \leq \omega^\omega\}$ which are not limit points of $K^{(u)}$, that is, $T_u = \{K[\omega^u(\alpha+1)] \colon 0 \leq \alpha \leq \omega^\omega\}$ (note that the ordinals $(1+\alpha)$ are successor or limits; we eliminate the limits by taking the successors). □

**Proposition 4.** *For integers $0 \leq \alpha < \omega^\omega$ and $u \in \mathbf{Z}_{\geq 0}$, we have*

$$\lim_{n\to\infty} T_u[\omega\alpha + n] = T_{u+1}[\alpha].$$

*When $u = \mu - 1$, we may write the above as*

$$\lim_{n\to\infty} T_{\mu-1}[\omega\alpha + n] = \frac{T_0[\alpha]}{\rho}.$$

*Proof.* Since $K$ is a compact set, its limit points are also limit points in the topological sense. So, for $u \geq 0$, we have

$$\lim_{n\to\infty} K[\omega\alpha + n] = K[\omega(\alpha+1)],$$
$$\lim_{n\to\infty} K[\omega^u(\omega\alpha + n)] = K[\omega^{u+1}(\alpha+1)].$$

Hence, for $u \geq 0$, we have

$$\lim_{n\to\infty} T_u[\omega\alpha + n] = \lim_{n\to\infty} K[\omega^u(\omega\alpha + n + 1)] = K[\omega^{u+1}(\alpha+1)] = T_{u+1}[\alpha].$$

In the case when $u = 0$ and $\alpha = 0$ we have

$$\lim_{n\to\infty} T_0[n] = \lim_{n\to\infty} K[n] = K[\omega] = T_1[0].$$

If $u = \mu - 1$, note that

$$T_\mu = K^{(\mu)} \smallsetminus K^{(\mu+1)} = \rho^{-1}(K \smallsetminus K') = \rho^{-1}T_0. \qquad \square$$

Next we define 'arithmetical' self-similar compact sets of module $\mu$ and ratio $\rho$. It is more convenient to present the definition in two steps. Firstly, we define 'quasi-arithmetical' compact set. This allows to define the 'stability gauge' $\|n\|_K$ before defining the (more interesting) concept of 'arithmetical compact sets'.

### 2.2. Quasi-Arithmetical Self-Similar Compact Sets

If $K$ is a self-similar compact set and $x \in K$ with $x \neq 0$, we have

$$\cdots \preccurlyeq x\rho^n \preccurlyeq x\rho^{n-1} \preccurlyeq \cdots \preccurlyeq x\rho \preccurlyeq x.$$

Since $K$ is well-ordered, there is some $n \geq 0$ such that $y := x\rho^n \in K$, but $y\rho \notin K$.

**Definition 2.** A self-similar compact set $K$ of module $\mu$ and ratio $\rho$ is called *quasi-arithmetical* if $\rho$ is a natural number with $\rho \geq 2$, and there is a function $\kappa\colon \mathbf{N} \smallsetminus (\rho\mathbf{N}) \to \mathbf{Z}_{\geq 0}$ such that

$$K \smallsetminus \{0\} = \Big\{\frac{n}{\rho^k} \colon \rho \nmid n \text{ and } k \geq \kappa(n)\Big\}.$$

**Proposition 5.** *Let $K$ be a quasi-arithmetical compact set of module $\mu$ and ratio $\rho$. Assume that $\frac{n}{\rho^p} \in T_v$ and $\frac{n}{\rho^q} \in T_w$. Then we have: $p\mu - v = q\mu - w$.*

*Proof.* Let $m \in \mathbf{N}$ be such that $\rho \nmid m$ and set $k = \kappa(m)$. Equation (2) implies that $\frac{m}{\rho^k} \in T_0 \cup T_1 \cup \cdots \cup T_{\mu-1}$. Since these sets are disjoint, there is a unique $u \in \{0, 1, \dots, \mu-1\}$ with $\frac{m}{\rho^k} \in T_u$. Then $\frac{m}{\rho^{k+\ell}} \in T_{u+\ell\mu}$. Since the sets $T_v$ are disjoint, a fraction $m/\rho^a$ is contained in $T_v$ if and only if $a \geq k$ with $a = k + \ell$ and $v = u + \ell\mu$.

Suppose that $\frac{n}{\rho^p} \in T_v$ and $\frac{n}{\rho^q} \in T_w$. Let $n = m\rho^r$, with $\rho \nmid m$ and $r \geq 0$. In the notation used above, we have, for some $\ell$, that $p - r = k + \ell$ and $v = u + \ell\mu$. Therefore,

$$p\mu - v = (r + k + \ell)\mu - (u + \ell\mu) = (r + k)\mu - u = q\mu - w. \qquad \square$$

**Definition 3.** Let $K$ be a quasi-arithmetical compact set $K$ of module $\mu$ and ratio $\rho$. For any natural number $n$, the *stability gauge* $\|n\|_K$ of $n$ is

$$\|n\|_K := \ell\mu + d - u, \text{ when } \frac{n}{\rho^\ell} \in T_u,$$

where the *displacement* $d$ is chosen so that $\|1\|_K = 0$.

Note, by Proposition 5, that $\|n\|_K$ does not depend on the choice of $\ell$ and $u$ satisfying $\frac{n}{\rho^\ell} \in T_u$. The definition of $\|n\|_K$ above may seem arbitrary. However, we know of only two examples of arithmetical self-similar sets. In both cases the definition seems to be a correct choice.

**Proposition 6.** *For any natural number $n$, we have $\|\rho n\|_K = \|n\|_K + \mu$.*

*Proof.* Determine $\ell$ so that $n/\rho^\ell \in T_u$. Then $\rho n/\rho^{\ell+1} \in T_u$ and so

$$\|\rho n\|_K = (\ell+1)\mu + d - u = \|n\|_K + \mu. \qquad \square$$

### 2.3. Arithmetical Self-Similar Compact Sets

**Definition 4.** An *arithmetical compact set* is a quasi-arithmetical self-similar compact set $K$ satisfying the following condition. For an ordinal $0 \leq \alpha < \omega^\omega$ and $u \in \mathbf{Z}_{\geq 0}$, set $T_{u+1}[\alpha] = \frac{n}{\rho^k}$, with $\rho \nmid n$. Then the sets

$$\{T_u[\omega\alpha + r] \colon r \in \mathbf{Z}_{\geq 0}\}, \tag{4}$$

$$\left\{\frac{b(a\rho^r+1)}{\rho^{r+k}} \colon n = ab, \quad \|n\|_K = \|a\|_K + \|b\|_K, \quad r \in \mathbf{Z}_{\geq 0}\right\}, \tag{5}$$

have a finite symmetric difference.

**Remark 1.** Definition 4 does not apply to the other arithmetical self-similar compact set satisfying $2K' = K$. We do not have a definition that holds in both cases. An ideal definition should describe the set $\{T_u[\omega\alpha + r] \colon r \in \mathbf{Z}_{\geq 0}\}$ independently, at least except for a finite set.

### 2.4. Properties of Arithmetical Self-Similar Compact Sets

**Proposition 7.** *Let $K$ be an arithmetical compact set of module $\mu$ and ratio $\rho$. Then, for any natural number $n$, the $n$-th derived set $K^{(n)}$ is also an arithmetical compact set of the same module and ratio.*

*Proof.* It suffices to show that $K'$ is an arithmetical compact set. Taking the derived sets of $\rho K^{(\mu)} = K$, we get $\rho(K')^{(\mu)} = K'$. The derived set $K'$ is a compact set because the set of limit points of a compact set is compact. Since it is contained in $K$, it is well-ordered by $\preccurlyeq$. Since $K' = \{K[\omega\beta] \colon 1 \leq \beta \leq \omega^\omega\}$, its order type is just $\omega^\omega + 1$. So $K'$ is a self-similar compact set of module $\mu$ and ratio $\rho$.

Next, we prove that $\|n\|_K = \|n\|_{K'}$. Assume first that

$$1/\rho^a \in T_u = K^{(u)} \smallsetminus K^{(u+1)} = K'^{(u-1)} \smallsetminus K'^{(u)}$$

with $0 < u$ (there is always $u > 0$ satisfying this condition). Then

$$\|1\|_{K'} = a\mu + d' - (u-1) = 0 = \|1\|_K = a\mu + d - u.$$

It follows that the displacements $d'$ for $K'$ and $d$ for $K$ are related by $d' = d - 1$.

For any natural number $n$ not divisible by $\rho$, there is an integer $\ell \geq 0$ and $u > 0$ such that

$$n/\rho^{\ell} \in K^{(u)} \smallsetminus K^{(u+1)} = K'^{(u-1)} \smallsetminus K'^{(u)}.$$

Therefore,

$$\|n\|_{K'} = \ell\mu + d' - (u-1) = \ell\mu + d - u = \|n\|_K.$$

Denote by $T_u(L)$ the set $T_u$ corresponding to a self-similar compact set $L$. Then

$$T_u(K') = (K')^u \smallsetminus (K')^{(u+1)} = T_{u+1}(K).$$

Hence, the assertion of $K'$ being arithmetical is just part of the assertion of $K$ being arithmetical, due to the equality of the stable complexity associated with both compact sets. $\square$

## 3. The Arithmetical Compact Set Associated with the Complexity of Natural Numbers

Recall that $\|n\|$ and $\|n\|_{\text{st}}$ are, respectively, the complexity and the stable complexity of a natural number $n$. When $K$ is a quasi-arithmetical compact set, $\|n\|_K$ is the corresponding stability gauge.

In [9], the followings sets, for $u \in \{0, 1, 2\}$, were considered:

$$S_u = \{n/3^{\ell} \colon n \text{ a stable number and } \|n\| = 3\ell + u\}.$$

On the other hand, in [1], [2], [7], the following sets were considered

$$\mathscr{D}^u_{\text{st}} = \{\delta(n) \colon\ n \text{ a stable number and } \|n\| \equiv u \bmod 3\},$$

where $\delta(n) = \delta_{\text{st}}(n) = \|n\| - 3\log_3 n$ is the (stable) defect of $n$.

The difference between these sets is just due to the different notation used, since there is a bijective correspondence between them which sends $x \in S_u$ to $u - 3\log_3 x \in \mathscr{D}^u_{\text{st}}$. This is established by the following argument.

Let $y \in \mathscr{D}^u_{\text{st}}$. Then $y$ is the defect of a stable number $n$, that is, $y = \delta(n)$. This means that $y = \|n\| - 3\log_3 n$ with $\|n\| = 3\ell + u$, for the given $u \in \{0, 1, 2\}$. Therefore, $x := \frac{n}{3^{\ell}} \in S_u$. Then

$$u - 3\log_3 x = u - 3\log_3 \frac{n}{3^{\ell}} = 3\ell + u - 3\log_3 n = \delta(n) = y.$$

This shows that to any stable number $n$ with $\|n\| \equiv u \bmod 3$ there are associated $y \in \mathscr{D}^u_{\text{st}}$ and $x \in S_u$ satysfying $u - 3\log_3 x = y$. The sets $\mathscr{D}^u_{\text{st}}$ and $S_u$ are both well-ordered, and the relation $u - 3\log_3 x = y$ respects order. Therefore, for any ordinal $0 \le \alpha \le \omega^\omega$, we have

$$\mathscr{D}_{\text{st}}^u[\alpha] = u - 3\log_3 S_u[\alpha]. \tag{6}$$

Theorem 4.19 in [7] can be stated using this notation as follows.

**Theorem 1.** *For any ordinal $\beta < \omega^\omega$, let $S_u[\beta] = \frac{n}{3^k}$ with $\|n\|_{\text{st}} = 3k + u$, for an integer $u \ge 0$. Then, for each $u = 0, 1, 2$, the sets*

$$\{S_{u+1}[\omega\beta + r] \colon r \in \mathbf{Z}_{\ge 0}\},$$
$$\Big\{\frac{b(a3^r+1)}{3^{r+k+\varepsilon}} \colon n = ab, \quad \|n\|_{\text{st}} = \|a\|_{\text{st}} + \|b\|_{\text{st}}, \quad r \in \mathbf{Z}_{\ge 0}\Big\},$$

*have finite symmetric difference. (We consider $u + 1 \bmod 3$, so $u + 1 = 0$ when $u = 2$; and $\varepsilon = 0$ for $u = 0, 1$, and $\varepsilon = 1$ for $u = 2$.)*

Consider the sets $S_i$, for $i = 0, 1, 2$, defined in Theorem 1. We label them in a different way. For $u \in \{0, 1, 2\}$, set

$$T_u := \Big\{\frac{n}{3^k} \colon n \text{ stable with } \|n\| = 3k + 2 - u\Big\}. \tag{7}$$

Then $T_0 = S_2$, $T_1 = S_1$ and $T_2 = S_0$. We will show that $K = \overline{T_0}$ is an arithmetical compact set with module $\mu = 3$ and ratio $\rho = 3$. Note that the sets $T_u$ coincide with the sets $T_u(K)$ defined in Proposition 2. In this section, we will only consider the sets $T_u$ defined via (7).

**Theorem 2.** *The set $K = \overline{T_0}$ (with $T_0$ defined as in (7)) is an arithmetical compact set with module $\mu = 3$ and ratio $\rho = 3$. The stability gauge $\|n\|_K$ coincides with the stable complexity $\|n\|_{\text{st}}$.*

*Proof.* The largest element with $\|n\| = 3k + 2$ is $E(k) = 2 \cdot 3^k$ [6, Thmeorem 1.1]. Hence, if $n \in T_0$ with $\|n\| = 3k + 2$, we have $\frac{n}{3^k} \le 2$. Therefore, $T_0 \subset [0, 2]$ and hence, the set $K$ is compact.

The set $T_u$ is related, to the set $\mathscr{D}_{\text{st}}^{2-u}$ in [2, Def. 2.4], namely

$$\mathscr{D}_{\text{st}}^{2-u} := \{\delta(n) : n \text{ is a stable number with } \|n\| \equiv 2 - u \bmod 3\}.$$

To see this, let $f(x) = 3^{(2-x)/3}$. Then $f(\mathscr{D}_{\text{st}}^{2-u} + u) = T_u$. It follows that $f(\overline{\mathscr{D}_{\text{st}}^2})$ is a closed set in $(0, \infty)$, with 0 as an accumulation point, because $\sup \mathscr{D}_{\text{st}}^2 = +\infty$. Consequently,

$$K \smallsetminus \{0\} = f(\overline{\mathscr{D}_{\text{st}}^2}).$$

Due to [2, Theorem 7.4 (4)], the set $\overline{\mathscr{D}_{\text{st}}^2}$ is well-ordered, for the usual order, with order type $\omega^\omega$. Since the map $f$ is monotonically decreasing, the set $K$ is well-ordered by $\preccurlyeq$ with order type $\omega^\omega + 1$, as it is obtained by joining 0, the maximal element in the order $\preccurlyeq$, to the image of $\overline{\mathscr{D}_{\text{st}}^2}$.

We have $(\overline{\mathscr{D}^2_{\rm st}})''' = \overline{\mathscr{D}^2_{\rm st}} + 3$ (cf. [7, Prop. 4.6]). Based on the above argument, the previous equality is equivalent to $(K \smallsetminus \{0\})''' = \frac{1}{3}(K \smallsetminus \{0\})$. Adding to these sets the element $\{0\}$, we obtain $K''' = \frac{1}{3}K$. Hence, $K$ is a self-similar compact set with module $\mu = 3$ and ratio $\rho = 3$.

In [7, Corollary 4.7] it is shown, for $u \in \{0, 1, 2\}$, that

$$\begin{aligned}
\overline{\mathscr{D}^2_{\rm st}} &= (\mathscr{D}^2_{\rm st} + 3\mathbf{Z}_{\geq 0}) \cup (\mathscr{D}^1_{\rm st} + 3\mathbf{Z}_{\geq 0} + 1) \cup (\mathscr{D}^0_{\rm st} + 3\mathbf{Z}_{\geq 0} + 2),\\
\overline{\mathscr{D}^1_{\rm st}} &= (\mathscr{D}^1_{\rm st} + 3\mathbf{Z}_{\geq 0}) \cup (\mathscr{D}^0_{\rm st} + 3\mathbf{Z}_{\geq 0} + 1) \cup (\mathscr{D}^2_{\rm st} + 3\mathbf{Z}_{\geq 0} + 2),\\
\overline{\mathscr{D}^0_{\rm st}} &= (\mathscr{D}^0_{\rm st} + 3\mathbf{Z}_{\geq 0}) \cup (\mathscr{D}^2_{\rm st} + 3\mathbf{Z}_{\geq 0} + 1) \cup (\mathscr{D}^1_{\rm st} + 3\mathbf{Z}_{\geq 0} + 2).
\end{aligned}$$

By [7, Prop. 4.6] we have

$$(\overline{\mathscr{D}^2_{\rm st}})' = \overline{\mathscr{D}^1_{\rm st}} + 1, \quad (\overline{\mathscr{D}^2_{\rm st}})'' = \overline{\mathscr{D}^0_{\rm st}} + 2.$$

Combining the above equalities, we obtain

$$\begin{aligned}
\overline{\mathscr{D}^2_{\rm st}} &= (\mathscr{D}^2_{\rm st} + 3\mathbf{Z}_{\geq 0}) \cup (\mathscr{D}^1_{\rm st} + 3\mathbf{Z}_{\geq 0} + 1) \cup (\mathscr{D}^0_{\rm st} + 3\mathbf{Z}_{\geq 0} + 2),\\
(\overline{\mathscr{D}^2_{\rm st}})' &= (\mathscr{D}^1_{\rm st} + 3\mathbf{Z}_{\geq 0} + 1) \cup (\mathscr{D}^0_{\rm st} + 3\mathbf{Z}_{\geq 0} + 2) \cup (\mathscr{D}^2_{\rm st} + 3\mathbf{Z}_{\geq 0} + 3),\\
(\overline{\mathscr{D}^2_{\rm st}})'' &= (\mathscr{D}^0_{\rm st} + 3\mathbf{Z}_{\geq 0} + 2) \cup (\mathscr{D}^2_{\rm st} + 3\mathbf{Z}_{\geq 0} + 3) \cup (\mathscr{D}^1_{\rm st} + 3\mathbf{Z}_{\geq 0} + 4).
\end{aligned}$$

Applying $f$ to these equalities, we get the first three of the following equalities:

$$\begin{aligned}
K \smallsetminus \{0\} &= \bigcup_{n\geq 0} (T_0 \cup T_1 \cup T_2)/3^n, \qquad (8)\\
K' \smallsetminus \{0\} &= T_1 \cup T_2 \cup \bigcup_{n\geq 1} (T_0 \cup T_1 \cup T_2)/3^n,\\
K'' \smallsetminus \{0\} &= T_2 \cup \bigcup_{n\geq 1} (T_0 \cup T_1 \cup T_2)/3^n,\\
K''' \smallsetminus \{0\} &= \bigcup_{n\geq 1} (T_0 \cup T_1 \cup T_2)/3^n.
\end{aligned}$$

The last equality follows from $K''' \smallsetminus \{0\} = (K \smallsetminus \{0\})/3$, together with the first equality in (8). It is easy to see that the sets $T_u/3^n$ are disjoint. From this and the above equalities, we easily get the following

$$T_0(K) = T_0, \quad T_1(K) = T_1, \quad T_2(K) = T_2.$$

To see that $K$ is quasi-arithmetical, we need to show (by (8)) that there is a function $\kappa\colon (\mathbf{N} \smallsetminus 3\mathbf{N}) \to \mathbf{Z}_{\geq 0}$ such that

$$A := \bigcup_{n\geq 0} (T_0 \cup T_1 \cup T_2)/3^n = \left\{\frac{m}{3^k} : 3 \nmid m \text{ and } k \geq \kappa(m)\right\} := B. \qquad (9)$$

Let us define $\kappa$. Let $m$ be a natural number with $3 \nmid m$. By [1, Theorem 13] there exists $a \geq 0$ such that $n = m3^a$ is stable. If $\|n\| = 3\ell + 2 - u$, with $u \in \{0, 1, 2\}$, then $\frac{m}{3^{\ell-a}} \in T_u$. In this case, we define $\kappa(m) = \ell - a$. It is easy to see that this value does not depend on the chosen value of $a$, making $m3^a$ stable. If $\|n\| = 3\ell + 2 - u$, we have $n < 3^{\ell+1}$ or $1 \leq m < 3^{\ell-a+1}$; hence $\kappa(m) = \ell - a \geq 0$.

Next we prove (9). With the same notation as above and for any $k \geq \kappa(m)$, we have $m/3^k$ on the left-hand side of (9). So, $B \subset A$. On the other hand, if $\frac{n}{3^k} \in A$, then $\frac{n}{3^k} \in T_u/3^p$. It follows that $\frac{n}{3^{k-p}} \in T_u$, that is, $\frac{n}{3^{k-p}} = \frac{t}{3^s}$, where $t$ is stable and $\|t\| = 3s + 2 - u$. Consequently, $t = m3^a$ with $3 \nmid m$ and $\kappa(m) = s - a$. It follows that

$$\frac{n}{3^k} = \frac{1}{3^p}\frac{t}{3^s} = \frac{m3^a}{3^p 3^s} = \frac{m}{3^{p+s-a}} = \frac{m}{3^{p+\kappa(m)}},$$

with $p+\kappa(m) \geq \kappa(m)$. Therefore, $\frac{n}{3^k} \in B$. This shows that $K$ is a quasi-arithmetical compact set.

Next we prove that $\|n\|_K = \|n\|_{\mathrm{st}}$. Since 3 is a stable number with $\|3\|_{\mathrm{st}} = 3$, we have $1 \in T_2(K) = T_2 = S_0$. Hence, $0 = \|1\|_K = 0 \cdot 3 + d - 2$, which implies that the displacement is $d = 2$. For any $n$, there is always $\ell$ and $u$ such that $n/3^\ell \in T_u(K)$, and then, $\|n\|_K = 3\ell + 2 - u$. If $n$ is stable, then $\|n\| = 3\ell + 2 - u$, so $\|n\|_K = \|n\|_{\mathrm{st}}$. When $n$ is not stable, there is $k$ with $n3^k$ stable, and so

$$\|n\|_K + 3k = \|n3^k\|_K = \|n3^k\|_{\mathrm{st}} = \|n\|_{\mathrm{st}} + 3k.$$

Showing that $K$ is arithmetical, given the equality of the gauge of $K$ and the stable complexity, is equivalent to showing that the two sets

$$\{T_u[\omega\beta + r] \colon r \in \mathbf{Z}_{\geq 0}\}, \tag{10}$$

$$\Big\{\frac{b(a3^r + 1)}{3^{r+k}} \colon n = ab, \quad \|n\|_{\mathrm{st}} = \|a\|_{\mathrm{st}} + \|b\|_{\mathrm{st}}, \quad r \in \mathbf{Z}_{\geq 0}\Big\}, \tag{11}$$

where $0 \leq \beta < \omega^\omega$, $u \in \mathbf{Z}_{\geq 0}$ and $T_{u+1}[\beta] = \frac{n}{3^k}$ with $3 \nmid n$, have a finite symmetric difference.

The core of the definition that $K$ is arithmetical is the fact that the sequence $\{T_u[\omega\beta + r] \colon r \in \mathbf{Z}_{\geq 0}\}$ is determined by its limit $T_{u+1}[\beta]$. Since $T_{u+3k} = T_u/3^k$ we only need to consider the cases of $T_0$, $T_1$ and $T_2$. Consider, for example, the case $u = 2$. The hypothesis $T_3[\beta] = n/3^k$ is equivalent to $T_0[\beta] = n/3^{k-1}$ because $T_3 = T_0/3$. This, in turn, is equivalent to $S_2[\beta] = n/3^{k-1}$, so that $\|n\|_{\mathrm{st}} = 3(k-1) + 2 = 3k - 1$ and $\delta_{\mathrm{st}}(n) = \mathscr{D}^2_{\mathrm{st}}[\beta]$ by (6). Proposition 4.19 in [7] implies that the symmetric difference of the two sets

$$A := \{\mathscr{D}^0_{\mathrm{st}}[\omega\beta + r] \colon r \geq 0\},$$
$$B := \{\delta_{\mathrm{st}}(b(a3^r + 1)) : r \geq 0, n = ab, \|n\|_{\mathrm{st}} = \|a\|_{\mathrm{st}} + \|b\|_{\mathrm{st}}\},$$

is finite.

The map $g(x) = 3^{-x/3}$ transforms set $A$ into $g(A) = \{S_0[\omega\beta + r] : r \geq 0\}$. By Corollary 5.2 in [7], the numbers $b(a3^r + 1)$, except for a finite set, satisfy

$$\|b(a3^r + 1)\|_{\rm st} = \|a\|_{\rm st} + \|b\|_{\rm st} + 3r + 1 = 3k + 3r.$$

Hence, the numbers in $B$ are of the form $\delta_{\rm st}(b(a3^r+1)) = 3k+3r-3\log_3(b(a3^r+1))$. It follows that $g$ maps the set $B$ into the set in (11), except for a finite set. Hence, we have proved that the symmetric difference between the sets (10) and (11) is finite in the case $u = 2$. The cases $u = 0$ and $u = 1$ are proven in a similar way. □

## 4. The Uniqueness Question

We have seen that there is an arithmetical compact set $K$ of ratio and module equal to 3. By Proposition 7 each derived set $K^{(n)}$ of $K$, for $n \in \mathbf{N}$, is also an arithmetical compact set. Essentially, there are three different sets: $K$, $K'$, and $K''$, and the other sets $K^{(n)}$, for $n \geq 3$, are homothetic to one of $K$, $K'$, $K''$. These three sets have a distinct first element. The question may be posed: *Is there a unique arithmetical compact set of ratio* 3 *and module* 3 *having* $K[0] = 2$*?*

For each ordinal $\beta < \omega^\omega$, consider $T_1[\beta] = K[\omega(\beta + 1)]$. This is the limit point of the sequence $K[\omega\beta + k]$. Since $K[\omega(\beta + 1)] > K[\omega\beta]$, we will denote by $I_\beta$ the interval in $\mathbf{R}$ given by $I_\beta := (K[\omega(\beta + 1)], K[\omega\beta])$, except when $\beta = 0$, in which case we set $I_0 := (K[\omega], K[0]]$. Thus, all elements in $T_0 = K \smallsetminus K'$ are contained in some of these intervals. The sets $I_\beta \cap T_0$ are called the $K$ *sections*. The section $I_\beta \cap T_0$ is a sequence with limit point $T_1[\beta]$.

Since $K$ is an arithmetical compact set, if $T_1[\beta] = \frac{n}{3^k}$ then the terms of the sequence $I_\beta \cap T_0$ of the section $I_\beta \cap T_0$ are given by (5).

The definition of the $I_\beta$ and the sections can be extended to $K'$ and $K''$ in relation to $T_1$ and $T_2$. The corresponding intervals will be called $I^u_\beta$ for $u = 1, 2$.

**Definition 5.** Let $T_{u+1}[\beta] = \frac{n}{3^k}$. The numbers in the section $I^u_\beta \cap T_u$ that are not contained in the set

$$A_\beta := I^u_\beta \cap \Bigl\{\frac{b(a3^r + 1)}{3^{r+k}} : n = ab, \quad \|n\|_{\rm st} = \|a\|_{\rm st} + \|b\|_{\rm st}, \quad r \in \mathbf{Z}_{\geq 0}, \quad a^2 + r^2 > 1\Bigr\},$$

will be called *sporadics*.

We consider that the definition of the sporadics given in Definition 5 is adequate. There are, however, other possible definitions leading to slightly different concepts.[1]

[1] For example, in the tables at the end of the paper, the fractions are arranged setting $\frac{b(a3^r+1)}{3^{r+k}} = \frac{n3^{r+v}+b3^v}{3^{r+k+v}}$, and taking $v$ the greatest value satisfying $b3^v < n$, and ordering the fractions according to decreasing values of $b3^v$. This is the natural order of the fractions. In this case, we could also consider the values of $\frac{n3^\ell+b3^v}{3^{\ell+k}}$ for all $\ell \geq 0$.

In the tables at the end of the paper the fractions corresponding to sporadics are written in boldface. For all the sections given in those tables we have $A_\beta \subset I_\beta \cap T_u$. This is not ensured by Theorem 1.

At the time of writing [9], uniqueness seemed very likely. The sporadics appeared to have some structure, as quite frequently they are powers of two. Theorem 1 and knowledge of the sporadics determine the sets $T_u$. Conjectures 9, 10, and 11 in [9] were unsuccessfully trying to determine these sporadic numbers. If these numbers were to follow a certain rule, it seemed probable that the compact set could be constructed in a unique way. The conjectures in [9] were aimed at giving exact rules for constructing $K$. To this end, a rule for determining the sporadics was needed. In this sense, it can be considered that the conjectures on the sporadics already proved are not completely satisfactory.

Two relevant questions arise. Is it true that $A_\beta \subset I_\beta \cap T_u$? Can we give a rule to determine the sporadics? Both questions seem to have a negative answer. This, however, is unclear.

### 4.1. Sporadics

From Definition 5 we have that $A_\beta \subset I_\beta$. The sporadics appear to be generated by the values of the fractions $a(b3^r+1)/3^{r+k}$ to the left of the interval $I_\beta$. For example, section $K \cap (76/81, 26/27) = (T_0[\omega^2+\omega 2], T_0[\omega^2+\omega 3])$ contains many sporadics. Most of them appear in the sequences with limit 2048/2187. Of the 19 sporadics in this section, 17 arise as follows:

$$\frac{18944}{19683} = \frac{512(4\cdot 3^2+1)}{3^{2+7}}, \quad \frac{6272}{6561} = \frac{128(16\cdot 3+1)}{3^{1+7}}, \quad \frac{56320}{59049} = \frac{1024(2\cdot 3^3+1)}{3^{3+7}},$$
$$\frac{2080}{2187} = \frac{32(64\cdot 3^0+1)}{3^{0+7}}, \quad \frac{18688}{19683} = \frac{256(8\cdot 3^2+1)}{3^{2+7}}, \quad \frac{167936}{177147} = \frac{2048(3^4+1)}{3^{4+7}}\dots$$

The only sporadics that remain to be understood are $\frac{700}{729} = \frac{25(3^3+1)}{3^{3+3}}$ and $\frac{2072}{2187} = \frac{56(4\cdot 3^2+1)}{3^{2+5}}$.

The facts above explain why the powers of two are usually sporadic. If $\|2^n\| = 2n = 3k+2-u$, then $\frac{2^n}{3^k} \in T_u$, and we have

$$\frac{2^n}{3^k} = \frac{2^{n-2}(1\cdot 3+1)}{3^{(u-1)+1}}, \quad \text{with} \quad \frac{2^{n-2}}{3^{u-1}} \in T_{u+1}.$$

Since the defects of the powers of two are relatively small, $\frac{2^n}{3^k}$ is usually sporadic.

### 4.2. Altman's Almost Counterexample to Uniqueness

Altman thinks that there is no uniqueness. He has constructed a different example of an arithmetical compact set $H$ with ratio and module 3. Although it seems very unlikely that $H = K$, we are not able to prove that $H \neq K$. We present Altman's example and discuss why $H = K$ appears unlikely.

Altman [8] considers a modified complexity. Instead of requiring that

$$\|ab\| \leq \|a\| + \|b\|, \quad \|a+b\| \leq \|a\| + \|b\|,$$

his modified complexity $\|\cdot\|_1$ only requires that

$$\|ab\|_1 \leq \|a\|_1 + \|b\|_1, \quad \|a+1\|_1 \leq \|a\|_1 + 1.$$

The whole theory goes through replacing $\|n\|$ with $\|n\|_1$. In this case, there are fewer low-defect polynomials (precisely the ones with +1s only). This is precisely what allows to define the compact set $H$.

Then, the equality $H = K$ would hold if and only if the stable complexities are equal, that is, $\|n\|_{\mathrm{st}} = \|n\|_{1,\mathrm{st}}$.

A *solid number* (see [1] and OEIS:A195101) is a number $n$ such that $n = a + b$ implies $\|n\| < \|a\| + \|b\|$. There are infinitely many solid numbers; the first ones are $1, 6, 9, \ldots$ And there are numbers $n$ whose complexity can only be computed with the decomposition $n = (n-b) + b$ where $b$ is a solid number. For the numbers $n$ whose complexity can only be obtained via $\|n\| = \|n-b\| + \|b\|$ where $b > 1$ is a solid number, we have $\|n\| < \|n\|_1$. Therefore, $H = K$ implies that such $n$ (with $\|n\| < \|n\|_1$) is not stable, so that $\|n\|_{\mathrm{st}} = \|n\|_{1_{\mathrm{st}}}$. Usually, numbers of this form can be rewritten more efficiently. For example, numbers of the form $(4 \cdot 3^k + 1) + 6$ do not have complexity equal to $3k + 10$ because

$$(4 \cdot 3^k + 1) + 6 = 6(2 \cdot 3^{k-1} + 1) + 1, \qquad k \geq 1,$$

giving a complexity $\leq 3k + 6$. The first example of a number which we do not know how to rewrite efficiently is $73(3^k + 1) + 6$. Since $\|73\| = 13$, we expect that $\|73(3^k + 1) + 6\| = 3k + 19$. In we proved, for $k \geq k_0$ that this is the case.

If $73(3^k + 1) + 6$ were not stable, $\|73(3^k + 1) + 6\|_{\mathrm{st}} \leq 3k + 18$, we would have, for $\ell$ large,

$$\begin{aligned}\delta((73 \cdot 3^k + 79)3^\ell) &\leq 3k + 3\ell + 18 - 3\log_3(73 \cdot 3^k + 79)3^\ell) \\ &= 18 + 3\log_3 \frac{3^{k+\ell}}{73 \cdot 3^{k+\ell} + 79 \cdot 3^\ell} < 18 + 3\log_3 \frac{1}{73} \\ &\leq 6.284 < 58.614 \cdot \delta(2).\end{aligned}$$

If we compute a covering set of $B_{59\delta(2)}$, we can determine if this is true or not. Using Altman's algorithm we have computed a good covering for $B_{42\delta(2)}$[2] (but using programs in Python, we have failed to compute $B_{43\delta(2)}$).

[2] Altman's version of his program is available on his website.

For $k = 20$, the number is $\|73(3^{20}+1)+6\| = 79$. We have another representation,

$$73(3^{20}+1)+6 = 2^3((((3^8+1)(3^6+1)3^2+1)(3^4+1)+1)3^2+1+1),$$

not using any solid number but 1, and giving its complexity.

Using the computations of Iraids et al. [11], for $k = 21$ we have $\|73(3^{21}+1)+6\| = 82$. This number is

$$73(3^{21}+1)+6 = 763605783898 = 2\cdot 381802891949, \text{ with } \|2\|+\|381802891949\| = 83,$$

with 381802891949 prime and $\|763605783898-1\|+\|1\| = 83$. Therefore, in this case, the complexity cannot be obtained by any other decomposition that does not use $b = 6$. We do not know if $73(3^{21}+1)+6$ is stable or not. Its stability can only be proved by computing first a good covering set of $B_{59\delta(2)}$. We do not know how to do this.

**Remark 2.** After writing this, Qizheng He [12] has uploaded a paper to arXiv giving a nice program that computes $\|n\|$ without computing the previous $\|k\|$ for $k < n$. We asked Qizheng He to consider the numbers $(73(3^{21}+1)+6)3^k$ to see whether $\|(73(3^{21}+1)+6)3^k\| = 82+3k$. He was able to check this for $0 \le k \le 54$, [13]. This makes harder to believe that $73(3^{21}+1)+6$ is not stable.

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## Appendix

In this Appendix we include tables for the compact set $K$. The tables are constructed from the good covering of $B_{24\delta(2)}$. Using this covering we can construct all pairs $[m, 3^{\lfloor \|m\|_{\rm st}\rfloor/3}]$ with $3\nmid m$ such that $\delta_{\rm st}(n) < 24\delta(2)$. This means, for $\|m\|_{\rm st} = 3k$, that $m/3^k > 2^{24}/3^{16}$. So, we know all fractions $x$ in $T_u$ such that $x > 3^{(2-u)/3}2^{24}/3^{16}$. In each case, we determine the sporadics and print them in boldface. We end each section with the fractions in $3^k$, generating all non-sporadic terms ordered as explained in the footnote of page 11. In each case, the last numerical fractions given are the values of the fractions in $3^k$ for a given value of $k$. This will make it easy to extend each section as needed.

### A. Table of Some Elements of the Compact Set $K$: $T_0$

$K[0] = 2, \qquad K\cap(\frac{4}{3}, 2] = [K[0], K[\omega]) = [T_0[0], T_0[\omega])$

$2,\ \frac{16}{9},\ \frac{5}{3},\ \mathbf{\frac{128}{81}},\ \frac{14}{9},\ \frac{40}{27},\ \frac{13}{9},\ \frac{38}{27},\ \mathbf{\frac{1024}{729}},\ \frac{112}{81},\ \frac{37}{27},\ \frac{110}{81},\ \dots\ \frac{4\cdot 3^k+4}{3^{k+1}},\ \frac{4\cdot 3^k+3}{3^{k+1}},\ \frac{4\cdot 3^k+2}{3^{k+1}},\ \dots$

$K[\omega] = \frac{4}{3}, \qquad K\cap(\frac{32}{27}, \frac{4}{3}) = [K[\omega+1], K[\omega 2]) = [T_0[\omega], T_0[\omega 2])$

$\frac{320}{243},\ \mathbf{\frac{35}{27}},\ \frac{104}{81},\ \frac{34}{27},\ \frac{304}{243},\ \mathbf{\frac{8192}{6561}},\ \frac{100}{81},\ \frac{896}{729},\ \frac{11}{9},\ \frac{296}{243},\ \frac{98}{81},\ \frac{880}{729},\ \frac{292}{243},\ \dots\ \frac{32\cdot 3^k+32}{3^{k+3}},\ \frac{32\cdot 3^k+27}{3^{k+3}},\ \frac{32\cdot 3^k+24}{3^{k+3}},\ \frac{32\cdot 3^k+18}{3^{k+3}},\ \frac{32\cdot 3^k+16}{3^{k+3}},\ \frac{32\cdot 3^k+12}{3^{k+3}},\ \dots$

$K[\omega 2] = \frac{32}{27}, \qquad K\cap(\frac{10}{9}, \frac{32}{27}) = [K[\omega 2+1], K[\omega 3]) = [T_0[\omega 2], T_0[\omega 3])$

$\frac{95}{81},\ \mathbf{\frac{2560}{2187}},\ \frac{280}{243},\ \frac{31}{27},\ \mathbf{\frac{832}{729}},\ \frac{92}{81},\ \frac{275}{243},\ \frac{820}{729},\ \frac{91}{81},\ \frac{272}{243},\ \frac{815}{729},\ \dots\ \frac{10\cdot 3^k+10}{3^{k+2}},\ \frac{10\cdot 3^k+9}{3^{k+2}},\ \frac{10\cdot 3^k+6}{3^{k+2}},\ \frac{10\cdot 3^k+5}{3^{k+2}},$

$\dots$

$K[\omega 3] = \frac{10}{9}, \qquad K\cap(\frac{256}{243}, \frac{10}{9}) = [K[\omega 3+1], K[\omega 4]) = [T_0[\omega 3], T_0[\omega 4])$

$\mathbf{\frac{65536}{59049}},\ \frac{800}{729},\ \mathbf{\frac{266}{243}},\ \frac{7168}{6561},\ \frac{88}{81},\ \frac{2368}{2187},\ \frac{784}{729},\ \mathbf{\frac{29}{27}},\ \frac{7040}{6561},\ \frac{260}{243},\ \frac{2336}{2187},\ \frac{20992}{19683},\ \frac{259}{243},\ \frac{776}{729},\ \frac{6976}{6561},\ \frac{86}{81},\ \frac{2320}{2187},\ \frac{20864}{19683},\ \frac{772}{729},\ \frac{6944}{6561},\ \dots\ \frac{256\cdot 3^k+256}{3^{k+5}},\ \frac{256\cdot 3^k+243}{3^{k+5}},\ \frac{256\cdot 3^k+216}{3^{k+5}},\ \frac{256\cdot 3^k+192}{3^{k+5}},\ \frac{256\cdot 3^k+162}{3^{k+5}},\ \frac{256\cdot 3^k+144}{3^{k+5}},\ \frac{256\cdot 3^k+128}{3^{k+5}},\ \frac{256\cdot 3^k+108}{3^{k+5}},\ \frac{256\cdot 3^k+96}{3^{k+5}},\ \dots$

$K[\omega 4] = \frac{256}{243}, \qquad K\cap(\frac{28}{27}, \frac{256}{243}) = [K[\omega 4+1], K[\omega 5]) = [T_0[\omega 4], T_0[\omega 5])$

$\frac{2296}{2187},\ \frac{85}{81},\ \frac{763}{729},\ \frac{254}{243},\ \frac{2282}{2187},\ \frac{760}{729},\ \frac{6832}{6561},\ \frac{253}{243},\ \mathbf{\frac{20480}{19683}},\ \frac{2275}{2187},\ \frac{758}{729},\ \frac{6818}{6561},\ \frac{2272}{2187},\ \frac{20440}{19683},\ \frac{757}{729},\ \frac{6811}{6561},\ \frac{2270}{2187},\ \frac{20426}{19683},\ \frac{6808}{6561},\ \dots\ \frac{28\cdot 3^k+28}{3^{k+3}},\ \frac{28\cdot 3^k+27}{3^{k+3}},\ \frac{28\cdot 3^k+21}{3^{k+3}},\ \frac{28\cdot 3^k+18}{3^{k+3}},\ \frac{28\cdot 3^k+14}{3^{k+3}},\ \frac{28\cdot 3^k+12}{3^{k+3}},\ \dots$

$K[\omega 5] = \frac{28}{27}, \qquad K\cap(\frac{82}{81}, \frac{28}{27}) = [K[\omega 5+1], K[\omega 6]) = [T_0[\omega 5], T_0[\omega 6])$

$\mathbf{\frac{250}{243}},\ \frac{6724}{6561},\ \frac{83}{81},\ \mathbf{\frac{2240}{2187}},\ \mathbf{\frac{248}{243}},\ \frac{20008}{19683},\ \frac{247}{243},\ \mathbf{\frac{740}{729}},\ \mathbf{\frac{6656}{6561}},\ \frac{59860}{59049},\ \frac{739}{729},\ \dots\ \frac{82\cdot 3^k+82}{3^{k+4}},\ \frac{82\cdot 3^k+81}{3^{k+4}},\ \dots$

$K[\omega 6] = \frac{82}{81}, \qquad K\cap(\frac{244}{243}, \frac{82}{81}) = [K[\omega 6+1], K[\omega 7]) = [T_0[\omega 6], T_0[\omega 7])$

$\mathbf{\frac{736}{729}},\ \frac{59536}{59049},\ \frac{245}{243},\ \mathbf{\frac{2200}{2187}},\ \frac{178120}{177147},\ \frac{733}{729},\ \dots\ \frac{244\cdot 3^k+244}{3^{k+5}},\ \frac{244\cdot 3^k+243}{3^{k+5}},\ \dots$

$K[\omega 7] = \frac{244}{243}, \qquad K\cap(\frac{730}{729}, \frac{244}{243}) = [K[\omega 7+1], K[\omega 8]) = [T_0[\omega 7], T_0[\omega 8])$

$\frac{532900}{531441},\ \frac{731}{729},\ \dots\ \frac{730\cdot 3^k+730}{3^{k+6}},\ \frac{730\cdot 3^k+729}{3^{k+6}},\ \dots$

$\dots$

$K[\omega^2] = 1, \qquad K \cap (\frac{80}{81}, 1) = [K[\omega^2+1], K[\omega^2+\omega]) = [T_0[\omega^2], T_0[\omega^2+\omega])$

$\frac{6560}{6561}, \frac{728}{729}, \frac{2180}{2187}, \frac{242}{243}, \frac{2176}{2187}, \frac{725}{729}, \frac{6520}{6561}, \frac{724}{729}, \frac{2170}{2187}, \frac{241}{243}, \dots \frac{80\cdot 3^k+80}{3^{k+4}}, \frac{80\cdot 3^k+72}{3^{k+4}}, \frac{80\cdot 3^k+60}{3^{k+4}}, \frac{80\cdot 3^k+54}{3^{k+4}}, \frac{80\cdot 3^k+48}{3^{k+4}}, \frac{80\cdot 3^k+45}{3^{k+4}}, \frac{80\cdot 3^k+40}{3^{k+4}}, \frac{80\cdot 3^k+36}{3^{k+4}}, \frac{80\cdot 3^k+30}{3^{k+4}}, \frac{80\cdot 3^k+27}{3^{k+4}}, \dots$

$K[\omega^2+\omega] = \frac{80}{81}, \qquad K \cap (\frac{26}{27}, \frac{80}{81}) = [K[\omega^2+\omega+1], K[\omega^2+\omega 2]) = [T_0[\omega^2+\omega], T_0[\omega^2+\omega 2])$

$\mathbf{\frac{524288}{531441}}, \frac{715}{729}, \mathbf{\frac{238}{243}}, \mathbf{\frac{6400}{6561}}, \frac{79}{81}, \frac{2132}{2187}, \mathbf{\frac{2128}{2187}}, \frac{236}{243}, \mathbf{\frac{57344}{59049}}, \frac{2119}{2187}, \frac{235}{243}, \frac{6344}{6561}, \frac{704}{729}, \frac{6331}{6561}, \frac{703}{729}, \dots \frac{26\cdot 3^k+26}{3^{k+3}}, \frac{26\cdot 3^k+18}{3^{k+3}}, \frac{26\cdot 3^k+13}{3^{k+3}}, \frac{26\cdot 3^k+9}{3^{k+3}}, \dots$

$K[\omega^2+\omega 2] = \frac{26}{27}, \qquad K\cap(\frac{76}{81}, \frac{26}{27}) = [K[\omega^2+\omega 2+1], K[\omega^2+\omega 3]) = [T_0[\omega^2+\omega 2], T_0[\omega^2+\omega 3])$

$\mathbf{\frac{18944}{19683}}, \mathbf{\frac{700}{729}}, \mathbf{\frac{6272}{6561}}, \frac{2090}{2187}, \frac{232}{243}, \mathbf{\frac{56320}{59049}}, \mathbf{\frac{2080}{2187}}, \frac{77}{81}, \frac{6232}{6561}, \mathbf{\frac{18688}{19683}}, \mathbf{\frac{167936}{177147}}, \mathbf{\frac{2072}{2187}}, \frac{2071}{2187}, \frac{230}{243}, \mathbf{\frac{6208}{6561}}, \mathbf{\frac{55808}{59049}}, \frac{6194}{6561}, \frac{688}{729}, \mathbf{\frac{18560}{19683}}, \frac{229}{243}, \mathbf{\frac{166912}{177147}}, \frac{18544}{19683}, \mathbf{\frac{6176}{6561}}, \frac{6175}{6561}, \frac{686}{729}, \mathbf{\frac{55552}{59049}}, \mathbf{\frac{499712}{531441}}, \frac{18506}{19683}, \frac{2056}{2187}, \mathbf{\frac{18496}{19683}}, \frac{685}{729}, \frac{55480}{59049}, \mathbf{\frac{166400}{177147}}, \frac{18487}{19683}, \frac{2054}{2187}, \frac{55442}{59049}, \frac{6160}{6561}, \frac{2053}{2187}, \frac{166288}{177147}, \mathbf{\frac{55424}{59049}}, \frac{55423}{59049}, \frac{6158}{6561}, \frac{166250}{177147}, \frac{18472}{19683}, \frac{6157}{6561}, \frac{498712}{531441}, \frac{166231}{177147}, \frac{18470}{19683}, \mathbf{\frac{498688}{531441}}, \frac{498674}{531441}, \frac{55408}{59049}, \frac{18469}{19683}, \frac{1495984}{1594323}, \frac{498655}{531441}, \frac{55406}{59049}, \frac{1495946}{1594323}, \frac{166216}{177147}, \frac{55405}{59049}, \dots \frac{76\cdot 3^k+76}{3^{k+4}}, \frac{76\cdot 3^k+57}{3^{k+4}}, \frac{76\cdot 3^k+54}{3^{k+4}}, \frac{76\cdot 3^k+38}{3^{k+4}}, \frac{76\cdot 3^k+36}{3^{k+4}}, \frac{76\cdot 3^k+27}{3^{k+4}}, \dots$

$K[\omega^2+\omega 3] = \frac{76}{81}, \quad K\cap(\frac{2048}{2187}, \frac{76}{81}) = [K[\omega^2+\omega 3+1], K[\omega^2+\omega 4]) = [T_0[\omega^2+\omega 3], T_0[\omega^2+\omega 4])$

$\frac{18464}{19683}, \frac{166144}{177147}, \frac{1495040}{1594323}, \frac{6152}{6561}, \frac{55360}{59049}, \frac{498176}{531441}, \frac{2050}{2187}, \frac{18448}{19683}, \frac{166016}{177147}, \frac{1494016}{1594323}, \frac{6148}{6561}, \frac{55328}{59049}, \frac{497920}{531441}, \frac{683}{729}, \dots \frac{2048\cdot 3^k+2048}{3^{k+7}}, \frac{2048\cdot 3^k+1944}{3^{k+7}}, \frac{2048\cdot 3^k+1728}{3^{k+7}}, \frac{2048\cdot 3^k+1536}{3^{k+7}}, \frac{2048\cdot 3^k+1458}{3^{k+7}}, \frac{2048\cdot 3^k+1296}{3^{k+7}}, \frac{2048\cdot 3^k+1152}{3^{k+7}}, \frac{2048\cdot 3^k+1024}{3^{k+7}}, \frac{2048\cdot 3^k+972}{3^{k+7}}, \frac{2048\cdot 3^k+864}{3^{k+7}}, \frac{2048\cdot 3^k+768}{3^{k+7}}, \frac{2048\cdot 3^k+729}{3^{k+7}}, \dots$

$K[\omega^2+\omega 4] = \frac{2048}{2187}, K\cap(\frac{25}{27}, \frac{2048}{2187}) = [K[\omega^2+\omega 4+1], K[\omega^2+\omega 5]) = [T_0[\omega^2+\omega 4], T_0[\omega^2+\omega 5])$

$\mathbf{\frac{2044}{2187}}, \mathbf{\frac{18368}{19683}}, \frac{680}{729}, \mathbf{\frac{679}{729}}, \mathbf{\frac{2035}{2187}}, \mathbf{\frac{6104}{6561}}, \frac{226}{243}, \frac{6100}{6561}, \mathbf{\frac{2032}{2187}}, \frac{2030}{2187}, \mathbf{\frac{18256}{19683}}, \frac{676}{729}, \frac{18250}{19683}, \frac{6080}{6561}, \frac{2026}{2187}, \dots \frac{25\cdot 3^k+25}{3^{k+3}}, \frac{25\cdot 3^k+15}{3^{k+3}}, \frac{25\cdot 3^k+9}{3^{k+3}}, \dots$

$\dots$

$K[\omega^2 2] = \frac{8}{9}, \qquad K \cap (\frac{640}{729}, \frac{8}{9}) = [K[\omega^2 2+1], K[\omega^2 2+\omega]) = [T_0[\omega^2 2], T_0[\omega^2 2+\omega])$

$\frac{52480}{59049}, \frac{5824}{6561}, \frac{1940}{2187}, \mathbf{\frac{646}{729}}, \frac{17440}{19683}, \frac{1936}{2187}, \frac{215}{243}, \frac{17408}{19683}, \frac{5800}{6561}, \frac{644}{729}, \frac{52160}{59049}, \frac{5792}{6561}, \frac{1930}{2187}, \frac{17360}{19683}, \frac{1928}{2187}, \frac{156160}{177147}, \frac{17344}{19683}, \frac{5780}{6561}, \frac{214}{243}, \frac{52000}{59049}, \frac{5776}{6561}, \frac{1925}{2187}, \frac{51968}{59049}, \frac{17320}{19683}, \frac{1924}{2187}, \frac{155840}{177147}, \frac{17312}{19683}, \frac{5770}{6561}, \frac{641}{729}, \frac{51920}{59049}, \frac{5768}{6561}, \dots \frac{640\cdot 3^k+640}{3^{k+6}}, \frac{640\cdot 3^k+576}{3^{k+6}}, \frac{640\cdot 3^k+540}{3^{k+6}}, \frac{640\cdot 3^k+486}{3^{k+6}}, \frac{640\cdot 3^k+480}{3^{k+6}}, \frac{640\cdot 3^k+432}{3^{k+6}}, \frac{640\cdot 3^k+405}{3^{k+6}}, \frac{640\cdot 3^k+384}{3^{k+6}}, \frac{640\cdot 3^k+360}{3^{k+6}}, \frac{640\cdot 3^k+324}{3^{k+6}}, \frac{640\cdot 3^k+320}{3^{k+6}}, \frac{640\cdot 3^k+288}{3^{k+6}}, \frac{640\cdot 3^k+270}{3^{k+6}}, \frac{640\cdot 3^k+243}{3^{k+6}}, \frac{640\cdot 3^k+240}{3^{k+6}}, \frac{640\cdot 3^k+216}{3^{k+6}}, \dots$

$K[\omega^2 2+\omega] = \frac{640}{729}, K\cap(\frac{70}{81}, \frac{640}{729}) = [K[\omega^2 2+\omega+1], K[\omega^2 2+\omega 2]) = [T_0[\omega^2 2+\omega], T_0[\omega^2 2+\omega 2])$

$\mathbf{\frac{4194304}{4782969}}, \frac{71}{81}, \frac{5740}{6561}, \frac{637}{729}, \frac{212}{243}, \mathbf{\frac{5720}{6561}}, \frac{635}{729}, \frac{1904}{2187}, \frac{5705}{6561}, \frac{1900}{2187}, \frac{211}{243}, \mathbf{\frac{1898}{2187}}, \frac{17080}{19683}, \frac{1897}{2187}, \mathbf{\frac{51200}{59049}}, \frac{632}{729}, \mathbf{\frac{17056}{19683}}, \frac{1895}{2187}, \frac{5684}{6561}, \frac{17045}{19683}, \frac{5680}{6561}, \frac{631}{729}, \frac{51100}{59049}, \frac{5677}{6561}, \frac{1892}{2187}, \frac{5675}{6561}, \frac{17024}{19683}, \frac{51065}{59049}, \frac{17020}{19683}, \frac{1891}{2187}, \dots \frac{70\cdot 3^k+70}{3^{k+4}}, \frac{70\cdot 3^k+63}{3^{k+4}}, \frac{70\cdot 3^k+54}{3^{k+4}}, \frac{70\cdot 3^k+45}{3^{k+4}}, \frac{70\cdot 3^k+42}{3^{k+4}}, \frac{70\cdot 3^k+35}{3^{k+4}}, \frac{70\cdot 3^k+30}{3^{k+4}}, \frac{70\cdot 3^k+27}{3^{k+4}}, \dots$

$K[\omega^2 2+\omega 2] = \frac{70}{81},$

$K \cap (\frac{208}{243}, \frac{70}{81}) = [K[\omega^2 2+\omega 2+1], K[\omega^2 2+\omega 3]) = [T_0[\omega^2 2+\omega 2], T_0[\omega^2 2+\omega 3])$

$\frac{5668}{6561}, \frac{1888}{2187}, \frac{\mathbf{458752}}{\mathbf{531441}}, \frac{\mathbf{629}}{\mathbf{729}}, \frac{1885}{2187}, \frac{628}{729}, \frac{16952}{19683}, \frac{209}{243}, \frac{5642}{6561}, \frac{1880}{2187}, \frac{50752}{59049}, \frac{626}{729}, \frac{16900}{19683}, \frac{5632}{6561}, \frac{5629}{6561},$
$\frac{1876}{2187}, \frac{50648}{59049}, \frac{625}{729}, \frac{16874}{19683}, \frac{5624}{6561}, \ldots \frac{208\cdot 3^k+208}{3^{k+5}}, \frac{208\cdot 3^k+162}{3^{k+5}}, \frac{208\cdot 3^k+156}{3^{k+5}}, \frac{208\cdot 3^k+144}{3^{k+5}}, \frac{208\cdot 3^k+117}{3^{k+5}},$
$\frac{208\cdot 3^k+108}{3^{k+5}}, \frac{208\cdot 3^k+104}{3^{k+5}}, \frac{208\cdot 3^k+81}{3^{k+5}}, \frac{208\cdot 3^k+78}{3^{k+5}}, \frac{208\cdot 3^k+72}{3^{k+5}}, \ldots$

$K[\omega^2 2+\omega 3] = \frac{208}{243},$

$K \cap (\frac{68}{81}, \frac{208}{243}) = [K[\omega^2 2+\omega 3+1], K[\omega^2 2+\omega 4]) = [T_0[\omega^2 2+\omega 3], T_0[\omega^2 2+\omega 4])$

$\frac{\mathbf{151552}}{\mathbf{177147}}, \frac{1870}{2187}, \frac{\mathbf{5600}}{\mathbf{6561}}, \frac{23}{27}, \frac{\mathbf{1862}}{\mathbf{2187}}, \frac{\mathbf{620}}{\mathbf{729}}, \frac{5576}{6561}, \frac{\mathbf{50176}}{\mathbf{59049}}, \frac{\mathbf{16720}}{\mathbf{19683}}, \frac{\mathbf{1856}}{\mathbf{2187}}, \frac{\mathbf{450560}}{\mathbf{531441}}, \frac{206}{243}, \frac{1853}{2187}, \frac{\mathbf{1850}}{\mathbf{2187}}, \frac{\mathbf{5548}}{\mathbf{6561}},$
$\frac{\mathbf{16640}}{\mathbf{19683}}, \frac{616}{729}, \frac{5542}{6561}, \frac{\mathbf{49856}}{\mathbf{59049}}, \frac{\mathbf{149504}}{\mathbf{177147}}, \frac{205}{243}, \frac{16592}{19683}, \frac{\mathbf{1843}}{\mathbf{2187}}, \frac{\mathbf{1343488}}{\mathbf{1594323}}, \frac{614}{729}, \frac{\mathbf{16576}}{\mathbf{19683}}, \frac{5525}{6561}, \frac{\mathbf{16568}}{\mathbf{19683}}, \frac{1840}{2187},$
$\frac{16558}{19683}, \frac{\mathbf{49664}}{\mathbf{59049}}, \frac{613}{729}, \frac{49640}{59049}, \frac{1838}{2187}, \frac{16541}{19683}, \frac{5512}{6561}, \frac{\mathbf{446464}}{\mathbf{531441}}, \frac{49606}{59049}, \frac{1837}{2187}, \frac{148784}{177147}, \frac{5510}{6561}, \frac{49589}{59049}, \frac{16528}{19683},$
$\frac{148750}{177147}, \frac{5509}{6561}, \ldots \frac{68\cdot 3^k+68}{3^{k+4}}, \frac{68\cdot 3^k+54}{3^{k+4}}, \frac{68\cdot 3^k+51}{3^{k+4}}, \frac{68\cdot 3^k+36}{3^{k+4}}, \frac{68\cdot 3^k+34}{3^{k+4}}, \frac{68\cdot 3^k+27}{3^{k+4}}, \ldots$

$K[\omega^2 2+\omega 4] = \frac{68}{81}$

$K \cap (\frac{608}{729}, \frac{68}{81}) = [K[\omega^2 2+\omega 4+1], K[\omega^2 2+\omega 5]) = [T_0[\omega^2 2+\omega 4], T_0[\omega^2 2+\omega 5])$

$\frac{49552}{59049}, \frac{5504}{6561}, \frac{\mathbf{5500}}{\mathbf{6561}}, \frac{\mathbf{148480}}{\mathbf{177147}}, \frac{16492}{19683}, \frac{1832}{2187}, \frac{\mathbf{1335296}}{\mathbf{1594323}}, \frac{148352}{177147}, \frac{5491}{6561}, \frac{610}{729}, \frac{\mathbf{49408}}{\mathbf{59049}}, \frac{49400}{59049}, \frac{5488}{6561},$
$\frac{\mathbf{444416}}{\mathbf{531441}}, \frac{16454}{19683}, \frac{1828}{2187}, \frac{\mathbf{3997696}}{\mathbf{4782969}}, \frac{148048}{177147}, \frac{16448}{19683}, \frac{203}{243}, \frac{49324}{59049}, \frac{\mathbf{147968}}{\mathbf{177147}}, \frac{5480}{6561}, \frac{443840}{531441}, \frac{16435}{19683}, \frac{\mathbf{1331200}}{\mathbf{1594323}},$
$\frac{1826}{2187}, \frac{147896}{177147}, \frac{16432}{19683}, \frac{49286}{59049}, \frac{5476}{6561}, \frac{443536}{531441}, \frac{49280}{59049}, \frac{1825}{2187}, \frac{147820}{177147}, \frac{16424}{19683}, \frac{1330304}{1594323}, \frac{49267}{59049}, \frac{5474}{6561},$
$\frac{\mathbf{443392}}{\mathbf{531441}}, \frac{443384}{531441}, \frac{49264}{59049}, \frac{147782}{177147}, \frac{16420}{19683}, \frac{1330000}{1594323}, \frac{147776}{177147}, \frac{5473}{6561}, \frac{443308}{531441}, \frac{49256}{59049}, \frac{3989696}{4782969}, \frac{147763}{177147},$
$\frac{16418}{19683}, \frac{1329848}{1594323}, \frac{147760}{177147}, \frac{\mathbf{3989504}}{\mathbf{4782969}}, \frac{443270}{531441}, \frac{49252}{59049}, \frac{3989392}{4782969}, \frac{443264}{531441}, \frac{16417}{19683}, \frac{1329772}{1594323}, \frac{147752}{177147},$
$\frac{11967872}{14348907}, \frac{443251}{531441}, \frac{49250}{59049}, \frac{3989240}{4782969}, \frac{443248}{531441}, \frac{1329734}{1594323}, \frac{147748}{177147}, \frac{11967568}{14348907}, \frac{1329728}{1594323}, \frac{49249}{59049}, \frac{3989164}{4782969},$
$\frac{443240}{531441}, \ldots \frac{608\cdot 3^k+608}{3^{k+6}}, \frac{608\cdot 3^k+513}{3^{k+6}}, \frac{608\cdot 3^k+486}{3^{k+6}}, \frac{608\cdot 3^k+456}{3^{k+6}}, \frac{608\cdot 3^k+432}{3^{k+6}}, \frac{608\cdot 3^k+342}{3^{k+6}}, \frac{608\cdot 3^k+324}{3^{k+6}},$
$\frac{608\cdot 3^k+304}{3^{k+6}}, \frac{608\cdot 3^k+288}{3^{k+6}}, \frac{608\cdot 3^k+243}{3^{k+6}}, \frac{608\cdot 3^k+228}{3^{k+6}}, \frac{608\cdot 3^k+216}{3^{k+6}}, \ldots$

$K[\omega^2 2+\omega 5] = \frac{608}{729},$

$K \cap (\frac{16384}{19683}, \frac{608}{729}) = [K[\omega^2 2+\omega 5+1], K[\omega^2 2+\omega 6]) = [T_0[\omega^2 2+\omega 5], T_0[\omega^2 2+\omega 6])$

$\frac{147712}{177147}, \frac{1329152}{1594323}, \frac{11960320}{14348907}, \frac{49216}{59049}, \frac{442880}{531441}, \frac{3985408}{4782969}, \frac{16400}{19683}, \frac{147584}{177147}, \frac{1328128}{1594323}, \frac{11952128}{14348907}, \frac{49184}{59049},$
$\frac{442624}{531441}, \frac{3983360}{4782969}, \frac{5464}{6561}, \frac{35848192}{43046721}, \frac{147520}{177147}, \frac{1327616}{1594323}, \frac{11948032}{14348907}, \frac{49168}{59049}, \frac{442496}{531441}, \frac{3982336}{4782969}, \frac{16388}{19683},$
$\frac{35840000}{43046721}, \frac{147488}{177147}, \frac{1327360}{1594323}, \frac{11945984}{14348907}, \frac{49160}{59049}, \frac{107511808}{129140163}, \frac{442432}{531441}, \frac{3981824}{4782969}, \frac{5462}{6561}, \frac{35835904}{43046721}, \frac{147472}{177147},$
$\frac{1327232}{1594323}, \frac{11944960}{14348907}, \frac{49156}{59049}, \frac{107503616}{129140163}, \frac{442400}{531441}, \frac{3981568}{4782969}, \frac{16385}{19683}, \frac{35833856}{43046721}, \frac{147464}{177147}, \ldots \frac{16384\cdot 3^k+16384}{3^{k+9}},$
$\frac{16384\cdot 3^k+15552}{3^{k+9}}, \frac{16384\cdot 3^k+13824}{3^{k+9}}, \frac{16384\cdot 3^k+13122}{3^{k+9}}, \frac{16384\cdot 3^k+12288}{3^{k+9}}, \frac{16384\cdot 3^k+11664}{3^{k+9}}, \frac{16384\cdot 3^k+10368}{3^{k+9}},$
$\frac{16384\cdot 3^k+9216}{3^{k+9}}, \frac{16384\cdot 3^k+8748}{3^{k+9}}, \frac{16384\cdot 3^k+8192}{3^{k+9}}, \frac{16384\cdot 3^k+7776}{3^{k+9}}, \frac{16384\cdot 3^k+6912}{3^{k+9}}, \frac{16384\cdot 3^k+6561}{3^{k+9}},$
$\frac{16384\cdot 3^k+6144}{3^{k+9}}, \frac{16384\cdot 3^k+5832}{3^{k+9}}, \ldots$

$K[\omega^2 2+\omega 6] = \frac{16384}{19683},$

$K \cap (\frac{200}{243}, \frac{16384}{19683}) = [K[\omega^2 2+\omega 6+1], K[\omega^2 2+\omega 7]) = [T_0[\omega^2 2+\omega 6], T_0[\omega^2 2+\omega 7])$

$\frac{1820}{2187}, \frac{202}{243}, \frac{\mathbf{16352}}{\mathbf{19683}}, \frac{5450}{6561}, \frac{605}{729}, \frac{\mathbf{146944}}{\mathbf{177147}}, \frac{5440}{6561}, \frac{\mathbf{1813}}{\mathbf{2187}}, \frac{604}{729}, \frac{16300}{19683}, \frac{\mathbf{5432}}{\mathbf{6561}}, \frac{1810}{2187}, \frac{67}{81}, \frac{\mathbf{16280}}{\mathbf{19683}}, \frac{\mathbf{48832}}{\mathbf{59049}},$
$\frac{5425}{6561}, \frac{1808}{2187}, \frac{48800}{59049}, \frac{5420}{6561}, \frac{\mathbf{16256}}{\mathbf{19683}}, \frac{602}{729}, \frac{16250}{19683}, \frac{1805}{2187}, \frac{16240}{19683}, \frac{1804}{2187}, \frac{48700}{59049}, \frac{5410}{6561}, \frac{\mathbf{146048}}{\mathbf{177147}}, \frac{601}{729}, \frac{16225}{19683},$
$\frac{5408}{6561}, \frac{146000}{177147}, \frac{16220}{19683}, \frac{1802}{2187}, \frac{48650}{59049}, \frac{5405}{6561}, \frac{48640}{59049}, \frac{5404}{6561}, \frac{145900}{177147}, \frac{16210}{19683}, \frac{1801}{2187}, \frac{48625}{59049}, \frac{16208}{19683}, \ldots,$

$\frac{200 \cdot 3^k+200}{3^{k+5}}, \frac{200 \cdot 3^k+180}{3^{k+5}}, \frac{200 \cdot 3^k+162}{3^{k+5}}, \frac{200 \cdot 3^k+150}{3^{k+5}}, \frac{200 \cdot 3^k+135}{3^{k+5}}, \frac{200 \cdot 3^k+120}{3^{k+5}}, \frac{200 \cdot 3^k+108}{3^{k+5}}, \frac{200 \cdot 3^k+100}{3^{k+5}}, \frac{200 \cdot 3^k+90}{3^{k+5}}, \frac{200 \cdot 3^k+81}{3^{k+5}}, \frac{200 \cdot 3^k+75}{3^{k+5}}, \frac{200 \cdot 3^k+72}{3^{k+5}}, \ldots$

$\ldots$

$K[\omega^2 5 + \omega 3] = \frac{532}{729},$

$K \cap (\frac{14336}{19683}, \frac{532}{729}) = [K[\omega^2 5 + \omega 3 + 1], K[\omega^2 5 + \omega 4]]) = [T_0[\omega^2 5 + \omega 3], T_0[\omega^2 5 + \omega 4])$

$\frac{1163264}{1594323}, \frac{129248}{177147}, \frac{1163008}{1594323}, \frac{43072}{59049}, \frac{10465280}{14348907}, \frac{387584}{531441}, \frac{43064}{59049}, \frac{387520}{531441}, \frac{4784}{6561}, \frac{3487232}{4782969}, \frac{129152}{177147}, \frac{14350}{19683}, \frac{1162240}{1594323}, \frac{129136}{177147}, \frac{1162112}{1594323}, \frac{43040}{59049}, \frac{10458112}{14348907}, \frac{387328}{531441}, \frac{43036}{59049}, \frac{3485696}{4782969}, \frac{387296}{531441}, \frac{14344}{19683}, \frac{3485440}{4782969}, \frac{129088}{177147}, \frac{4781}{6561}, \frac{31367168}{43046721}, \frac{1161728}{1594323}, \frac{129080}{177147}, \frac{1161664}{1594323}, \frac{43024}{59049}, \frac{10454528}{14348907}, \frac{387200}{531441}, \frac{43022}{59049}, \frac{3484672}{4782969}, \frac{387184}{531441}, \frac{4780}{6561}, \frac{3484544}{4782969}, \frac{129056}{177147}, \frac{31360000}{43046721}, \frac{1161472}{1594323}, \frac{129052}{177147}, \frac{10452992}{14348907}, \frac{1161440}{1594323}, \frac{43016}{59049}, \frac{10452736}{14348907}, \frac{387136}{531441}, \frac{43015}{59049}, \frac{94072832}{129140163}, \frac{3484160}{4782969}, \frac{387128}{531441}, \frac{14338}{19683}, \frac{3484096}{4782969}, \frac{129040}{177147}, \frac{31356416}{43046721}, \frac{1161344}{1594323}, \frac{129038}{177147}, \frac{10451968}{14348907}, \frac{1161328}{1594323}, \frac{43012}{59049}, \frac{10451840}{14348907}, \frac{387104}{531441}, \frac{94065664}{129140163}, \frac{3483904}{4782969}, \frac{387100}{531441}, \frac{59}{81}, \frac{31354880}{43046721}, \frac{3483872}{4782969}, \frac{129032}{177147}, \frac{31354624}{43046721}, \frac{1161280}{1594323}, \frac{129031}{177147}, \ldots \frac{14336 \cdot 3^k+14336}{3^{k+9}}, \frac{14336 \cdot 3^k+13824}{3^{k+9}}, \frac{14336 \cdot 3^k+13608}{3^{k+9}}, \frac{14336 \cdot 3^k+13122}{3^{k+9}}, \frac{14336 \cdot 3^k+12096}{3^{k+9}}, \frac{14336 \cdot 3^k+11664}{3^{k+9}}, \frac{14336 \cdot 3^k+10752}{3^{k+9}}, \frac{14336 \cdot 3^k+10368}{3^{k+9}}, \frac{14336 \cdot 3^k+10206}{3^{k+9}}, \frac{14336 \cdot 3^k+9216}{3^{k+9}}, \frac{14336 \cdot 3^k+9072}{3^{k+9}}, \frac{14336 \cdot 3^k+8748}{3^{k+9}}, \frac{14336 \cdot 3^k+8064}{3^{k+9}}, \frac{14336 \cdot 3^k+7776}{3^{k+9}}, \frac{14336 \cdot 3^k+7168}{3^{k+9}}, \frac{14336 \cdot 3^k+6912}{3^{k+9}}, \frac{14336 \cdot 3^k+6804}{3^{k+9}}, \frac{14336 \cdot 3^k+6561}{3^{k+9}}, \frac{14336 \cdot 3^k+6144}{3^{k+9}}, \frac{14336 \cdot 3^k+6048}{3^{k+9}}, \frac{14336 \cdot 3^k+5832}{3^{k+9}}, \frac{14336 \cdot 3^k+5376}{3^{k+9}}, \frac{14336 \cdot 3^k+5184}{3^{k+9}}, \frac{14336 \cdot 3^k+5103}{3^{k+9}}, \ldots$

$\ldots$

## B. Some Elements of the Compact Set $K$: $T_1$

$K[\omega] = K'[0] = \frac{4}{3}, \qquad K' \cap (1, \frac{4}{3}] = [K'[0], K'[\omega]) = [T_1[0], T_1[\omega])$

$\frac{4}{3}, \mathbf{\frac{32}{27}}, \frac{10}{9}, \mathbf{\frac{256}{243}}, \frac{28}{27}, \frac{82}{81}, \frac{244}{243}, \frac{730}{729}, \ldots \frac{3^k+1}{3^k}, \ldots$

$K[\omega^2] = K'[\omega] = 1, \qquad K' \cap (\frac{8}{9}, 1) = (K'[\omega], K'[\omega 2]) = [T_1[\omega], T_1[\omega 2])$

$\frac{80}{81}, \frac{26}{27}, \frac{76}{81}, \mathbf{\frac{2048}{2187}}, \frac{25}{27}, \frac{224}{243}, \frac{74}{81}, \frac{220}{243}, \frac{73}{81}, \ldots \frac{8 \cdot 3^k+8}{3^{k+2}}, \frac{8 \cdot 3^k+6}{3^{k+2}}, \frac{8 \cdot 3^k+4}{3^{k+2}}, \frac{8 \cdot 3^k+3}{3^{k+2}}, \ldots$

$K[\omega^2 2] = K'[\omega 2] = \frac{8}{9}, \qquad K' \cap (\frac{64}{81}, \frac{8}{9}) = (K'[\omega 2], K'[\omega 3]) = [T_1[\omega 2], T_1[\omega 3])$

$\frac{640}{729}, \mathbf{\frac{70}{81}}, \frac{208}{243}, \frac{68}{81}, \frac{608}{729}, \mathbf{\frac{16384}{19683}}, \frac{200}{243}, \frac{1792}{2187}, \frac{22}{27}, \frac{592}{729}, \frac{196}{243}, \frac{1760}{2187}, \frac{65}{81}, \frac{584}{729}, \ldots \frac{64 \cdot 3^k+64}{3^{k+4}}, \frac{64 \cdot 3^k+54}{3^{k+4}}, \frac{64 \cdot 3^k+48}{3^{k+4}}, \frac{64 \cdot 3^k+36}{3^{k+4}}, \frac{64 \cdot 3^k+32}{3^{k+4}}, \frac{64 \cdot 3^k+27}{3^{k+4}}, \frac{64 \cdot 3^k+24}{3^{k+4}}, \ldots$

$K[\omega^2 3] = K'[\omega 3] = \frac{64}{81}, \qquad K' \cap (\frac{7}{9}, \frac{64}{81}) = (K'[\omega 3], K'[\omega 4]) = [T_1[\omega 3], T_1[\omega 4])$

$\frac{574}{729}, \frac{190}{243}, \frac{1708}{2187}, \mathbf{\frac{5120}{6561}}, \frac{568}{729}, \frac{5110}{6561}, \frac{1702}{2187}, \ldots \frac{7 \cdot 3^k+7}{3^{k+2}}, \frac{7 \cdot 3^k+3}{3^{k+2}}, \ldots$

$K[\omega^2 4] = K'[\omega 4] = \frac{7}{9}, \qquad K' \cap (\frac{20}{27}, \frac{7}{9}) = (K'[\omega 4], K'[\omega 5]) = [T_1[\omega 4], T_1[\omega 5])$

$\frac{560}{729}, \frac{62}{81}, \frac{185}{243}, \mathbf{\frac{1664}{2187}}, \frac{184}{243}, \frac{550}{729}, \frac{61}{81}, \frac{1640}{2187}, \frac{182}{243}, \frac{545}{729}, \frac{544}{729}, \frac{1630}{2187}, \frac{181}{243}, \ldots \frac{20 \cdot 3^k+20}{3^{k+3}}, \frac{20 \cdot 3^k+18}{3^{k+3}}, \frac{20 \cdot 3^k+15}{3^{k+3}}, \frac{20 \cdot 3^k+12}{3^{k+3}}, \frac{20 \cdot 3^k+10}{3^{k+3}}, \frac{20 \cdot 3^k+9}{3^{k+3}}, \ldots$

$K[\omega^2 5] = K'[\omega 5] = \frac{20}{27}, \qquad K' \cap (\frac{19}{27}, \frac{20}{27}) = (K'[\omega 5], K'[\omega 6]) = [T_1[\omega 5], T_1[\omega 6])$

$\mathbf{\frac{131072}{177147}}, \mathbf{\frac{1600}{2187}}, \frac{532}{729}, \mathbf{\frac{14336}{19683}}, \mathbf{\frac{176}{243}}, \mathbf{\frac{4736}{6561}}, \mathbf{\frac{175}{243}}, \mathbf{\frac{1568}{2187}}, \frac{58}{81}, \mathbf{\frac{14080}{19683}}, \mathbf{\frac{520}{729}}, \frac{1558}{2187}, \mathbf{\frac{4672}{6561}}, \mathbf{\frac{41984}{59049}}, \mathbf{\frac{518}{729}}, \mathbf{\frac{1552}{2187}}, \mathbf{\frac{13952}{19683}}, \frac{172}{243}, \mathbf{\frac{4640}{6561}}, \mathbf{\frac{41728}{59049}}, \frac{4636}{6561}, \mathbf{\frac{1544}{2187}}, \mathbf{\frac{13888}{19683}}, \mathbf{\frac{124928}{177147}}, \frac{514}{729}, \mathbf{\frac{4624}{6561}}, \frac{13870}{19683}, \mathbf{\frac{41600}{59049}}, \frac{1540}{2187}, \frac{41572}{59049}, \mathbf{\frac{13856}{19683}}, \frac{4618}{6561}, \frac{124678}{177147}, \mathbf{\frac{124672}{177147}}, \frac{13852}{19683}, \frac{373996}{531441}, \frac{41554}{59049}, \dots \frac{19\cdot 3^k+19}{3^{k+3}}, \frac{19\cdot 3^k+9}{3^{k+3}}, \dots$

$K[\omega^2 6] = K'[\omega 6] = \frac{19}{27}, \qquad K' \cap (\frac{512}{729}, \frac{19}{27}) = (K'[\omega 6], K'[\omega 7]) = [T_1[\omega 6], T_1[\omega 7])$

$\frac{4616}{6561}, \frac{41536}{59049}, \frac{373760}{531441}, \frac{1538}{2187}, \frac{13840}{19683}, \frac{124544}{177147}, \frac{4612}{6561}, \frac{41504}{59049}, \frac{373504}{531441}, \frac{1537}{2187}, \frac{13832}{19683}, \frac{124480}{177147}, \dots \frac{512\cdot 3^k+512}{3^{k+6}}, \frac{512\cdot 3^k+486}{3^{k+6}}, \frac{512\cdot 3^k+432}{3^{k+6}}, \frac{512\cdot 3^k+384}{3^{k+6}}, \frac{512\cdot 3^k+324}{3^{k+6}}, \frac{512\cdot 3^k+288}{3^{k+6}}, \frac{512\cdot 3^k+256}{3^{k+6}}, \frac{512\cdot 3^k+243}{3^{k+6}}, \frac{512\cdot 3^k+216}{3^{k+6}}, \frac{512\cdot 3^k+192}{3^{k+6}}, \dots$

$K[\omega^2 7] = K'[\omega 7] = \frac{512}{729}, \qquad K' \cap (\frac{56}{81}, \frac{512}{729}) = (K'[\omega 7], K'[\omega 8]) = [T_1[\omega 7], T_1[\omega 8])$

$\frac{511}{729}, \frac{4592}{6561}, \frac{170}{243}, \frac{1526}{2187}, \frac{508}{729}, \frac{4564}{6561}, \frac{169}{243}, \frac{1520}{2187}, \frac{1519}{2187}, \frac{13664}{19683}, \frac{506}{729}, \mathbf{\frac{40960}{59049}}, \frac{4550}{6561}, \frac{1516}{2187}, \frac{13636}{19683}, \frac{505}{729}, \frac{4544}{6561}, \frac{4543}{6561}, \frac{40880}{59049}, \frac{1514}{2187}, \frac{13622}{19683}, \frac{4540}{6561}, \frac{40852}{59049}, \frac{1513}{2187}, \frac{13616}{19683}, \frac{13615}{19683}, \dots \frac{56\cdot 3^k+56}{3^{k+4}}, \frac{56\cdot 3^k+54}{3^{k+4}}, \frac{56\cdot 3^k+42}{3^{k+4}}, \frac{56\cdot 3^k+36}{3^{k+4}}, \frac{56\cdot 3^k+28}{3^{k+4}}, \frac{56\cdot 3^k+27}{3^{k+4}}, \frac{56\cdot 3^k+24}{3^{k+4}}, \frac{56\cdot 3^k+21}{3^{k+4}}, \dots$

$\dots$

$K[\omega^3] = K'[\omega^2] = \frac{2}{3}, \qquad K' \cap (\frac{160}{243}, \frac{2}{3}) = (K'[\omega^2], K'[\omega^2+\omega]) = [T_1[\omega^2], T_1[\omega^2+\omega])$

$\frac{13120}{19683}, \frac{1456}{2187}, \frac{485}{729}, \frac{4360}{6561}, \frac{484}{729}, \frac{4352}{6561}, \frac{1450}{2187}, \frac{161}{243}, \frac{13040}{19683}, \frac{1448}{2187}, \frac{4340}{6561}, \frac{482}{729}, \dots \frac{160\cdot 3^k+160}{3^{k+5}}, \frac{160\cdot 3^k+144}{3^{k+5}}, \frac{160\cdot 3^k+135}{3^{k+5}}, \frac{160\cdot 3^k+120}{3^{k+5}}, \frac{160\cdot 3^k+108}{3^{k+5}}, \frac{160\cdot 3^k+96}{3^{k+5}}, \frac{160\cdot 3^k+90}{3^{k+5}}, \frac{160\cdot 3^k+81}{3^{k+5}}, \frac{160\cdot 3^k+80}{3^{k+5}}, \frac{160\cdot 3^k+72}{3^{k+5}}, \frac{160\cdot 3^k+60}{3^{k+5}}, \frac{160\cdot 3^k+54}{3^{k+5}}, \dots$

$K[\omega^3+\omega^2] = K'[\omega^2+\omega] = \frac{160}{243}, \qquad K' \cap (\frac{52}{81}, \frac{160}{243}) = (K'[\omega^2+\omega], K'[\omega^2+\omega 2]) = [T_1[\omega^2+\omega], T_1[\omega^2+\omega 2])$

$\mathbf{\frac{1048576}{1594323}}, \frac{53}{81}, \frac{1430}{2187}, \mathbf{\frac{476}{729}}, \mathbf{\frac{475}{729}}, \mathbf{\frac{12800}{19683}}, \frac{158}{243}, \frac{4264}{6561}, \mathbf{\frac{4256}{6561}}, \frac{1417}{2187}, \frac{472}{729}, \mathbf{\frac{114688}{177147}}, \frac{157}{243}, \frac{4238}{6561}, \frac{470}{729}, \frac{12688}{19683}, \frac{4225}{6561}, \frac{1408}{2187}, \frac{469}{729}, \frac{12662}{19683}, \frac{1406}{2187}, \dots \frac{52\cdot 3^k+52}{3^{k+4}}, \frac{52\cdot 3^k+39}{3^{k+4}}, \frac{52\cdot 3^k+36}{3^{k+4}}, \frac{52\cdot 3^k+27}{3^{k+4}}, \frac{52\cdot 3^k+26}{3^{k+4}}, \frac{52\cdot 3^k+18}{3^{k+4}}, \dots$

$K[\omega^3+\omega^2 2] = K'[\omega^2+\omega 2] = \frac{52}{81}, \qquad K' \cap (\frac{17}{27}, \frac{52}{81}) = (K'[\omega^2+\omega 2], K'[\omega^2+\omega 3]) = [T_1[\omega^2+\omega 2], T_1[\omega^2+\omega 3])$

$\mathbf{\frac{37888}{59049}}, \mathbf{\frac{1400}{2187}}, \mathbf{\frac{155}{243}}, \frac{1394}{2187}, \mathbf{\frac{12544}{19683}}, \mathbf{\frac{4180}{6561}}, \mathbf{\frac{464}{729}}, \mathbf{\frac{112640}{177147}}, \mathbf{\frac{1387}{2187}}, \mathbf{\frac{4160}{6561}}, \frac{154}{243}, \mathbf{\frac{12464}{19683}}, \mathbf{\frac{37376}{59049}}, \frac{4148}{6561}, \mathbf{\frac{335872}{531441}}, \mathbf{\frac{4144}{6561}}, \mathbf{\frac{4142}{6561}}, \frac{460}{729}, \mathbf{\frac{12416}{19683}}, \frac{12410}{19683}, \frac{1378}{2187}, \mathbf{\frac{111616}{177147}}, \frac{37196}{59049}, \frac{4132}{6561}, \dots \frac{17\cdot 3^k+17}{3^{k+3}}, \frac{17\cdot 3^k+9}{3^{k+3}}, \dots$

$K[\omega^3+\omega^2 3] = K'[\omega^2+\omega 3] = \frac{17}{27}, \qquad K' \cap (\frac{152}{243}, \frac{17}{27}) = (K'[\omega^2+\omega 3], K'[\omega^2+\omega 4]) = [T_1[\omega^2+\omega 3], T_1[\omega^2+\omega 4])$

$\frac{12388}{19683}, \frac{1376}{2187}, \mathbf{\frac{1375}{2187}}, \mathbf{\frac{37120}{59049}}, \frac{4123}{6561}, \frac{458}{729}, \mathbf{\frac{333824}{531441}}, \frac{37088}{59049}, \mathbf{\frac{12352}{19683}}, \frac{12350}{19683}, \frac{1372}{2187}, \mathbf{\frac{111104}{177147}}, \frac{457}{729}, \mathbf{\frac{999424}{1594323}}, \frac{37012}{59049}, \frac{4112}{6561}, \frac{12331}{19683}, \mathbf{\frac{36992}{59049}}, \frac{1370}{2187}, \frac{110960}{177147}, \mathbf{\frac{332800}{531441}}, \frac{36974}{59049}, \frac{4108}{6561}, \frac{1369}{2187}, \frac{110884}{177147}, \frac{12320}{19683}, \frac{36955}{59049}, \frac{4106}{6561}, \frac{332576}{531441}, \mathbf{\frac{110848}{177147}}, \frac{110846}{177147}, \frac{12316}{19683}, \frac{4105}{6561}, \frac{332500}{531441}, \frac{36944}{59049}, \frac{110827}{177147}, \frac{12314}{19683}, \frac{997424}{1594323}, \frac{332462}{531441}, \frac{36940}{59049}, \mathbf{\frac{997376}{1594323}}, \frac{12313}{19683}, \frac{997348}{1594323}, \frac{110816}{177147}, \frac{332443}{531441}, \frac{36938}{59049}, \frac{2991968}{4782969}, \frac{997310}{1594323}, \frac{110812}{177147}, \frac{36937}{59049}, \frac{2991892}{4782969}, \frac{332432}{531441}, \frac{997291}{1594323}, \frac{110810}{177147}, \dots \frac{152\cdot 3^k+152}{3^{k+5}}, \frac{152\cdot 3^k+114}{3^{k+5}}, \frac{152\cdot 3^k+108}{3^{k+5}}, \frac{152\cdot 3^k+81}{3^{k+5}}, \frac{152\cdot 3^k+76}{3^{k+5}}, \frac{152\cdot 3^k+72}{3^{k+5}}, \frac{152\cdot 3^k+57}{3^{k+5}}, \frac{152\cdot 3^k+54}{3^{k+5}}, \dots$

$K[\omega^3+\omega^2 4] = K'[\omega^2+\omega 4] = \frac{152}{243}, \qquad K' \cap (\frac{4096}{6561}, \frac{152}{243}) = (K'[\omega^2+\omega 4], K'[\omega^2+\omega 5]) = [T_1[\omega^2+\omega 4], T_1[\omega^2+\omega 5])$

$\frac{36928}{59049}, \frac{332288}{531441}, \frac{2990080}{4782969}, \frac{12304}{19683}, \frac{110720}{177147}, \frac{996352}{1594323}, \frac{4100}{6561}, \frac{36896}{59049}, \frac{332032}{531441}, \frac{2988032}{4782969}, \frac{12296}{19683}, \frac{110656}{177147}, \frac{995840}{1594323}, \frac{1366}{2187}, \frac{8962048}{14348907}, \frac{36880}{59049}, \frac{331904}{531441}, \frac{2987008}{4782969}, \frac{12292}{19683}, \frac{110624}{177147}, \frac{995584}{1594323}, \frac{4097}{6561}, \frac{8960000}{14348907}, \frac{36872}{59049}, \frac{331840}{531441}, \frac{2986496}{4782969}, \frac{12290}{19683}, \ldots \frac{4096\cdot 3^k+4096}{3^{k+8}}, \frac{4096\cdot 3^k+3888}{3^{k+8}}, \frac{4096\cdot 3^k+3456}{3^{k+8}}, \frac{4096\cdot 3^k+3072}{3^{k+8}}, \frac{4096\cdot 3^k+2916}{3^{k+8}}, \frac{4096\cdot 3^k+2592}{3^{k+8}}, \frac{4096\cdot 3^k+2304}{3^{k+8}}, \frac{4096\cdot 3^k+2187}{3^{k+8}}, \frac{4096\cdot 3^k+2048}{3^{k+8}}, \frac{4096\cdot 3^k+1944}{3^{k+8}}, \frac{4096\cdot 3^k+1728}{3^{k+8}}, \frac{4096\cdot 3^k+1536}{3^{k+8}}, \frac{4096\cdot 3^k+1458}{3^{k+8}}, \ldots$

$\ldots$

$K[\omega^3 2] = K'[\omega^2 2] = \frac{16}{27}, \quad K' \cap (\frac{1280}{2187}, \frac{16}{27}) = (K'[\omega^2 2], K'[\omega^2 2+\omega]) = [T_1[\omega^2 2], T_1[\omega^2 2+\omega])$

$\frac{104960}{177147}, \mathbf{\frac{1295}{2187}}, \frac{11648}{19683}, \frac{3880}{6561}, \mathbf{\frac{1292}{2187}}, \frac{34880}{59049}, \frac{3872}{6561}, \frac{430}{729}, \frac{34816}{59049}, \frac{11600}{19683}, \frac{1288}{2187}, \frac{104320}{177147}, \frac{11584}{19683}, \mathbf{\frac{143}{243}}, \frac{3860}{6561}, \frac{34720}{59049}, \frac{3856}{6561}, \frac{312320}{531441}, \frac{1285}{2187}, \frac{34688}{59049}, \frac{11560}{19683}, \frac{428}{729}, \frac{104000}{177147}, \frac{11552}{19683}, \frac{3850}{6561}, \frac{103936}{177147}, \frac{34640}{59049}, \frac{3848}{6561}, \frac{311680}{531441}, \frac{34624}{59049}, \frac{11540}{19683}, \frac{1282}{2187}, \frac{103840}{177147}, \frac{11536}{19683}, \frac{934400}{1594323}, \frac{3845}{6561}, \frac{103808}{177147}, \frac{34600}{59049}, \frac{3844}{6561}, \frac{311360}{531441}, \frac{34592}{59049}, \frac{11530}{19683}, \frac{311296}{531441}, \frac{427}{729}, \frac{103760}{177147}, \frac{11528}{19683}, \ldots \frac{1280\cdot 3^k+1280}{3^{k+7}}, \frac{1280\cdot 3^k+1215}{3^{k+7}}, \frac{1280\cdot 3^k+1152}{3^{k+7}}, \frac{1280\cdot 3^k+1080}{3^{k+7}}, \frac{1280\cdot 3^k+972}{3^{k+7}}, \frac{1280\cdot 3^k+960}{3^{k+7}}, \frac{1280\cdot 3^k+864}{3^{k+7}}, \frac{1280\cdot 3^k+810}{3^{k+7}}, \frac{1280\cdot 3^k+768}{3^{k+7}}, \frac{1280\cdot 3^k+729}{3^{k+7}}, \frac{1280\cdot 3^k+720}{3^{k+7}}, \frac{1280\cdot 3^k+648}{3^{k+7}}, \frac{1280\cdot 3^k+640}{3^{k+7}}, \frac{1280\cdot 3^k+576}{3^{k+7}}, \frac{1280\cdot 3^k+540}{3^{k+7}}, \frac{1280\cdot 3^k+486}{3^{k+7}}, \frac{1280\cdot 3^k+480}{3^{k+7}}, \frac{1280\cdot 3^k+432}{3^{k+7}}, \ldots$

$K[\omega^3 2+\omega^2] = K'[\omega^2 2+\omega] = \frac{1280}{2187}, \qquad K' \cap (\frac{140}{243}, \frac{1280}{2187}) = (K'[\omega^2 2+\omega], K'[\omega^2 2+\omega 2]) = [T_1[\omega^2 2+\omega], T_1[\omega^2 2+\omega 2])$

$\mathbf{\frac{8388608}{14348907}}, \frac{142}{243}, \frac{11480}{19683}, \frac{425}{729}, \frac{1274}{2187}, \frac{424}{729}, \frac{3815}{6561}, \mathbf{\frac{11440}{19683}}, \frac{1270}{2187}, \frac{3808}{6561}, \frac{47}{81}, \frac{11410}{19683}, \frac{1267}{2187}, \frac{3800}{6561}, \frac{422}{729}, \mathbf{\frac{3796}{6561}}, \frac{34160}{59049}, \frac{1265}{2187}, \frac{3794}{6561}, \mathbf{\frac{102400}{177147}}, \frac{1264}{2187}, \frac{11375}{19683}, \mathbf{\frac{34112}{59049}}, \frac{3790}{6561}, \frac{11368}{19683}, \frac{421}{729}, \frac{34090}{59049}, \frac{3787}{6561}, \frac{11360}{19683}, \frac{1262}{2187}, \frac{102200}{177147}, \frac{3785}{6561}, \frac{11354}{19683}, \frac{3784}{6561}, \frac{34055}{59049}, \frac{11350}{19683}, \frac{34048}{59049}, \frac{1261}{2187}, \frac{102130}{177147}, \frac{11347}{19683}, \frac{34040}{59049}, \frac{3782}{6561}, \ldots \frac{140\cdot 3^k+140}{3^{k+5}}, \frac{140\cdot 3^k+135}{3^{k+5}}, \frac{140\cdot 3^k+126}{3^{k+5}}, \frac{140\cdot 3^k+108}{3^{k+5}}, \frac{140\cdot 3^k+105}{3^{k+5}}, \frac{140\cdot 3^k+90}{3^{k+5}}, \frac{140\cdot 3^k+84}{3^{k+5}}, \frac{140\cdot 3^k+81}{3^{k+5}}, \frac{140\cdot 3^k+70}{3^{k+5}}, \frac{140\cdot 3^k+63}{3^{k+5}}, \frac{140\cdot 3^k+60}{3^{k+5}}, \frac{140\cdot 3^k+54}{3^{k+5}}, \ldots$

## C. Some Elements of the Compact Set $K$: $T_2$

$K[\omega^2] = K''[0] = 1, \qquad K'' \cap (\frac{2}{3}, 1] = [K''[0], K''[\omega]) = [T_2[0], T_2[\omega])$

$1, \frac{8}{9}, \mathbf{\frac{64}{81}}, \frac{7}{9}, \frac{20}{27}, \frac{19}{27}, \mathbf{\frac{512}{729}}, \frac{56}{81}, \frac{55}{81}, \frac{164}{243}, \frac{163}{243}, \frac{488}{729}, \frac{487}{729}, \ldots \frac{2\cdot 3^k+2}{3^{k+1}}, \frac{2\cdot 3^k+1}{3^{k+1}}, \ldots$

$K[\omega^3] = K''[\omega] = \frac{2}{3}, \qquad K'' \cap (\frac{16}{27}, \frac{2}{3}) = (K''[\omega], K''[\omega 2]) = [T_2[\omega], T_2[\omega 2])$

$\frac{160}{243}, \frac{52}{81}, \frac{17}{27}, \frac{152}{243}, \mathbf{\frac{4096}{6561}}, \frac{50}{81}, \frac{448}{729}, \frac{148}{243}, \frac{49}{81}, \frac{440}{729}, \frac{146}{243}, \ldots \frac{16\cdot 3^k+16}{3^{k+3}}, \frac{16\cdot 3^k+12}{3^{k+3}}, \frac{16\cdot 3^k+9}{3^{k+3}}, \frac{16\cdot 3^k+8}{3^{k+3}}, \frac{16\cdot 3^k+6}{3^{k+3}}, \ldots$

$K[\omega^3 2] = K''[\omega 2] = \frac{16}{27}, \qquad K'' \cap (\frac{5}{9}, \frac{16}{27}) = (K''[\omega 2], K''[\omega 3]) = [T_2[\omega 2], T_2[\omega 3])$

$\mathbf{\frac{1280}{2187}}, \frac{140}{243}, \mathbf{\frac{416}{729}}, \frac{46}{81}, \frac{410}{729}, \frac{136}{243}, \ldots \frac{5\cdot 3^k+5}{3^{k+2}}, \frac{5\cdot 3^k+3}{3^{k+2}}, \ldots$

$K[\omega^3 3] = K''[\omega 3] = \frac{5}{9}, \qquad K'' \cap (\frac{128}{243}, \frac{5}{9}) = (K''[\omega 3], K''[\omega 4]) = [T_2[\omega 3], T_2[\omega 4])$

$\mathbf{\frac{32768}{59049}}, \frac{400}{729}, \mathbf{\frac{133}{243}}, \frac{3584}{6561}, \frac{44}{81}, \frac{1184}{2187}, \frac{392}{729}, \frac{3520}{6561}, \frac{130}{243}, \frac{1168}{2187}, \frac{10496}{19683}, \frac{388}{729}, \frac{3488}{6561}, \frac{43}{81}, \frac{1160}{2187}, \frac{10432}{19683}, \frac{386}{729}, \frac{3472}{6561}, \ldots \frac{128\cdot 3^k+128}{3^{k+5}}, \frac{128\cdot 3^k+108}{3^{k+5}}, \frac{128\cdot 3^k+96}{3^{k+5}}, \frac{128\cdot 3^k+81}{3^{k+5}}, \frac{128\cdot 3^k+72}{3^{k+5}}, \frac{128\cdot 3^k+64}{3^{k+5}}, \frac{128\cdot 3^k+54}{3^{k+5}}, \frac{128\cdot 3^k+48}{3^{k+5}}, \ldots$

$K[\omega^3 4] = K''[\omega 4] = \frac{128}{243}, \qquad K'' \cap (\frac{14}{27}, \frac{128}{243}) = (K''[\omega 4], K''[\omega 5]) = [T_2[\omega 4], T_2[\omega 5])$

$\frac{1148}{2187}, \frac{127}{243}, \frac{1141}{2187}, \frac{380}{729}, \frac{3416}{6561}, \mathbf{\frac{10240}{19683}}, \frac{379}{729}, \frac{3409}{6561}, \frac{1136}{2187}, \frac{10220}{19683}, \frac{1135}{2187}, \frac{10213}{19683}, \frac{3404}{6561}, \ldots \frac{14\cdot 3^k+14}{3^{k+3}}, \frac{14\cdot 3^k+9}{3^{k+3}}, \frac{14\cdot 3^k+7}{3^{k+3}}, \frac{14\cdot 3^k+6}{3^{k+3}}, \ldots$

$K[\omega^3 5] = K''[\omega 5] = \frac{14}{27}, \qquad K'' \cap (\frac{40}{81}, \frac{14}{27}) = (K''[\omega 5], K''[\omega 6]) = [T_2[\omega 5], T_2[\omega 6])$

$\frac{125}{243}, \frac{1120}{2187}, \frac{124}{243}, \frac{370}{729}, \mathbf{\frac{3328}{6561}}, \frac{41}{81}, \frac{368}{729}, \frac{1100}{2187}, \frac{122}{243}, \frac{365}{729}, \frac{3280}{6561}, \frac{364}{729}, \frac{1090}{2187}, \frac{121}{243}, \frac{1088}{2187}, \frac{3260}{6561}, \frac{362}{729}, \frac{1085}{2187}, \ldots \frac{40\cdot 3^k+40}{3^{k+4}}, \frac{40\cdot 3^k+36}{3^{k+4}}, \frac{40\cdot 3^k+30}{3^{k+4}}, \frac{40\cdot 3^k+27}{3^{k+4}}, \frac{40\cdot 3^k+24}{3^{k+4}}, \frac{40\cdot 3^k+20}{3^{k+4}}, \frac{40\cdot 3^k+18}{3^{k+4}}, \frac{40\cdot 3^k+15}{3^{k+4}}, \ldots$

$K[\omega^3 6] = K''[\omega 6] = \frac{40}{81}, \qquad K'' \cap (\frac{13}{27}, \frac{40}{81}) = (K''[\omega 6], K''[\omega 7]) = [T_2[\omega 6], T_2[\omega 7])$

$\mathbf{\frac{262144}{531441}}, \mathbf{\frac{119}{243}}, \mathbf{\frac{3200}{6561}}, \frac{1066}{2187}, \mathbf{\frac{1064}{2187}}, \frac{118}{243}, \mathbf{\frac{28672}{59049}}, \frac{3172}{6561}, \frac{352}{729}, \ldots \frac{13\cdot 3^k+13}{3^{k+3}}, \frac{13\cdot 3^k+9}{3^{k+3}}, \ldots$

$K[\omega^3 7] = K''[\omega 7] = \frac{13}{27}, \qquad K'' \cap (\frac{38}{81}, \frac{13}{27}) = (K''[\omega 7], K''[\omega 8]) = [T_2[\omega 7], T_2[\omega 8])$

$\mathbf{\frac{9472}{19683}}, \mathbf{\frac{350}{729}}, \mathbf{\frac{3136}{6561}}, \frac{1045}{2187}, \frac{116}{243}, \mathbf{\frac{28160}{59049}}, \mathbf{\frac{1040}{2187}}, \frac{3116}{6561}, \mathbf{\frac{9344}{19683}}, \mathbf{\frac{83968}{177147}}, \mathbf{\frac{1036}{2187}}, \frac{115}{243}, \mathbf{\frac{3104}{6561}}, \mathbf{\frac{27904}{59049}}, \frac{3097}{6561}, \frac{344}{729}, \mathbf{\frac{9280}{19683}}, \mathbf{\frac{83456}{177147}}, \frac{9272}{19683}, \mathbf{\frac{3088}{6561}}, \frac{343}{729}, \mathbf{\frac{27776}{59049}}, \mathbf{\frac{249856}{531441}}, \frac{9253}{19683}, \frac{1028}{2187}, \mathbf{\frac{9248}{19683}}, \frac{27740}{59049}, \mathbf{\frac{83200}{177147}}, \frac{1027}{2187}, \frac{27721}{59049}, \frac{3080}{6561}, \frac{83144}{177147}, \mathbf{\frac{27712}{59049}}, \frac{3079}{6561}, \frac{83125}{177147}, \frac{9236}{19683}, \frac{249356}{531441}, \frac{9235}{19683}, \mathbf{\frac{249344}{531441}}, \frac{249337}{531441}, \frac{27704}{59049}, \frac{747992}{1594323}, \frac{27703}{59049}, \frac{747973}{1594323}, \frac{83108}{177147}, \ldots \frac{38\cdot 3^k+38}{3^{k+4}}, \frac{38\cdot 3^k+27}{3^{k+4}}, \frac{38\cdot 3^k+19}{3^{k+4}}, \frac{38\cdot 3^k+18}{3^{k+4}}, \ldots$

$K[\omega^3 8] = K''[\omega 8] = \frac{38}{81}, \qquad K'' \cap (\frac{1024}{2187}, \frac{38}{81}) = (K''[\omega 8], K''[\omega 9]) = [T_2[\omega 8], T_2[\omega 9])$

$\frac{9232}{19683}, \frac{83072}{177147}, \frac{747520}{1594323}, \frac{3076}{6561}, \frac{27680}{59049}, \frac{249088}{531441}, \frac{1025}{2187}, \frac{9224}{19683}, \frac{83008}{177147}, \frac{747008}{1594323}, \frac{3074}{6561}, \frac{27664}{59049}, \frac{248960}{531441}, \ldots \frac{1024\cdot 3^k+1024}{3^{k+7}}, \frac{1024\cdot 3^k+972}{3^{k+7}}, \frac{1024\cdot 3^k+864}{3^{k+7}}, \frac{1024\cdot 3^k+768}{3^{k+7}}, \frac{1024\cdot 3^k+729}{3^{k+7}}, \frac{1024\cdot 3^k+648}{3^{k+7}}, \frac{1024\cdot 3^k+576}{3^{k+7}}, \frac{1024\cdot 3^k+512}{3^{k+7}}, \frac{1024\cdot 3^k+486}{3^{k+7}}, \frac{1024\cdot 3^k+432}{3^{k+7}}, \frac{1024\cdot 3^k+384}{3^{k+7}}, \ldots$

$\ldots$

$K[\omega^4] = K''[\omega^2] = \frac{4}{9}, \qquad K'' \cap (\frac{320}{729}, \frac{4}{9}) = (K''[\omega^2], K''[\omega^2+\omega]) = [T_2[\omega^2], T_2[\omega^2+\omega])$

$\frac{26240}{59049}, \frac{2912}{6561}, \frac{970}{2187}, \mathbf{\frac{323}{729}}, \frac{8720}{19683}, \frac{968}{2187}, \frac{8704}{19683}, \frac{2900}{6561}, \frac{322}{729}, \frac{26080}{59049}, \frac{2896}{6561}, \frac{965}{2187}, \frac{8680}{19683}, \frac{964}{2187}, \frac{78080}{177147}, \frac{8672}{19683}, \frac{2890}{6561}, \frac{107}{243}, \frac{26000}{59049}, \frac{2888}{6561}, \frac{25984}{59049}, \frac{8660}{19683}, \frac{962}{2187}, \frac{77920}{177147}, \frac{8656}{19683}, \frac{2885}{6561}, \frac{25960}{59049}, \frac{2884}{6561}, \ldots \frac{320\cdot 3^k+320}{3^{k+6}}, \frac{320\cdot 3^k+288}{3^{k+6}}, \frac{320\cdot 3^k+270}{3^{k+6}}, \frac{320\cdot 3^k+243}{3^{k+6}}, \frac{320\cdot 3^k+240}{3^{k+6}}, \frac{320\cdot 3^k+216}{3^{k+6}}, \frac{320\cdot 3^k+192}{3^{k+6}}, \frac{320\cdot 3^k+180}{3^{k+6}}, \frac{320\cdot 3^k+162}{3^{k+6}}, \frac{320\cdot 3^k+160}{3^{k+6}}, \frac{320\cdot 3^k+144}{3^{k+6}}, \frac{320\cdot 3^k+135}{3^{k+6}}, \frac{320\cdot 3^k+120}{3^{k+6}}, \frac{320\cdot 3^k+108}{3^{k+6}}, \ldots$

$K[\omega^4+\omega^3] = K''[\omega^2+\omega] = \frac{320}{729},$

$K'' \cap (\frac{35}{81}, \frac{320}{729}) = (K''[\omega^2+\omega], K''[\omega^2+\omega 2]) = [T_2[\omega^2+\omega], T_2[\omega^2+\omega 2])$

$\mathbf{\frac{2097152}{4782969}}, \frac{2870}{6561}, \frac{106}{243}, \mathbf{\frac{2860}{6561}}, \frac{952}{2187}, \frac{950}{2187}, \mathbf{\frac{949}{2187}}, \frac{8540}{19683}, \mathbf{\frac{25600}{59049}}, \frac{316}{729}, \mathbf{\frac{8528}{19683}}, \frac{2842}{6561}, \frac{2840}{6561}, \frac{25550}{59049}, \frac{946}{2187}, \frac{8512}{19683}, \frac{8510}{19683}, \ldots \frac{35\cdot 3^k+35}{3^{k+4}}, \frac{35\cdot 3^k+27}{3^{k+4}}, \frac{35\cdot 3^k+21}{3^{k+4}}, \frac{35\cdot 3^k+15}{3^{k+4}}, \ldots$

$K[\omega^4+\omega^3 2] = K''[\omega^2+\omega 2] = \frac{35}{81},$

$K'' \cap (\frac{104}{243}, \frac{35}{81}) = (K''[\omega^2+\omega 2], K''[\omega^2+\omega 3]) = [T_2[\omega^2+\omega 2], T_2[\omega^2+\omega 3])$

$\frac{2834}{6561}, \frac{944}{2187}, \mathbf{\frac{229376}{531441}}, \frac{314}{729}, \frac{8476}{19683}, \frac{2821}{6561}, \frac{940}{2187}, \frac{25376}{59049}, \frac{313}{729}, \frac{8450}{19683}, \frac{2816}{6561}, \frac{938}{2187}, \frac{25324}{59049}, \frac{8437}{19683}, \frac{2812}{6561},$

$\dots \frac{104\cdot 3^k+104}{3^{k+5}}, \frac{104\cdot 3^k+81}{3^{k+5}}, \frac{104\cdot 3^k+78}{3^{k+5}}, \frac{104\cdot 3^k+72}{3^{k+5}}, \frac{104\cdot 3^k+54}{3^{k+5}}, \frac{104\cdot 3^k+52}{3^{k+5}}, \frac{104\cdot 3^k+39}{3^{k+5}}, \frac{104\cdot 3^k+36}{3^{k+5}},$

$\dots$

$K[\omega^4+\omega^3 3] = K''[\omega^2+\omega 3] = \frac{104}{243},$

$K'' \cap (\frac{34}{81}, \frac{104}{243}) = (K''[\omega^2+\omega 3], K''[\omega^2+\omega 4]) = [T_2[\omega^2+\omega 3], T_2[\omega^2+\omega 4])$

$\mathbf{\frac{75776}{177147}}, \frac{935}{2187}, \mathbf{\frac{2800}{6561}}, \mathbf{\frac{931}{2187}}, \mathbf{\frac{310}{729}}, \frac{2788}{6561}, \mathbf{\frac{25088}{59049}}, \mathbf{\frac{8360}{19683}}, \mathbf{\frac{928}{2187}}, \mathbf{\frac{225280}{531441}}, \frac{103}{243}, \mathbf{\frac{925}{2187}}, \mathbf{\frac{2774}{6561}}, \mathbf{\frac{8320}{19683}},$
$\frac{308}{729}, \frac{2771}{6561}, \mathbf{\frac{24928}{59049}}, \mathbf{\frac{74752}{177147}}, \frac{8296}{19683}, \mathbf{\frac{671744}{1594323}}, \frac{307}{729}, \mathbf{\frac{8288}{19683}}, \mathbf{\frac{8284}{19683}}, \frac{920}{2187}, \frac{8279}{19683}, \mathbf{\frac{24832}{59049}}, \frac{24820}{59049}, \frac{919}{2187},$
$\frac{2756}{6561}, \mathbf{\frac{223232}{531441}}, \frac{24803}{59049}, \frac{74392}{177147}, \frac{2755}{6561}, \frac{8264}{19683}, \frac{74375}{177147}, \dots \frac{34\cdot 3^k+34}{3^{k+4}}, \frac{34\cdot 3^k+27}{3^{k+4}}, \frac{34\cdot 3^k+18}{3^{k+4}}, \frac{34\cdot 3^k+17}{3^{k+4}},$

$\dots$

$K[\omega^4+\omega^3 4] = K''[\omega^2+\omega 4] = \frac{34}{81},$

$K'' \cap (\frac{304}{729}, \frac{34}{81}) = (K''[\omega^2+\omega 4], K''[\omega^2+\omega 5]) = [T_2[\omega^2+\omega 4], T_2[\omega^2+\omega 5])$

$\frac{24776}{59049}, \frac{2752}{6561}, \mathbf{\frac{2750}{6561}}, \mathbf{\frac{74240}{177147}}, \frac{8246}{19683}, \frac{916}{2187}, \mathbf{\frac{667648}{1594323}}, \frac{74176}{177147}, \frac{305}{729}, \mathbf{\frac{24704}{59049}}, \frac{24700}{59049}, \frac{2744}{6561}, \mathbf{\frac{222208}{531441}},$
$\frac{8227}{19683}, \frac{914}{2187}, \mathbf{\frac{1998848}{4782969}}, \frac{74024}{177147}, \frac{8224}{19683}, \frac{24662}{59049}, \mathbf{\frac{73984}{177147}}, \frac{2740}{6561}, \frac{221920}{531441}, \mathbf{\frac{665600}{1594323}}, \frac{913}{2187}, \frac{73948}{177147}, \frac{8216}{19683},$
$\frac{24643}{59049}, \frac{2738}{6561}, \frac{221768}{531441}, \frac{24640}{59049}, \frac{73910}{177147}, \frac{8212}{19683}, \frac{665152}{1594323}, \frac{2737}{6561}, \mathbf{\frac{221696}{531441}}, \frac{221692}{531441}, \frac{24632}{59049}, \frac{73891}{177147}, \frac{8210}{19683},$
$\frac{665000}{1594323}, \frac{73888}{177147}, \frac{221654}{531441}, \frac{24628}{59049}, \frac{1994848}{4782969}, \frac{8209}{19683}, \frac{664924}{1594323}, \frac{73880}{177147}, \mathbf{\frac{1994752}{4782969}}, \frac{221635}{531441}, \frac{24626}{59049}, \frac{1994696}{4782969},$
$\frac{221632}{531441}, \frac{664886}{1594323}, \frac{73876}{177147}, \frac{5983936}{14348907}, \frac{24625}{59049}, \frac{1994620}{4782969}, \frac{221624}{531441}, \frac{664867}{1594323}, \frac{73874}{177147}, \frac{5983784}{14348907}, \frac{664864}{1594323},$
$\frac{1994582}{4782969}, \frac{221620}{531441}, \dots \frac{304\cdot 3^k+304}{3^{k+6}}, \frac{304\cdot 3^k+243}{3^{k+6}}, \frac{304\cdot 3^k+228}{3^{k+6}}, \frac{304\cdot 3^k+216}{3^{k+6}}, \frac{304\cdot 3^k+171}{3^{k+6}}, \frac{304\cdot 3^k+162}{3^{k+6}},$
$\frac{304\cdot 3^k+152}{3^{k+6}}, \frac{304\cdot 3^k+144}{3^{k+6}}, \frac{304\cdot 3^k+114}{3^{k+6}}, \frac{304\cdot 3^k+108}{3^{k+6}}, \dots$

$\dots$

$K[\omega^4 2] = K''[\omega^2 2] = \frac{32}{81}$, $K'' \cap (\frac{95}{243}, \frac{32}{81}) = (K''[\omega^2 2], K''[\omega^2 2+\omega]) = [T_2[\omega^2 2], T_2[\omega^2 2+\omega])$

$\mathbf{\frac{209920}{531441}}, \mathbf{\frac{2590}{6561}}, \mathbf{\frac{23296}{59049}}, \mathbf{\frac{7760}{19683}}, \frac{2584}{6561}, \mathbf{\frac{69760}{177147}}, \mathbf{\frac{287}{729}}, \mathbf{\frac{7744}{19683}}, \frac{860}{2187}, \mathbf{\frac{69632}{177147}}, \mathbf{\frac{23200}{59049}}, \mathbf{\frac{2576}{6561}}, \mathbf{\frac{208640}{531441}},$
$\frac{23180}{59049}, \mathbf{\frac{23168}{59049}}, \frac{286}{729}, \mathbf{\frac{7720}{19683}}, \mathbf{\frac{69440}{177147}}, \frac{7714}{19683}, \mathbf{\frac{7712}{19683}}, \mathbf{\frac{624640}{1594323}}, \frac{2570}{6561}, \mathbf{\frac{69376}{177147}}, \mathbf{\frac{23120}{59049}}, \frac{69350}{177147}, \frac{856}{2187},$
$\mathbf{\frac{208000}{531441}}, \frac{23104}{59049}, \frac{7700}{19683}, \mathbf{\frac{207872}{531441}}, \frac{207860}{531441}, \frac{2566}{6561}, \mathbf{\frac{69280}{177147}}, \frac{69274}{177147}, \frac{23090}{59049}, \frac{623390}{1594323}, \frac{7696}{19683}, \mathbf{\frac{623360}{1594323}},$
$\frac{207784}{531441}, \frac{69260}{177147}, \frac{1869980}{4782969}, \frac{23086}{59049}, \frac{623314}{1594323}, \frac{207770}{531441}, \dots \frac{95\cdot 3^k+95}{3^{k+5}}, \frac{95\cdot 3^k+81}{3^{k+5}}, \frac{95\cdot 3^k+57}{3^{k+5}}, \frac{95\cdot 3^k+45}{3^{k+5}}, \dots$

$K[\omega^4 2+\omega^3] = K''[\omega^2 2+\omega] = \frac{95}{243},$

$K'' \cap (\frac{2560}{6561}, \frac{95}{243}) = (K''[\omega^2 2+\omega], K''[\omega^2 2+\omega 2]) = [T_2[\omega^2 2+\omega], T_2[\omega^2 2+\omega 2])$

$\frac{69248}{177147}, \frac{23080}{59049}, \frac{2564}{6561}, \frac{207680}{531441}, \frac{23072}{59049}, \frac{1868800}{4782969}, \frac{7690}{19683}, \frac{207616}{531441}, \frac{69200}{177147}, \frac{7688}{19683}, \frac{622720}{1594323}, \frac{69184}{177147},$
$\frac{23060}{59049}, \frac{622592}{1594323}, \frac{854}{2187}, \frac{207520}{531441}, \frac{23056}{59049}, \frac{1867520}{4782969}, \frac{7685}{19683}, \frac{207488}{531441}, \frac{69160}{177147}, \frac{7684}{19683}, \frac{622400}{1594323}, \frac{69152}{177147},$
$\frac{5601280}{14348907}, \frac{23050}{59049}, \frac{622336}{1594323}, \frac{2561}{6561}, \frac{207440}{531441}, \frac{23048}{59049}, \frac{1866880}{4782969}, \frac{207424}{531441}, \frac{69140}{177147}, \frac{1866752}{4782969}, \frac{7682}{19683}, \frac{622240}{1594323},$
$\frac{69136}{177147}, \frac{5600000}{14348907}, \frac{23045}{59049}, \frac{622208}{1594323}, \frac{207400}{531441}, \frac{23044}{59049}, \frac{1866560}{4782969}, \frac{207392}{531441}, \dots \frac{2560\cdot 3^k+2560}{3^{k+8}}, \frac{2560\cdot 3^k+2430}{3^{k+8}},$
$\frac{2560\cdot 3^k+2304}{3^{k+8}}, \frac{2560\cdot 3^k+2187}{3^{k+8}}, \frac{2560\cdot 3^k+2160}{3^{k+8}}, \frac{2560\cdot 3^k+1944}{3^{k+8}}, \frac{2560\cdot 3^k+1920}{3^{k+8}}, \frac{2560\cdot 3^k+1728}{3^{k+8}},$
$\frac{2560\cdot 3^k+1620}{3^{k+8}}, \frac{2560\cdot 3^k+1536}{3^{k+8}}, \frac{2560\cdot 3^k+1458}{3^{k+8}}, \frac{2560\cdot 3^k+1440}{3^{k+8}}, \frac{2560\cdot 3^k+1296}{3^{k+8}}, \frac{2560\cdot 3^k+1280}{3^{k+8}},$
$\frac{2560\cdot 3^k+1215}{3^{k+8}}, \frac{2560\cdot 3^k+1152}{3^{k+8}}, \frac{2560\cdot 3^k+1080}{3^{k+8}}, \frac{2560\cdot 3^k+972}{3^{k+8}}, \frac{2560\cdot 3^k+960}{3^{k+8}}, \frac{2560\cdot 3^k+864}{3^{k+8}}, \dots$