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THE STRUCTURE OF BASE PHI EXPANSIONS

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Abstract

In the base phi expansion, a natural number is written uniquely as a sum of powers of the golden mean with coefficients 0 and 1, where it is required that the product of two consecutive digits is always 0. We tackle the problem of describing these expansions in detail. We classify the positive parts of the base phi expansions according to their suffixes, and the negative parts according to their prefixes, specifying the sequences of occurrences of these digit blocks. We prove that the positive parts of the base phi expansions are a subsequence of the sequence of Zeckendorf expansions, giving an explicit formula in terms of a generalized Beatty sequence. The negative parts of the base phi expansions no longer appear lexicographically. We prove that all allowed digit blocks appear, and determine the order in which they do appear.

1. Introduction

Let the golden mean be given by $\varphi := (1+\sqrt{5})/2$. Ignoring leading and trailing zeros, a natural number N can be written uniquely as

$$N = \sum_{i=-\infty}^{\infty} d_i \varphi^i,$$

with digits $d_i = 0$ or 1, and where $d_i d_{i+1} = 11$ is not allowed (see [2]). As usual, we denote the base phi expansion of N as $\beta(N)$, and these expansions are written with a radix point as

$$\beta(N) = d_L d_{L-1} \cdots d_1 d_0 \cdot d_{-1} d_{-2} \cdots d_{R+1} d_R.$$

We define

$$\beta^+(N) = d_L d_{L-1} \cdots d_1 d_0 \ \text{ and } \ \beta^-(N) = d_{-1} d_{-2} \cdots d_{R+1} d_R.$$

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So $\beta(N) = \beta^+(N) \cdot \beta^-(N)$. For example, $\beta(2) = 10 \cdot 01$, and $\beta(3) = 100 \cdot 01$.

This paper deals with the following question: what are the words of 0's and 1's that can occur as digit blocks in the base phi expansion N, and for which numbers N do they occur? In Section 6, we answer this question for the suffixes of the β^+-part of the base phi expansions, and in Section 7 for the complete β^--part of the base phi expansions, and the prefixes of the β^--part of length at most 3.

The first five sections establish relationships between the base phi expansions and Zeckendorf expansions, also known as Fibonacci representations. Recall that in the Zeckendorf expansions, a natural number is written uniquely as a sum of Fibonacci numbers with coefficients 0 and 1, where, again, it is required that the product of two consecutive digits is always 0. In a previous work [6] we have classified the Zeckendorf expansions according to their suffixes. It turned out that if we consider the suffixes as labels on the Fibonacci tree, then the numbers with a given suffix in their Zeckendorf expansion appear as generalized Beatty sequences in a natural way on this tree. The connection between base phi and Zeckendorf expansions, permits us to exploit these results in Section 6. See the paper [8] for a less direct approach, in terms of two-tape automata.

In Section 2 we give a formula for the positive part of a base phi expansion in terms of Zeckendorf expansions. In Section 3 we recall the recursive structure of base phi expansions, and derive some tools from this which are useful in the final two sections. In Section 4 we take a closer look at the Lucas intervals. In Section 5 we introduce generalized Beatty sequences, which for the base phi expansion take over the role played by arithmetic sequences in the classical expansions in base b, where b is an integer larger than 1.

We end this introduction by pointing out that there is a neat way to obtain N from the $\beta^+(N)$-part of $\beta(N)$, without knowing the $\beta^-(N)$-part. If $\beta(N) = \beta^+(N) \cdot \beta^-(N)$ is the base phi expansion of a natural number N, then $N = \lceil \beta^+(N) \rceil$. Here $\lceil \cdot \rceil$ is the ceiling function. For a proof, add the maximum number of powers corresponding to $\beta^-(N)$, taking into account that no 11 appears. This is bounded by the geometric series starting at φ^{-1} with common ratio φ^{-2}, i.e., by $\varphi^{-1}/(1-\varphi^{-2}) = 1$.

2. Embedding Base Phi into Zeckendorf

We define the Lekkerkerker-Zeckendorf expansion. Let (F_n) be the Fibonacci numbers. Let $\ddot{F}_0 = 1, \ddot{F}_1 = 2, \ddot{F}_2 = 3, \ldots$ be the twice-shifted Fibonacci numbers, defined by $\ddot{F}_i = F_{i+2}$. Ignoring leading and trailing zeros, a natural number N can be written uniquely as

$$N = \sum_{i=0}^{\infty} e_i \ddot{F}_i,$$

with digits $e_i = 0$ or 1, and where $e_i e_{i+1} = 11$ is not allowed. We denote the Zeckendorf expansion of N as $Z(N)$.

Let V be the generalized Beatty sequence (cf. [1]) defined by

$$V(n) = 3\lfloor n\varphi \rfloor + n + 1.$$

Here $\lfloor \cdot \rfloor$ denotes the floor function, and $(\lfloor n\varphi \rfloor)$ is the well-known lower Wythoff sequence.

We define the function S by

$$S(n) = \max\{k \in \mathbb{N} : V(k) \le n\} - 1.$$

Theorem 1. *For all $N \ge 0$,*

$$\beta^+(N) = Z(N + S(N)).$$

This theorem will be proved in Section 2.2.

The basis for the embedding of the $\beta^+(N)$ into the collection of Zeckendorf words is the following analysis.

2.1. The Art of Adding 1

It is essential to give ourselves the freedom to also write non-admissible expansions in the form

$$\beta(N) = d_L d_{L-1} \cdots d_1 d_0 \cdot d_{-1} d_{-2} \cdots d_{R+1} d_R.$$

For example, since $\beta(4) = 101.01$ and $\beta(2) = 10 \cdot 01$, we can write

$$\beta(5) \doteq \beta(4) + 1 \doteq 101 \cdot 01 + 1 \cdot 0 \doteq 102 \cdot 01 \doteq 110 \cdot 02 \doteq 1000 \cdot 1001. \tag{1}$$

Here the symbol $\doteq$ indicates that we consider a non-admissible expansion.

It is convenient to generate all Zeckendorf expansions and base phi expansions by repeatedly adding the number 1. To compute $\beta(N) + 1$ for some number N, then, in general, there is a carry both to the left and (two places) to the right. This is illustrated by the example in Equation (1). Note that there is not only a *double carry*, but that we also have to get rid of the 11's, by replacing them with 100's. This is allowed because of the equation $\varphi^{n+2} = \varphi^{n+1} + \varphi^n$. We call this operation a *golden mean flip*.

To compute $Z(N) + 1$ for some number N, a distinction between $e_0 = 0$ and $e_0 = 1$ has to be made:

$$Z(N) = e_L \cdots e_2 e_1 \, 0 \quad \text{gives} \quad Z(N) + 1 = e_L \cdots e_2 e_1 \, 1$$

and

$$Z(N) = e_L \cdots e_2 e_1 \, 1 \quad \text{gives} \quad Z(N) + 1 \doteq e_L \cdots e_2 \, 10.$$

Here we used the symbol $\doteq$ because (several) golden mean flips might follow, where for the Zeckendorf expansion these are justified by the equation $F_{n+2} = F_{n+1} + F_n$. Note that replacing $e_1 1 + 1$ by 10 follows from 1+1=2 (!).

For the convenience of the reader we provide in Table 1 a list of the Zeckendorf and base phi expansions of the first 18 natural numbers.

N	$Z(N)$	$\beta(N)$	N	$Z(N)$	$\beta(N)$
1	1	$1\cdot$	10	10010	$10100 \cdot 0101$
2	10	$10 \cdot 01$	11	10100	$10101 \cdot 0101$
3	100	$100 \cdot 01$	12	10101	$100000 \cdot 101001$
4	101	$101 \cdot 01$	13	100000	$100010 \cdot 001001$
5	1000	$1000 \cdot 1001$	14	100001	$100100 \cdot 001001$
6	1001	$1010 \cdot 0001$	15	100010	$100101 \cdot 001001$
7	1010	$10000 \cdot 0001$	16	100100	$101000 \cdot 100001$
8	10000	$10001 \cdot 0001$	17	100101	$101010 \cdot 000001$
9	10001	$10010 \cdot 0101$	18	101000	$1000000 \cdot 000001$

Table 1: Zeckendorf and base phi expansions

2.2. Proof of Theorem 1

The essential ingredient of the proof is the following result from [5], Theorem 5.1 and Remark 5.4. An alternative, short proof of the first part could be given with the Propagation Principle from Section 3.

Proposition 1. *Let $\beta(N) = (d_i(N))$ be the base phi expansion of N. Then*
$d_1 d_0 \cdot d_{-1}(N) = 10 \cdot 1$ *never occurs,*
$d_1 d_0 \cdot d_{-1}(N) = 00 \cdot 1$ *if and only if* $N = 3\lfloor n\varphi \rfloor + n + 1$ *for some natural number* n.

Proof of Theorem 1. One observes that there are many $\beta(N)$ such that $\beta^+(N) = Z(N')$ for some N'. Moreover, if this is the case, then also $\beta^+(N+1) = Z(N'+1)$, *except* if $d_{-1}(N) = 1$ in $\beta(N)$. Indeed, as long as $d_{-1}(N) = 0$, adding 1 gives the same result for both the Zeckendorf and the positive part of the base phi expansion, as seen in the previous section. However, suppose

$$Z(N') = \beta^+(N), \text{ and } d_{-1}(N) = 1.$$

Then, by Proposition 1, $d_1 d_0 \cdot d_{-1}(N) = 00\cdot 1$, and adding 1 to N gives the expansion $\beta(N+1)$ with digit block $d_1 d_0 \cdot d_{-1}(N+1) = 10\cdot 0$. So $\beta^+(N+1)$ ends in exactly the same two digits as $Z(N'+2)$, and in fact $\beta^+(N+1) = Z(N'+2)$. This means that one Zeckendorf expansion has been skipped: that of $N'+1$. Every time a $d_{-1}(N) = 1$ occurs, this skipping takes place. Since $Z(0) = \beta^+(0), \ldots, Z(5) = \beta^+(5)$, this gives the formula $\beta^+(N) = Z(N + S(N))$, with $S(n) = \max\{k \in \mathbb{N} : 3\lfloor k\varphi \rfloor + k \leq n\}$, by the second statement of Proposition 1. □

3. The Recursive Structure of Base Phi Expansions

The Lucas numbers $(L_n) = (2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots)$ are defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geq 2.$$

The Lucas numbers have a particularly simple base phi expansion. For all $n \geq 1$ we obtain from the well-known formula $L_{2n} = \varphi^{2n} + \varphi^{-2n}$, and the recursion formula $L_{2n+1} = L_{2n} + L_{2n-1}$ that

$$\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1}1, \quad \beta(L_{2n+1}) = 1(01)^n \cdot (01)^n. \tag{2}$$

By iterated application of the double carry and the golden mean flip to $\beta(L_{2n+1}) + \beta(1)$, and a similar operation for $\beta(L_{2n+2} - 1)$ (see also the last page of [7]) for all $n \geq 1$ one finds that

$$\beta(L_{2n+1} + 1) = 10^{2n+1} \cdot (10)^n 01, \quad \beta(L_{2n+2} - 1) = (10)^{n+1} \cdot 0^{2n+1}1. \tag{3}$$

As in [5] we partition the natural numbers into Lucas intervals

$$\Lambda_{2n} := [L_{2n},\, L_{2n+1}] \quad \text{and} \quad \Lambda_{2n+1} := [L_{2n+1} + 1,\, L_{2n+2} - 1].$$

The basic idea behind this partition is that if

$$\beta(N) = d_L d_{L-1} \cdots d_1 d_0 \cdot d_{-1} d_{-2} \cdots d_{R+1} d_R,$$

then the leftmost index $L = L(N)$ and the rightmost index $R = R(N)$ satisfy

$$\begin{aligned} &L(N) = |R(N)| = 2n \;\text{ if and only if }\; N \in \Lambda_{2n},\\ &L(N) = 2n+1,\; |R(N)| = 2n+2 \;\text{ if and only if }\; N \in \Lambda_{2n+1}. \end{aligned} \tag{4}$$

This is not hard to see from the simple expressions we have for the β-expansions of the Lucas numbers; see also Theorem 1 in [9].

To obtain recursive relations, the interval $\Lambda_{2n+1} = [L_{2n+1} + 1, L_{2n+2} - 1]$ has to be divided into three subintervals. These three intervals are

$$\begin{aligned} I_n :=& [L_{2n+1} + 1,\, L_{2n+1} + L_{2n-2} - 1],\\ J_n :=& [L_{2n+1} + L_{2n-2},\, L_{2n+1} + L_{2n-1}],\\ K_n :=& [L_{2n+1} + L_{2n-1} + 1,\, L_{2n+2} - 1]. \end{aligned}$$

It is very convenient to use the free group versions of words of 0's and 1's. So, for example, $(01)^{-1}0001 = 1^{-1}001$.

Theorem 2 (Recursive Structure Theorem)**.** *Let the odd and even Lucas intervals be given by*

$$\Lambda_{2n+1} = [L_{2n+1} + 1,\, L_{2n+2} - 1],\; \Lambda_{2n+2} = [L_{2n+2},\, L_{2n+3}].$$

(A) *For all $n \geq 2$ and $k = 1, \ldots, L_{2n} - 1$, we have*

$$\begin{aligned} I_n: &\quad \beta(L_{2n+1}+k) = 1000(10)^{-1}\beta(L_{2n-1}+k)(01)^{-1}1001, \\ K_n: &\quad \beta(L_{2n+1}+L_{2n-1}+k) = 1010(10)^{-1}\beta(L_{2n-1}+k)(01)^{-1}0001. \end{aligned}$$

Moreover, for all $n \geq 2$ and $k = 0, \ldots, L_{2n-1}$, we have

$$J_n: \quad \beta(L_{2n+1}+L_{2n-2}+k) = 10010(10)^{-1}\beta(L_{2n-2}+k)(01)^{-1}001001.$$

(B) *For all $n \geq 1$ and $k = 0, \ldots, L_{2n+1}$ one has*

$$\beta(L_{2n+2}+k) = \beta(L_{2n+2}) + \beta(k) = 10\cdots0\,\beta(k)\,0\cdots01.$$

See [7] for a proof of this theorem. As an illustration of the use of Theorem 2 we shall now prove a lemma that we need in Section 6.

Lemma 1. *Let $m \geq 1$ be an integer. There are* (a) *no expansions $\beta(N)$ with the digit block $d_{2m}\cdots d_0 \cdot d_{-1}(N) = 10^{2m}\cdot 1$, and there are* (b) *no expansions $\beta(N)$ with the digit block $d_{2m+1}\cdots d_0 \cdot d_{-1}(N) = 10^{2m+1}\cdot 0$.*

Proof. (a) The first time $d_{2m}\cdots d_0 = 10^{2m}$ occurs is for $N = L_{2m}$, and then $d_{-1}(N) = 0$, as follows from the $\beta(L_{2m})$ formula in Equation (2). With Equation (4) we see that this is also the only occurrence of the digit block $10^{2m}\cdot 1$ in the expansions of the numbers N in Λ_{2m}. Similarly, it is obvious that the digit block $10^{2m}\cdot 1$ does not appear in the expansions of the numbers N in Λ_{2m+1}.

From Part (B) of the Recursive Structure Theorem we see that the digit block $d_{2m}\cdots d_0 = 10^{2m}$ in the expansions of the numbers N in Λ_{2m+2} only occurs in combination with $d_{-1}(N) = 0$ (since we already proved this for the interval Λ_{2m}) .

From Part (A) of the Recursive Structure Theorem we will see that the digit block $d_{2m}\cdots d_0 = 10^{2m}$ in the expansions of the numbers N in Λ_{2m+3} only occurs in combination with $d_{-1}(N) = 0$. This is definitely more complicated than this observation for Λ_{2m+2}. We have to split Λ_{2m+3} into the three pieces I_{m+1}, J_{m+1} and K_{m+1}. The middle piece J_{m+1} corresponds to numbers in Λ_{2m}, from which we already know that $d_{2m}\cdots d_0(N) = 10^{2m}$ implies that $d_{-1}(N) = 0$. The numbers N in the first piece, I_{m+1}, correspond to numbers in Λ_{2m+1} from which the digits $d_{2m+1}d_{2m} = 10$ have been replaced by the digits $d_{2m+3}d_{2m+2}d_{2m+1}d_{2m} = 1000$. In particular $d_{2m} = 0$ excludes any occurrence of $d_{2m}\cdots d_0 = 10^{2m}$. In the same way occurrences of $d_{2m}\cdots d_0 = 10^{2m}$ in K_{m+1} are excluded.

The final conclusion is that both intervals Λ_{2m+2} and Λ_{2m+3} only contain numbers N for which the occurrence of $d_{2m}\cdots d_0(N) = 10^{2m}$ implies $d_{-1}(N) = 0$. In the same way, these properties of Λ_{2m+2} and Λ_{2m+3} carry over to the two Lucas intervals Λ_{2m+4} and Λ_{2m+5}, and we can finish the proof by induction.

(b) The first time $d_{2m+1}\cdots d_0 = 10^{2m+1}$ occurs is for $N = L_{2m+1}+1$ in Λ_{2m+1}, and then $d_{-1}(N) = 1$ (see Equation (3)). This is also the only occurrence of

$d_{2m+1}\cdots d_0 = 10^{2m+1}$ in numbers from Λ_{2m+1}. Moreover, in all numbers from Λ_{2m+2} the digit block $d_{2m+1}\cdots d_0 = 10^{2m+1}$ does not occur at all. We finish the proof as in part (a), by applying the Recursive Structure Theorem. □

It is convenient to have a second version of the Recursive Structure Theorem which involves a higher resemblance between the Λ-intervals with an odd index in Part (A), and the Λ-intervals with an even index in the Part (B) case. It is also convenient to have the Λ-intervals play a more visible role in the recursion. In fact, it is easy to check that the three intervals I_n, J_n and K_n in the Recursive Structure Theorem satisfy

$$\begin{aligned} I_n &= \Lambda^{(a)}_{2n+1} := \Lambda_{2n+1} + L_{2n+2},\\ J_n &= \Lambda^{(b)}_{2n} := \Lambda_{2n} + L_{2n+3},\\ K_n &= \Lambda^{(c)}_{2n+1} := \Lambda_{2n+1} + L_{2n+3}. \end{aligned}$$

In this equation we employ the usual notation $A + x := \{a + x : a \in A\}$ for a set of real numbers A and a real number x.

Theorem 3 (Recursive Structure Theorem: 2nd version)**.** *Let the odd and even Lucas intervals be given by*

$$\Lambda_{2n+1} = [L_{2n+1} + 1,\, L_{2n+2} - 1],\ \Lambda_{2n+2} = [L_{2n+2},\, L_{2n+3}].$$

(A) *For all $n \ge 1$ one has*

$$\Lambda_{2n+1} = \Lambda^{(a)}_{2n-1} \cup \Lambda^{(b)}_{2n-2} \cup \Lambda^{(c)}_{2n-1},$$

where $\Lambda^{(a)}_{2n-1} = \Lambda_{2n-1} + L_{2n}$, $\Lambda^{(b)}_{2n-2} = \Lambda_{2n-2} + L_{2n+1}$, and $\Lambda^{(c)}_{2n-1} = \Lambda_{2n-1} + L_{2n+1}$. We have

$$\beta(N) = \begin{cases} 1000(10)^{-1}\,\beta(N - L_{2n})\,(01)^{-1}1001, & \text{if } N \in \Lambda^{(a)}_{2n-1},\\ 100\,\beta(N - L_{2n+1})\,(01)^{-1}001001, & \text{if } N \in \Lambda^{(b)}_{2n-2},\\ 10\,\beta(N - L_{2n+1})\,(01)^{-1}0001, & \text{if } N \in \Lambda^{(c)}_{2n-1}. \end{cases} \tag{5}$$

(B) *For all $n \ge 1$ one has*

$$\Lambda_{2n+2} = \Lambda^{(a)}_{2n} \cup \Lambda^{(b)}_{2n-1} \cup \Lambda^{(c)}_{2n},$$

where $\Lambda^{(a)}_{2n} = \Lambda_{2n} + L_{2n+1}$, $\Lambda^{(b)}_{2n-1} = \Lambda_{2n-1} + L_{2n+2}$, and $\Lambda^{(c)}_{2n} = \Lambda_{2n} + L_{2n+2}$. We have

$$\beta(N) = \begin{cases} 1000(10)^{-1}\,\beta(N - L_{2n+1})\,(01)^{-1}0001, & \text{if } N \in \Lambda^{(a)}_{2n},\\ 100\,\beta(N - L_{2n+2})\,01 & \text{if } N \in \Lambda^{(b)}_{2n-1},\\ 10\,\beta(N - L_{2n+1})\,01 & \text{if } N \in \Lambda^{(c)}_{2n}. \end{cases} \tag{6}$$

Proof. (A) This is a rephrasing of Part (A) in Theorem 2.
(B) We start by showing that the three intervals $\Lambda^{(a)}_{2n}, \Lambda^{(b)}_{2n-1}, \Lambda^{(c)}_{2n}$ partition Λ_{2n+2}.

The first number in $\Lambda^{(a)}_{2n}$ is $L_{2n} + L_{2n+1} = L_{2n+2}$, which is the first number of Λ_{2n+2}. The last number in $\Lambda^{(a)}_{2n}$ is $L_{2n+1} + L_{2n+1} = 2L_{2n+1}$.

The first number in $\Lambda^{(b)}_{2n-1}$ is $L_{2n-1} + 1 + L_{2n+2} = L_{2n-1} + 1 + L_{2n} + L_{2n+1} = 2L_{2n+1} + 1$, which indeed, is the successor of the last number in $\Lambda^{(a)}_{2n}$.

The last number in $\Lambda^{(b)}_{2n-1}$ is $L_{2n} - 1 + L_{2n+2}$, which indeed has successor $L_{2n} + L_{2n+2}$, the first number in $\Lambda^{(c)}_{2n}$. Finally, the last number in $\Lambda^{(c)}_{2n}$ is $L_{2n+1} + L_{2n+2} = L_{2n+3}$, which is the last number in Λ_{2n+2}.

To prove the first case in Equation (5), we first show, using Equation (2) twice, that this equation is correct for $N = L_{2n+2}$, which is the first number of $\Lambda^{(a)}_{2n}$:

$$\begin{aligned}
\beta(L_{2n+2}) &= 10^{2n+2} \cdot 0^{2n+1}1 \\
&= 1000\,0^{2n-1} \cdot 0^{2n-2}\,0001 \\
&= 1000(10)^{-1}10^{2n} \cdot 0^{2n-1}1\,(01)^{-1}0001 \\
&= 1000(10)^{-1}\beta(L_{2n})\,(01)^{-1}0001 \\
&= 1000(10)^{-1}\beta(L_{2n+2} - L_{2n+1})\,(01)^{-1}0001.
\end{aligned}$$

The first case in Equation (5) is also correct for all other $N \in \Lambda^{(a)}_{2n}$, because as above, the digit block $d_L d_{L-1} d_{L-2} d_{L-3}(N)$ always is 1000, and the digit block $d_{L-2} d_{L-3}(N - L_{2n+1})$ always is 10. For the negative digits we have a similar property.

The second case in Equation (5) follows directly from the fact that if $N \in \Lambda^{(b)}_{2n-1}$, then

$$\begin{aligned}
\beta(N - L_{2n+2}) + \beta(L_{2n+2}) &= d_{2n-1} \cdots d_0 \cdot d_{-1} \cdots d_{-2n} + 10^{2n+2} \cdot 0^{2n+1}1 \\
&= d_{2n-1} \cdots d_0 \cdot d_{-1} \cdots d_{-2n} + 100\,0^{2n} \cdot 0^{2n}\,01 \\
&= 100 d_{2n-1} \cdots d_0 \cdot d_{-1} \cdots d_{-2n} 01,
\end{aligned}$$

since the numbers in Λ_{2n-1} have a β-expansion $d_{2n-1} \cdots d_0 \cdot d_{-1} \cdots d_{-2n}$ with $2n$ digits on the left and $2n$ digits on the right. Note that we do not have to use the symbol $\doteq$, as there are no double carries or golden mean flips.

The third case in Equation (5) follows in the same way. $\square$

Lemma 1 is an example of a general phenomenon, which we call the Propagation Principle. It has an extension to combinations of digit blocks that we give in Lemma 2. The Propagation Principle is closely connected to the following notion. We say an interval Γ and a union of intervals Δ of natural numbers are *β-congruent modulo q* for some natural number q if Δ is a disjoint union of translations of Λ-intervals, such that for all $j = 1, \ldots, |\Gamma|$, if N is the j^{th} element of Γ, and N' is the j^{th} element of Δ, then

$$d_{q-1}\cdots d_1 d_0 \cdot d_{-1}\cdots d_{-q}(N) = d_{q-1}\cdots d_1 d_0 \cdot d_{-1}\cdots d_{-q}(N').$$

We write this as $\Gamma \cong \Delta_1\Delta_2\dots\Delta_r \mod q$, when the number of translations of Λ-intervals in Δ equals r. Note that the definition implies that the r disjoint translations of Λ-intervals appear in the natural order, and that we refrain from indicating the translations. Simple examples are $\Lambda_5 \cong \Lambda_3\Lambda_2\Lambda_3 \mod 1$ and $\Lambda_6 \cong \Lambda_4\Lambda_3\Lambda_4 \mod 3$. Theorem 3 is a source of many more examples.

An important observation is that if

$$\Gamma \cong \Delta_1\Delta_2\dots\Delta_r \mod q \quad \text{and} \quad \Gamma' \cong \Delta_1'\Delta_2'\dots\Delta_{r'}' \mod q',$$

and $\Gamma \cup \Gamma'$ is an interval, then

$$\Gamma\Gamma' := \Gamma \cup \Gamma' \cong \Delta_1\Delta_2\dots\Delta_r\Delta_1'\Delta_2'\dots\Delta_{r'}' \mod \min\{q,q'\}. \tag{7}$$

To keep the formulation and the proof of the following lemma simple, we only formulate it for central digit blocks of length 8 (i.e., $q = 4$). In the following, occurrences of digit blocks in β-expansions have to be interpreted with additional 0's added to the left of the expansion.

Lemma 2 (Propagation Principle for β-expansions with length 8, i.e., $q = 4$)**.** (a) *Suppose the digit block $d_3\cdots d_0 \cdot d_{-1}\cdots d_{-4}$, does not occur in the β-expansions of the numbers $N = 1, 2, \dots, 17$. Then it does not occur in any β-expansion.*
(b) *Let D be an integer between 1 and 4. Suppose the digit block $d_3\cdots d_0 \cdot d_{-1}\cdots d_{-4}$ occurs in the β-expansion of N if and only if the digit block $e_3\cdots e_0 \cdot e_{-1}\cdots e_{-4}$ occurs in the β-expansion of the number $N-D$, for $N = D, D+1, \dots, D+17$. Then this paired occurrence holds for all N.*

Proof. (a) Let us say that a Lucas interval Λ_m satisfies property $\mathcal{D}$ if the digit block $d_3\cdots d_0 \cdot d_{-1}\cdots d_{-4}$ does not occur in the β-expansions of the numbers N from Λ_m. Note that $N = 17$ is the last number in Λ_5, so it is given that the intervals $\Lambda_1, \dots, \Lambda_5$ all satisfy property $\mathcal{D}$. The interval Λ_6 also satisfies property $\mathcal{D}$, by an application of Theorem 2 (B).

The interval $\Lambda_7 = \Lambda_5^{(a)} \cup \Lambda_4^{(b)} \cup \Lambda_5^{(c)}$ satisfies property $\mathcal{D}$. For $\Lambda_5^{(a)}$, this follows since Λ_5 satisfies property $\mathcal{D}$, and the first case in Equation (5) does not change the central block of length 8. The same argument applies to $\Lambda_5^{(c)}$. For the interval $\Lambda_4^{(b)}$, the second case in Equation (5) gives that the positive digit blocks $d_3\cdots d_0$ are the same as for the corresponding numbers in Λ_4, and that the negative digit blocks are $d_{-1}\cdots d_{-4}(7)(01)^{-1}00 = 0000$ and $d_{-1}\cdots d_{-4}(9)(01)^{-1}00 = 0100$, which already occurred in the expansions $\beta(0)$ and $\beta(3)$.

The interval $\Lambda_8 = \Lambda_6^{(a)} \cup \Lambda_5^{(b)} \cup \Lambda_6^{(c)}$ also satisfies property $\mathcal{D}$, since the word transformations in Equation (6) do not change the central blocks of length 8 in

Λ_6, nor in Λ_5. Another way to put this, is that $\Lambda_8 \cong \Lambda_6\Lambda_5\Lambda_6 \mod 4$. Since the β-expansions only get longer, we have in fact that $\Lambda_m \cong \Lambda_{m-2}\Lambda_{m-3}\Lambda_{m-2} \mod 4$ for all $m \geq 8$. Thus it follows by induction that Λ_m satisfies property $\mathcal{D}$ for all $m \geq 8$.

(b) Let us say that a Lucas interval Λ_m, $m \geq 1$ satisfies property $\mathcal{E}$ if the numbers N from Λ_m have the property that the digit block $d_3 \cdots d_0 \cdot d_{-1} \cdots d_{-4}$ occurs in the β-expansion of N if and only if the digit block $e_3 \cdots e_0 \cdot e_{-1} \cdots e_{-4}$ occurs in the β-expansion of $N - D$. Then it is given that $\Lambda_1, \ldots, \Lambda_5$ all satisfy property $\mathcal{E}$. The proof continues as in part (a), but we have to take into account that the numbers $N - D$ and N can be elements of different Lucas intervals. This 'boundary' problem is easily solved by induction: it is given for $\Lambda_4\Lambda_5$ and $\Lambda_5\Lambda_6$, and the equation used for induction is

$$\Lambda_{m+1}\Lambda_{m+2} \cong \Lambda_{m-1}\Lambda_{m-2}\Lambda_{m-1}\Lambda_m\Lambda_{m-1}\Lambda_m \mod 4.$$

This equation is an instance of Equation (7). □

4. A Closer Look at the Lucas Intervals

Here we say more on the idea of splitting Lucas intervals in unions of translated Lucas intervals. To keep the presentation simple, we start with showing how all the natural numbers can be split into translations of the three Lucas intervals Λ_3, Λ_4 and Λ_5. This can of course be done in many ways, but we consider a way derived from the Recursive Structure Theorem (Theorem 3). One has

$$\begin{aligned}
\Lambda_6 &= \Lambda_4^{(a)} \cup \Lambda_3^{(b)} \cup \Lambda_4^{(c)} = [\Lambda_4 + L_5] \cup [\Lambda_3 + L_6] \cup [\Lambda_4 + L_6],\\
\Lambda_7 &= \Lambda_5^{(a)} \cup \Lambda_4^{(b)} \cup \Lambda_5^{(c)} = [\Lambda_5 + L_5] \cup [\Lambda_4 + L_7] \cup [\Lambda_5 + L_7],\\
\Lambda_8 &= \Lambda_6^{(a)} \cup \Lambda_5^{(b)} \cup \Lambda_6^{(c)} = [\Lambda_6 + L_7] \cup [\Lambda_5 + L_8] \cup [\Lambda_6 + L_8]\\
&= [\Lambda_4 + L_5 + L_7] \cup [\Lambda_3 + L_6 + L_7] \cup [\Lambda_4 + L_6 + L_7] \cup [\Lambda_5 + L_8]\\
&\quad \cup [\Lambda_4 + L_5 + L_8] \cup [\Lambda_3 + L_6 + L_8] \cup [\Lambda_4 + L_6 + L_8].
\end{aligned}$$

Note how the splitting of Λ_6 was used in the splitting of Λ_8. Continuing in this fashion, we inductively obtain a splitting of all Lucas intervals Λ_n, which we call the *canonical* splitting.

What is the sequence of translated intervals Λ_3, Λ_4 and Λ_5 created in this way?

Let the word $C(\Lambda_n)$ code these successive intervals in Λ_n by their indices 3, 4 or 5. Let κ be the morphism on the monoid $\{3, 4, 5\}^*$ defined by

$$\kappa(3) = 5, \quad \kappa(4) = 434 \quad \kappa(5) = 545.$$

Theorem 4. *For all $n \geq 3$ the interval Λ_n is a union of adjacent translations of the three intervals Λ_3, Λ_4 and Λ_5. If $C(\cdot)$ is the coding function for the canonical* *splittings then for* $n \geq 0$

$$C(\Lambda_{2n+4}) = \kappa^n(4), \quad C(\Lambda_{2n+5}) = \kappa^n(5).$$

Proof. For $n = 0$ this is trivially true. We continue by induction. Suppose it is true for $k = 1, \ldots, n$. Then, by Theorem 3,

$$\begin{aligned}
C(\Lambda_{2n+6}) &= C(\Lambda_{2n+4})C(\Lambda_{2n+3})C(\Lambda_{2n+4}) = \kappa^n(4)\kappa^{n-1}(5)\kappa^n(4) \\
&= \kappa^{n-1}(\kappa(4)5\kappa(4)) = \kappa^{n-1}(4345434) = \kappa^{n-1}(\kappa^2(4)) = \kappa^{n+1}(4), \\
C(\Lambda_{2n+7}) &= C(\Lambda_{2n+5})C(\Lambda_{2n+4})C(\Lambda_{2n+5}) = \\
&= \kappa^n(5)\kappa^n(4)\kappa^n(5) = \kappa^n(545) = \kappa^{n+1}(5).
\end{aligned}$$

□

We continue this analysis, now focussing on the partition of the natural numbers by the intervals

$$\Xi_n := \Lambda_{2n-1} \cup \Lambda_{2n} = [L_{2n-1} + 1, L_{2n+1}].$$

The relevance of the Ξ_n is that these are exactly the intervals where $\beta^-(N)$ has length $2n$, for $n \geq 1$. The results in the sequel of this section will therefore be useful in Section 7.

There are three (Sturmian) morphisms f, g and h that play an important role in these results, where it is convenient to look at a and b both as integers and as abstract letters. The morphisms are given by

$$f : \begin{cases} a \to aba \\ b \to ab \end{cases}, \qquad g : \begin{cases} a \to baa \\ b \to ba \end{cases}, \qquad h : \begin{cases} a \to aab \\ b \to ab \end{cases}. \tag{8}$$

Theorem 5. *For all* $n \geq 2$ *the interval* Ξ_n *is a union of adjacent translations of the three intervals* Λ_3, Λ_4 *and* Λ_5*. If* $C(\cdot)$ *is the coding function for the canonical splittings, then for* $n \geq 0$

$$C(\Xi_{n+2}) = \delta(h^n(b)),$$

where δ *is the decoration morphism given by* $\delta(a) = 54$, $\delta(b) = 34$.

Proof. We first establish the commutation relation $\kappa\,\delta = \delta\,h$. It suffices to prove this for the generators, and indeed:

$$\begin{aligned}
\kappa(\delta(a)) &= \kappa(54) = 545434 = \delta(aab) = \delta(h(a)), \\
\kappa(\delta(b)) &= \kappa(34) = 5434 = \delta(ab) = \delta(h(b)).
\end{aligned}$$

Using Theorem 4 and the commutation relation, for $n \geq 1$ we obtain

$$\begin{aligned}
C(\Xi_{n+2}) &= C(\Lambda_{2n+3})C(\Lambda_{2n+4}) = \\
&= \kappa^{n-1}(5)\kappa^n(4) = \kappa^{n-1}(5434) = \kappa^{n-1}(\delta(ab)) = \delta(h^{n-1}(ab)) = \delta(h^n(b)).
\end{aligned}$$

For $n = 0$ we have $\Xi_2 = \Lambda_3 \cup \Lambda_4$, so $C(\Xi_2) = 34 = \delta(b)$. □

5. Generalized Beatty Sequences

Let α be an irrational number larger than 1. We call a sequence V with terms of the form $V_n = p\lfloor n\alpha\rfloor + qn + r$, $n \geq 1$ a *generalized Beatty sequence.* Here p, q and r are integers, called the *parameters* of V, and we write $V = V(p, q, r)$.

In this paper we only consider the case $\alpha = \varphi$, the golden mean, so any mention of a generalized Beatty sequence assumes that $\alpha = \varphi$. A prominent role is played by the lower Wythoff sequence $A := V(1, 0, 0)$ and the upper Wythoff sequence $B := V(1, 1, 0)$. These are complementary sequences, associated with the Beatty pair (φ, φ^2).

Here is the key lemma that tells us how generalized Beatty sequences behave under compositions. In its statement below, as Lemma 3, a typographical error in its source is corrected.

Lemma 3 ([1, Corollary 2])**.** *Let V be a generalized Beatty sequence with parameters (p, q, r). Then VA and VB are generalized Beatty sequences with parameters $(p_{VA}, q_{VA}, r_{VA}) = (p + q, p, r - p)$ and $(p_{VB}, q_{VB}, r_{VB}) = (2p + q, p + q, r)$.*

It will be useful later on to have a sort of converse of this lemma. If C and D are two $\mathbb{N}$-valued sequences, then we denote by $C \sqcup D$ the sequence whose terms give the set $C(\mathbb{N}) \cup D(\mathbb{N})$, in increasing order.

Lemma 4. *Let $V = V(p, q, r)$ be a generalized Beatty sequence. Let U and W be two disjoint sequences with union $V = U \sqcup W$:*

$$U(\mathbb{N}) \cap W(\mathbb{N}) = \emptyset, \quad U(\mathbb{N}) \cup W(\mathbb{N}) = V(\mathbb{N}).$$

Suppose U is a generalized Beatty sequence with parameters $(p + q, p, r - p)$. Then W is the generalized Beatty sequence with parameters $(2p + q, p + q, r)$.

Proof. According to Lemma 3, we have $U = VA$. Since A and B are disjoint with union $\mathbb{N}$, we must have $W = VB$, and Lemma 3 gives that W is a generalized Beatty sequence with parameters $(2p + q, p + q, r)$. □

Here is the key lemma to 'recognize' a generalized Beatty sequence, taken from [1]. If S is a sequence, we denote its sequence of first order differences as ΔS, i.e., ΔS is defined by

$$\Delta S(n) = S(n + 1) - S(n), \quad \text{for } n = 1, 2 \ldots.$$

Lemma 5 ([1])**.** *Let $V = (V_n)_{n\geq 1}$ be the generalized Beatty sequence defined by $V_n = p\lfloor n\varphi\rfloor + qn + r$, and let ΔV be the sequence of its first differences. Then ΔV is the Fibonacci word over the alphabet $\{2p + q, p + q\}$. Conversely, if $x_{a,b}$ is the Fibonacci word over the alphabet $\{a, b\}$, then every V with $\Delta V = x_{a,b}$ is a generalized Beatty sequence $V = V(a - b, 2b - a, r)$ for some integer r.*

6. The Positive Powers of the Golden Mean

For every digit block w we will determine the sequence R_w of those numbers N with digit block $w = d_{m-1}\cdots d_0$ as the suffix of $\beta^+(N)$. More generally, we are also interested in occurrence sequences of numbers N with $d_{m-1}\cdots d_0(N) = w$ and $d_{-1}\cdots d_{-m'}(N) = v$. We denote these as $R_{w\cdot v}$.

For a couple of small values of m, m', we have the following result from the paper [5, Theorem 5.1].

Theorem 6 ([5])**.** *Let $\beta(N) = (d_i(N))$ be the base phi expansion of a natural number N. Then:*

$$R_1 = V_0(1,2,1), \quad R_{10} = V(1,2,-1), \quad R_{00\cdot 0} = V_0(1,2,0), \quad R_{00\cdot 1} = V(3,1,1).$$

Here it made sense to add $N = 1$ to $V(1,2,1)$, and $N = 0$ to $R_{00\cdot 0}$. We accomplished this by adding the $n = 0$ term to the generalized Beatty sequence V: we define V_0 by

$$V_0(p,q,r) := (p\lfloor n\varphi\rfloor + qn + r)_{n\geq 0}.$$

The digit blocks $w = d_{m-1}\cdots d_1\,0$ behave rather differently from digit blocks $w = d_{m-1}\cdots d_1\,1$. We therefore analyse these cases separately, in Section 6.1 and Section 6.2.

6.1. Digit Blocks $w = d_{m-1}\cdots d_1 0$

We order the digit blocks w with $d_0 = 0$ in a Fibonacci tree. The first four levels of this tree are depicted in Figure 1. The first line gives w, the second line R_w.

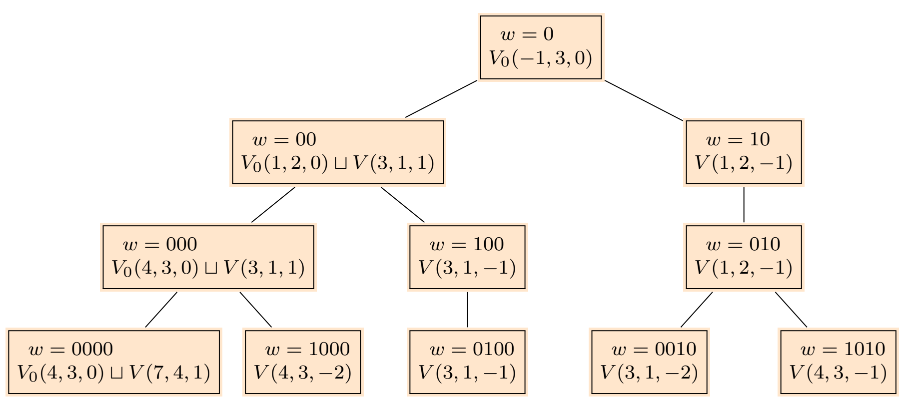

Figure 1: Tree of digit blocks $w = d_{m-1}\cdots d_1 0$

We start with the short words w.

Proposition 2. *The sequence of occurrences R_w of numbers N such that the digits $d_{m-1}\cdots d_0$ of the base phi expansion of N are equal to w, i.e., $d_{m-1}\cdots d_0(N) = w$, is given for the words w of length at most 3, and ending in 0, by*
(a) $R_0 = V(-1,3,0)$,
(b) $R_{00} = V_0(1,2,0) \sqcup V(3,1,1)$,
(c) $R_{10} = R_{010} = V(1,2,-1)$,
(d) $R_{000} = V_0(4,3,0) \sqcup V(3,1,1)$,
(e) $R_{100} = V(3,1,-1)$.

Proof. (a) $w = 0$: Since the numbers $\varphi + 2$ and $3 - \varphi$ form a Beatty pair, i.e.,

$$\frac{1}{\varphi+2} + \frac{1}{3-\varphi} = 1,$$

the sequences $V(1,2,0)$ and $V(-1,3,0)$ are complementary in the positive integers. It follows that $R_0 = V_0(-1,3,0)$ is the complement of $R_1 = V_0(1,2,1)$, by Theorem 6.

(b) $w = 00$: Theorem 6 gives that R_{00} is the union of the two GBS $V_0(1,2,0)$ and $V(3,1,1)$. These two sequences correspond to the numbers N with expansions containing $00\cdot 0$, coded B in [5], and those containing $00\cdot 1$, coded D in [5].

(c) $w = 10$ and $w = 010$: From Theorem 6 we obtain that R_{10} is equal to $V(1,2,-1)$.

(d) $w = 000$: By Lemma 1 there are no base phi expansions with $d_2d_1d_0d_{-1}(N) = 100\cdot 1$. This means that the numbers N from $V(3,1,1)$ in the last part of Theorem 6 do exactly correspond with the numbers N with $d_2d_1d_0d_{-1}(N) = 000\cdot 1$. This gives one part of the numbers N where $\beta^+(N)$ has the suffix 000.

The other part comes from the occurrences of N with $d_2d_1d_0d_{-1}(N) = 000\cdot 0$. The trick is to observe that the digit blocks 1010 and $000\cdot 0$ always occur in pairs of the expansions of $N-1$ and N, for $N = 7,\dots 18$. The Propagation Principle (Lemma 2 (b)) gives that this pairing holds for all positive integers N. From Theorem 7 we know that the digit block 1010 has occurrence sequence $R_{1010} = V(4,3,-1)$. So the pairing implies that the digit block $000\cdot 0$ has occurrence sequence $V_0(4,3,0)$. Here we should mention that Theorem 7 uses the proposition we are on the way of proving (via the formula $R_{1010} = R_{010}\,\mathrm{B}$), however, this only uses part (c), which we already proved above.

(e) $w = 100$: We already know that expansions with $100\cdot 1$ do not occur, and one checks that an expansion $\beta(N-2) = \cdots 100\cdot 0\cdots$ always occurs paired to an expansion $\beta(N) = \cdots 00\cdot 1\cdots$, for $N = 2,\dots,19$. The Propagation Principle (Lemma 2 (b)) then implies that this pairing occurs for all N. This gives that $R_{100} = R_{00\cdot 1} - 2 = V(3,1,-1)$, using the result of part (b). □

The sequences R_{010} and R_{100} are examples of what we call Lucas-Wythoff sequences: their parameters are given by $(L_1, L_0, -1)$ and $(L_2, L_1, -1)$, respectively.

In general, a *Lucas-Wythoff sequence* G is a generalized Beatty sequence defined for a natural number m by

$$G = V(L_{m+1}, L_m, r),$$

where r is an integer.

Theorem 7. *Fix a word $w = w_{m-1}\cdots w_0$ of 0's and 1's, containing no 11, where $m \geq 2$. Let $w_0 = 0$. Then—except if $w = 0^m$—the sequence R_w of occurrences of numbers N such that the digits $d_{m-1}\cdots d_0$ of the base phi expansion of N are equal to w, i.e., $d_{m-1}\cdots d_0(N) = w$, is a Lucas-Wythoff sequence of the form*

$$R_w = \begin{cases} V(L_{m-2}, L_{m-3}, \gamma_w), & \text{if } w_{m-1} = 0 \\ V(L_{m-1}, L_{m-2}, \gamma_w), & \text{if } w_{m-1} = 1. \end{cases}$$

where γ_w is a negative integer or 0.

If w consists entirely of 0's, this sequence of occurrences is given by a disjoint union of two Lucas-Wythoff sequences. We have

$$R_{0^{2m}} = V(L_{2m}, L_{2m-1}, 1) \sqcup V_0(L_{2m-1}, L_{2m-2}, 0),$$
$$R_{0^{2m+1}} = V_0(L_{2m+1}, L_{2m}, 0) \sqcup V(L_{2m}, L_{2m-1}, 1).$$

Proof. Suppose first that w is a word *not* equal to 0^m for some $m \geq 2$.

The proof is by induction on the length m of w. For $m = 2$ the statement of the theorem holds by Proposition 2 (c). Next, let w be a word of length m with $w_0 = 0$.

In the case that $w_{m-1} = 1$, w has a unique extension to $0w$, and $R_{0w} = R_w$ is equal to the correct Lucas-Wythoff sequence.

In the case that $w_{m-1} = 0$, the induction hypothesis is that R_w is a Lucas-Wythoff sequence $R_w = V(L_{m-2}, L_{m-3}, \gamma_w)$. By Theorem 1, the numbers N with a $\beta^+(N)$ ending with the digit block w are in one-to-one correspondence with numbers N' with a $Z(N')$ ending with the digit block w, and the same property holds for the digit blocks $0w$, and $1w$, respectively. Note that the correspondence is one-to-one, since the numbers 'skipped' in the Zeckendorf expansions all[1] have $d_0 = 1$. It therefore follows from Proposition 2.6 in [6] that

$$R_{0w} = R_w A \quad \text{and} \quad R_{1w} = R_w B.$$

By Lemma 3, $R_w A$ has parameters

$$(L_{m-2} + L_{m-3}, L_{m-2}, \gamma_w - L_{m-2}) = (L_{m-1}, L_{m-2}, \gamma_w - L_{m-2}),$$

and $R_w B$ has parameters

$$(2L_{m-2} + L_{m-1}, L_{m-2} + L_{m-3}, \gamma_w) = (L_m, L_{m-1}, \gamma_w).$$

[1]We have to follow a different strategy for the words $w = d_{m-1}\cdots d_1 1$ in the next section.

These are indeed the right expressions for the two words $0w$ and $1w$ of length $m+1$.

The words $w = 0^m$ with $m \geq 2$ are next. For all $m \geq 1$ we claim that

$$R_{0^{2m}\cdot 0} = V_0(L_{2m-1}, L_{2m-2}, 0), \qquad R_{0^{2m}\cdot 1} = V(L_{2m}, L_{2m-1}, 1), \tag{9}$$

$$R_{0^{2m+1}\cdot 0} = V_0(L_{2m+1}, L_{2m}, 0), \qquad R_{0^{2m+1}\cdot 1} = V(L_{2m}, L_{2m-1}, 1). \tag{10}$$

The proof is by induction. In the proof of Proposition 2 (b), we find that $R_{00\cdot 0} = V_0(1, 2, 0)$ and $R_{00\cdot 1} = V(3, 1, 1)$. Since $L_0 = 2, L_1 = 1$ and $L_2 = 3$, this is Equation (9) for $m = 1$. We find in the proof of Proposition 2 (d), that $R_{000\cdot 0} = V_0(4, 3, 0)$ and $R_{000\cdot 1} = V(3, 1, 1)$. This is Equation (10) for $m = 1$.

Suppose now that both Equation (9) and Equation (10) hold.

First, we do the induction step for Equation (9). Since $10^{2m+1} \cdot 0$ never occurs by Lemma 1, we must have

$$R_{0^{2m+2}\cdot 0} = R_{0^{2m+1}\cdot 0} = V_0(L_{2m+1}, L_{2m}, 0). \tag{11}$$

This is the left part of Equation (9) for $m + 1$ instead of m.

The fact that $10^{2m+1} \cdot 0$ never occurs also implies that

$$R_{10^{2m+1}\cdot 1} = R_{10^{2m+1}} = V(L_{2m+1}, L_{2m}, \gamma_{10^{2m+1}}) = V(L_{2m+1}, L_{2m}, -L_{2m}+1). \tag{12}$$

Here we used the first part of the proof, determining $\gamma_{10^{2m+1}}$ from the observation that the first occurrence of $d_{2m+1} \cdots d_0(N) = 10^{2m+1}$ is at $N = L_{2m+1} + 1$, the first element of the Lucas interval Λ_{2m+1}.

Next, we take $V = R_{0^{2m+1}\cdot 1}$, $U = R_{10^{2m+1}\cdot 1}$ and $W = R_{0^{2m+2}\cdot 1}$ in Lemma 4. According to Equation (10), we take $(p, q, r) = (L_{2m}, L_{2m-1}, 1)$. The parameters of the sequence U should be $(p+q, p, r-p) = (L_{2m+1}, L_{2m}, 1 - L_{2m})$, which conforms with Equation (12).

The conclusion of Lemma 4 is that $W = R_{0^{2m+2}\cdot 1}$ has parameters $(2p + q, p + q, r) = (2L_{2m} + L_{2m-1}, L_{2m} + L_{2m-1}, 1) = (L_{2m+2}, L_{2m+1}, 1)$. Therefore,

$$R_{0^{2m+2}\cdot 1} = V(L_{2m+2}, L_{2m+1}, 1). \tag{13}$$

This is the right part of Equation (9) for $m + 1$ instead of m.

Next, we do the induction step for Equation (10). Since $10^{2m+2} \cdot 1$ never occurs by Lemma 1, we must have, using Equation (13),

$$R_{0^{2m+3}\cdot 1} = R_{0^{2m+2}\cdot 1} = V(L_{2m+2}, L_{2m+1}, 1).$$

This is the right part of Equation (10) for $m + 1$ instead of m.

The fact that $10^{2m+2} \cdot 1$ never occurs also implies that

$$R_{10^{2m+2}\cdot 0} = R_{10^{2m+2}} = V(L_{2m+2}, L_{2m+1}, -L_{2m+1}). \tag{14}$$

Here we used the first part of the proof, determining $\gamma_{10^{2m+2}}$ from the observation that the first occurrence of $d_{2m+2}\cdots d_0(N) = 10^{2m+2}$ is at $N = L_{2m+2}$, the first element of the Lucas interval Λ_{2m+2}.

Next, we take $V = R_{0^{2m+2}.0}$, $U = R_{10^{2m+2}.0}$ and $W = R_{0^{2m+3}.0}$ in Lemma 4. According to Equation (11), we take $(p, q, r) = (L_{2m+1}, L_{2m}, 0)$. The parameters of the sequence U should be $(p + q, p, r - p) = (L_{2m+2}, L_{2m+1}, -L_{2m+1})$, which conforms with Equation (14).

The conclusion of Lemma 4 is that $W = R_{0^{2m+3}.0}$ has parameters $(2p + q, p + q, r) = (2L_{2m+1} + L_{2m}, L_{2m+1} + L_{2m}, 0) = (L_{2m+3}, L_{2m+2}, 0)$. Therefore,

$$R_{0^{2m+3}.0} = V(L_{2m+3}, L_{2m+2}, 0). \tag{15}$$

This is the left part of Equation (10) for $m + 1$ instead of m. $\square$

6.2. Digit Blocks $w = d_{m-1}\cdots d_1 1$

Here there are digit blocks that do not occur at all, like $w = 1001$. We denote this as $R_{1001} = \emptyset$. We order the digit blocks w with $d_0 = 1$ in a tree. The first four levels of this tree (taking into account that the node corresponding to R_{1001} has no offspring), are depicted in Figure 2.

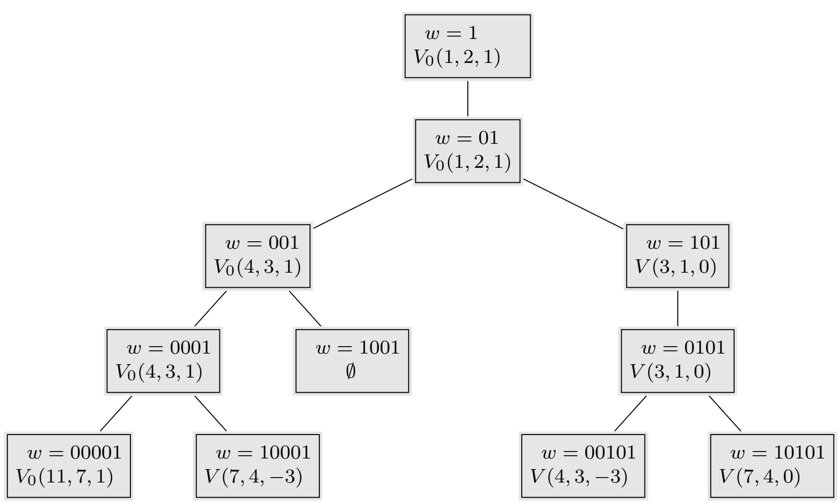

Figure 2: Tree of digit blocks $w = d_{m-1}\cdots d_1 1$

Here $R_{01} = R_1 = V_0(1, 2, 1)$ has been given in Theorem 6. The correctness of the other occurrence sequences follows from Theorem 8.

We next determine an infinite family of excluded blocks.

Lemma 6. *Let $m \geq 2$ be an integer. Expansions $\beta^+(N) = \cdots d_2 d_1 d_0 = \cdots 10^{2m}1$ do not occur for any N.*

Proof. Consider an N such that $\beta^+(N) = \cdots 10^{2m}1$. Such an N, of course, has $d_{-1}(N) = 0$, so we see that $\beta(N-1) = \cdots 10^{2m+1} \cdot 0 \cdots$. According to Lemma 1 this is not possible. $\square$

Next, we establish a connection with the previous section.

Lemma 7. *Let $m \geq 2$ be an integer. The digit block $w = d_{m-1} \cdots d_1 1 \cdot 0$ is a digit block of $\beta(N)$ if and only if the digit block $\breve{w} := d_{m-1} \cdots d_1 0 \cdot 0$ occurs in $\beta(N-1)$.*

Proof. This follows quickly from the Propagation Principle (Lemma 2) applied to the blocks $00 \cdot 0$ and $01 \cdot 0$. $\square$

Theorem 8. *Fix a word $w = w_{m-1} \cdots w_0$ of 0's and 1's, containing no 11, where $m \geq 3$. Let $w_0 = 1$. With exception of a family of words listed below, the sequence R_w of occurrences of numbers N such that the digits $d_{m-1} \cdots d_0$ of the base phi expansion of N are equal to w, i.e., $d_{m-1} \cdots d_0(N) = w$, is a Lucas-Wythoff sequence of the form*

$$R_w = \begin{cases} V(L_{m-2}, L_{m-3}, \gamma_w),\ \gamma_w \in \mathbb{Z} \setminus \mathbb{Z}^+ & \text{if } w_{m-1} = 0 \\ V(L_{m-1}, L_{m-2}, \gamma_w),\ \gamma_w \in \mathbb{Z} \setminus \mathbb{Z}^+ & \text{if } w_{m-1} = 1. \end{cases}$$

For $m \geq 1$ the following words are exceptional. In case $w = 0^{2m}1$ we have $R_w = V_0(L_{2m+1}, L_{2m}, 1)$, and this is also the sequence of occurrences of $w = 0^{2m+1}1$. In case $w = 10^{2m}1$ the word w does not occur at all as $\cdots d_2 d_1 d_0 = \cdots 10^{2m}1$. In case $w = 10^{2m+1}1$ we have $R_w = V(L_{2m+2}, L_{2m+1}, -L_{2m+1}+1)$.

Proof. It follows from Lemma 7 that $R_w = R_{\breve{w}} + 1$, if $R_w \neq \emptyset$. So the first part of Theorem 7 yields the statement of the theorem for all w not equal to 0^m1 or 10^m1.

In case $w = 0^{2m}1 \cdot 0$, we have $\breve{w} = 0^{2m+1} \cdot 0$, and the result follows from the left part of Equation (10).

In case $w = 10^{2m}1$ the word w does not occur at all as $\cdots d_2 d_1 d_0 = \cdots 10^{2m}1$, according to Lemma 6.

In case $w = 10^{2m+1}1 \cdot 0$ we have $\breve{w} = 10^{2m+2} \cdot 0$, and now Equation (14) gives that $R_w = R_{\breve{w}} + 1 = V(L_{2m+2}, L_{2m+1}, -L_{2m+1}+1)$. $\square$

7. The Negative Powers of the Golden Mean

Here we discuss what we can say about the words $\beta^-(N)$. These have an even more intricate structure than the $\beta^+(N)$.

7.1. The Words $\beta^-(N)$

We start by describing complete $\beta^-(N)$'s. Although at first sight these seem to appear in a random order, there is an order, dictated not by a coin toss, but by another dynamical system: the rotation over an angle φ. Moreover, they appear in singletons, or triples. This can be proved with the {ABC, D}–structure found in the paper [5]. For a more extensive analysis, partition the natural numbers larger than 1 into intervals

$$\Xi_n = \Lambda_{2n-1} \cup \Lambda_{2n} = [L_{2n-1}+1, L_{2n+1}].$$

The $\Xi_n, n = 1, 2, \dots$, introduced in Section 4, are exactly the intervals where $\beta^-(N)$ has length $2n$. The Ξ_n intervals have length

$$L_{2n+1} - L_{2n-1} = L_{2n+1} - L_{2n} + L_{2n} - L_{2n-1} = L_{2n-1} + L_{2n-2} = L_{2n}.$$

Call three consecutive numbers $N, N+1, N+2$ a *trident* if $\beta^-(N) = \beta^-(N+1) = \beta^-(N+2)$. For example: 2,3,4 and 6,7,8 are tridents. We shall always take the middle number N+1 as the representing number of a trident interval $[N, N{+}1, N{+}2]$. We call this number Π-*essential*. By definition the other Π-essential numbers are the singletons.

Lemma 8 (Trident Splitting). *In $\Lambda_{2n-1} \cup \Lambda_{2n}$ the last number of Λ_{2n-1} and the first two numbers in Λ_{2n} are in the same trident.*

Proof. This is true for $n = 1$ and $n = 2$: $\Lambda_1 \cup \Lambda_2 = \{2\} \cup [3, 4]$ is a trident, and $\Lambda_3 \cup \Lambda_4 = [5, 6] \cup [7, 8, \dots, 11]$ contains the trident $[6, 7, 8]$. The property then follows by induction, using Theorem 3. $\square$

The following lemma helps to count singletons and tridents.

Lemma 9. *The following relation between Lucas numbers and Fibonacci numbers holds: $F_n + 3F_{n+1} = L_{n+2}$ for $n = 0, 1, 2, \dots$.*

For a proof, note that $F_0{+}3F_1 = 3 = L_2$, and $F_2{+}3F_3 = 1{+}6 = L_4$, and then add these two equations, etc. The lemma describes the fact that the Ξ_n intervals contain F_{2n-2} singletons, and F_{2n-1} tridents, making a total number of L_{2n}. The collection of different $\beta^-(N)$-blocks of length $2n$ thus has cardinality $F_{2n-2} + F_{2n-1} = F_{2n}$. This implies that we have proved the following theorem.

Theorem 9. *All Zeckendorf words of even length ending in 1 appear as $\beta^-(N)$-blocks.*

Here we mean by a Zeckendorf word (or golden mean word), all words in which 11 does not occur. We denote by $\mathcal{Z}_m$ the set of Zeckendorf words of length m, for $m = 1, 2, \dots$. It is easily proved that the cardinality of $\mathcal{Z}_m$ equals F_{m+2}. So the cardinality of the set of words from $\mathcal{Z}_{2n}$ ending in 1 is equal to F_{2n}, implying the result of Theorem 9.

Since all $\beta^-(N)$ have the suffix 01, the essential information of these words is contained in

$$\gamma^-(N) := \beta^-(N)1^{-1}0^{-1}.$$

The words $\gamma^-(N)$ are Zeckendorf words, corresponding one-to-one to the natural numbers $Z^{-1}(\gamma^-(N))$. Obviously, the $\gamma^-(N)$ have the same ordering as the $\beta^-(N)$. According to Theorem 9 we then (after identifying tridents with their middle number) obtain a permutation of length F_{2n} of the Π-essential elements of Ξ_n by coding these numbers by $\mathrm{C}(N) := Z^{-1}(\gamma^-(N))$. We denote this permutation by Π^{β}_{2n}. The following Zeckendorf words and codes will be important in the sequel.

Lemma 10. *For all natural numbers n we have*

$$\gamma^-(L_{2n}) = 0^{2n-2},\ \gamma^-(L_{2n+1}) = [01]^{n-1}, \tag{16}$$

$$\gamma^-(L_{2n+1}+1) = [10]^{n},\ \gamma^-(L_{2n+2}-1) = 0^{2n}, \tag{17}$$

$$\mathrm{C}(L_{2n}) = 0,\ \mathrm{C}(L_{2n+1}) = F_{2n-1}-1, \tag{18}$$

$$\mathrm{C}(L_{2n+1}+1) = F_{2n+2}-1,\ \mathrm{C}(L_{2n+2}-1) = 0. \tag{19}$$

Proof. The correctness of Equations (16) and (17) follows from Equations (2) and (3). So $\gamma^-(L_{2n})$ is the first word in $\mathcal{Z}_{2n-2}$, $\gamma^-(L_{2n+1})$ is 0 followed by the last word in $\mathcal{Z}_{2n-3}$, $\gamma^-(L_{2n+1}+1)$ is the last word in $\mathcal{Z}_{2n}$, and $\gamma^-(L_{2n+2}-1)$ is the first word in $\mathcal{Z}_{2n-2}$. Since $\mathcal{Z}_m$ has cardinality F_{m+2}, Equations (18) and (19) follow. $\square$

We have to determine the codings of all natural numbers N. For this, it is useful to translate Theorem 3 to the γ^--blocks.

Theorem 10 (Recursive Structure Theorem: γ^--version)**.** *Let the odd and even Lucas intervals be given by*

$$\Lambda_{2n+1} = [L_{2n+1}+1,\, L_{2n+2}-1],\ \Lambda_{2n+2} = [L_{2n+2},\, L_{2n+3}].$$

(A) *For all $n \geq 1$ one has $\Lambda_{2n+1} = \Lambda^{(a)}_{2n-1} \cup \Lambda^{(b)}_{2n-2} \cup \Lambda^{(c)}_{2n-1}$, where $\Lambda^{(a)}_{2n-1} = \Lambda_{2n-1}+L_{2n}$, $\Lambda^{(b)}_{2n-2} = \Lambda_{2n-2}+L_{2n+1}$, and $\Lambda^{(c)}_{2n-1} = \Lambda_{2n-1}+L_{2n+1}$. We have*

$$\gamma^-(N) = \begin{cases} \gamma^-(N-L_{2n})\,10, & \text{if } N \in \Lambda^{(a)}_{2n-1},\\ \gamma^-(N-L_{2n+1})\,0010, & \text{if } N \in \Lambda^{(b)}_{2n-2},\\ \gamma^-(N-L_{2n+1})\,00, & \text{if } N \in \Lambda^{(c)}_{2n-1}. \end{cases} \tag{20}$$

(B) *For all $n \geq 1$ one has $\Lambda_{2n+2} = \Lambda^{(a)}_{2n} \cup \Lambda^{(b)}_{2n-1} \cup \Lambda^{(c)}_{2n}$, where $\Lambda^{(a)}_{2n} = \Lambda_{2n}+L_{2n+1}$, $\Lambda^{(b)}_{2n-1} = \Lambda_{2n-1}+L_{2n+2}$, and $\Lambda^{(c)}_{2n} = \Lambda_{2n}+L_{2n+2}$. We have*

$$\gamma^-(N) = \begin{cases} \gamma^-(N-L_{2n+1})\,00, & \text{if } N \in \Lambda^{(a)}_{2n},\\ \gamma^-(N-L_{2n+2})\,01 & \text{if } N \in \Lambda^{(b)}_{2n-1},\\ \gamma^-(N-L_{2n+1})\,01 & \text{if } N \in \Lambda^{(c)}_{2n}. \end{cases} \tag{21}$$

Table 2 gives the situation for $n = 2$.

N	Λ-int.	$\cdot\beta^-(N)$	$\cdot\gamma^-(N)$	C(N)
5	Λ_3	·1001	·10	2
6	Λ_3	·0001	·00	0
7	Λ_4	·0001	·00	0
8	Λ_4	·0001	·00	0
9	Λ_4	·0101	·01	1
10	Λ_4	·0101	·01	1
11	Λ_4	·0101	·01	1

Table 2: The case $n = 2$, where $\Xi_2 = \Lambda_3 \cup \Lambda_4 = [5, 6, \ldots, 11]$.

We see that $\Pi_4^\beta = (2\,0\,1)$.

For $n = 3$, the situation is displayed in Table 3.

N	Λ-int.	$\cdot\beta^-(N)$	$\cdot\gamma^-(N)$	C(N)
12	Λ_5	·101001	·1010	7
13	Λ_5	·001001	·0010	2
14	Λ_5	·001001	·0010	2
15	Λ_5	·001001	·0010	2
16	Λ_5	·100001	·1000	5
17	Λ_5	·000001	·0000	0
18	Λ_6	·000001	·0000	0
19	Λ_6	·000001	·0000	0
20	Λ_6	·010001	·0100	3

N	Λ-int.	$\cdot\beta^-(N)$	$\cdot\gamma^-(N)$	C(N)
21	Λ_6	·010001	·0100	3
22	Λ_6	·010001	·0100	3
23	Λ_6	·100101	·1001	6
24	Λ_6	·000101	·0001	1
25	Λ_6	·000101	·0001	1
26	Λ_6	·000101	·0001	1
27	Λ_6	·010101	·0101	4
28	Λ_6	·010101	·0101	4
29	Λ_6	·010101	·0101	4

Table 3: The case $n = 3$, where $\Xi_3 = \Lambda_5 \cup \Lambda_6 = [12, 13, \ldots, 29]$.

We see that $\Pi_6^\beta = (7\,2\,5\,0\,3\,6\,1\,4)$.

What are these permutations?

Theorem 11. *For all natural numbers n, consider the F_{2n} Zeckendorf words of length $2n$ occurring as $\beta^-(N)$ in the β-expansions of the numbers in Ξ_n. Then these occur in an order given by a permutation Π_{2n}^β, which is the orbit of the element $F_{2n} - 1$ under addition by the element F_{2n-2} on the cyclic group $\mathbb{Z}/F_{2n}\mathbb{Z}$.*

Proof. We have to show for all n that

$$\Pi_{2n}^\beta(1) = F_{2n} - 1, \tag{22}$$

$$\Pi_{2n}^\beta(j+1) = \Pi_{2n}^\beta(j) + F_{2n-2} \mod F_{2n}, \text{ for } j = 1, \ldots, F_{2n} - 1. \tag{23}$$

It is easily checked that the cases $n = 2$ and $n = 3$ given above conform with this. For $n = 3$ one has $F_6 = 8$, $F_4 = 3$, and $\Pi_6^\beta(1) \equiv 7$, $\Pi_6^\beta(j+1) \equiv \Pi_6^\beta(j) + 3 \mod 8$ for $j = 1, \ldots 7$.

The claim in Equation (22) follows from Lemma 10 for all n: since the interval $\Xi_n = [L_{2n-1}+1, L_{2n+1}]$, we have $\Pi^{\beta}_{2n}(1) = F_{2n} - 1$ by Equation (19). The proof proceeds by induction, based on Theorem 10, the γ^--version of the Recursive Structure Theorem.

For Equation (23) with n replaced by $n+1$, we have to split the permutation Π^{β}_{2n+2} into six pieces, and then we have to glue the expressions together to obtain the full permutation on the set $\Xi_{n+1} = \Lambda_{2n+1} \cup \Lambda_{2n+2} = [L_{2n+1}+1, L_{2n+2}-1] \cup [L_{2n+2}, L_{2n+3}]$. According to the Recursive Structure Theorem

$$\Xi_{n+1} = \Lambda^{(a)}_{2n-1} \cup \Lambda^{(b)}_{2n-2} \cup \Lambda^{(c)}_{2n-1} \cup \Lambda^{(a)}_{2n} \cup \Lambda^{(b)}_{2n-1} \cup \Lambda^{(c)}_{2n}. \tag{24}$$

We start with the first interval, $\Lambda^{(a)}_{2n-1}$. From Theorem 10, for $N \in \Lambda^{(a)}_{2n-1}$, we have that

$$\gamma^-(N) = \gamma^-(N - L_{2n})\,10. \tag{25}$$

What does this imply for the codes?

Let $Z(\mathrm{C}(N-L_{2n})) = \gamma^-(N-L_{2n}) = d_{2n-3}\cdots d_0$, so $\mathrm{C}(N-L_{2n}) = \sum_{i=0}^{2n-3} d_i \ddot{F}_i$. Then Equation (25) leads to

$$\mathrm{C}(N) = \sum_{i=0}^{2n-3} d_i \ddot{F}_{i+2} + 1\cdot\ddot{F}_1 + 0\cdot\ddot{F}_0 = \sum_{i=0}^{2n-3} d_i \ddot{F}_{i+2} + 2.$$

This implies, in particular, that the differences between the codes of two consecutive Π-essential numbers within the interval Λ_{2n-1} have increased from $F_{2n-2} \mod F_{2n}$ to $F_{2n} \mod F_{2n+2}$ for the corresponding numbers in the interval $\Lambda^{(a)}_{2n-1}$. We pass to the second interval, $\Lambda^{(b)}_{2n-2}$. From Theorem 10 we have that for N from $\Lambda^{(b)}_{2n-2}$,

$$\gamma^-(N) = \gamma^-(N - L_{2n+1})\,0010. \tag{26}$$

What does this imply for the codes?

Let $Z(\mathrm{C}(N-L_{2n+1})) = \gamma^-(N-L_{2n+1}) = d_{2n-4}\cdots d_0$, so $\mathrm{C}(N-L_{2n+1}) = \sum_{i=0}^{2n-4} d_i \ddot{F}_i$. Then Equation (26) leads to

$$\mathrm{C}(N) = \sum_{i=0}^{2n-4} d_i \ddot{F}_{i+4} + 0\cdot\ddot{F}_3 + 0\cdot\ddot{F}_2 + 1\cdot\ddot{F}_1 + 0\cdot\ddot{F}_0 = \sum_{i=0}^{2n-4} d_i \ddot{F}_{i+4} + 2.$$

This implies that the differences between the codes of two consecutive numbers within the interval Λ_{2n-2} have increased from $F_{2n-4} \mod F_{2n-2}$ to $F_{2n} \mod F_{2n+2}$ for the corresponding numbers in the interval $\Lambda^{(b)}_{2n-2}$. Similar computations give that for the next 4 intervals $\Lambda^{(c)}_{2n-1}, \Lambda^{(a)}_{2n}, \Lambda^{(b)}_{2n-1}$, and $\Lambda^{(c)}_{2n}$ there always is an addition of $F_{2n} \mod F_{2n+2}$.

The remaining task is to check that the same holds on the five boundaries between the translated Λ-intervals. We number these boundaries with the roman numerals I, II, III, IV, V.

$\boxed{\text{III \& V:}}$ For the third boundary between the intervals $\Lambda^{(c)}_{2n-1}$ and $\Lambda^{(a)}_{2n}$, and the fifth boundary between the intervals $\Lambda^{(b)}_{2n-1}$ and $\Lambda^{(c)}_{2n}$, this follows from the Trident Splitting Lemma (Lemma 8). The reason is that if $[N, N+1, N+2]$ is the trident which is splitted, then the difference between $\mathrm{C}(N-1)$ and $\mathrm{C}(N)$ is equal to F_{2n} mod F_{2n+2}, as these two numbers are both from the first translated Λ-interval, and not from the same trident. But then the difference between the codes of the last Π-essential number $N-1$ in the first translated Λ-interval, and the first Π-essential number $N+1$ in the second translated Λ-interval is also equal to $F_{2n} \mod F_{2n+2}$.

$\boxed{\text{I:}}$ The last number in the first interval $\Lambda^{(a)}_{2n-1}$ is $2L_{2n}-1$ with associated γ^--block

$$\gamma^-(2L_{2n}-1) = \gamma^-(2L_{2n}-1-L_{2n})\,10 = \gamma^-(L_{2n}-1)\,10 = 0^{2n-1}\,10.$$

Here we used the first case of Equation (20) in the first step, and Equation (17) in the last step. It follows directly that $\mathrm{C}(2L_{2n}-1) = 2$.

The first number in the second interval $\Lambda^{(b)}_{2n-2}$ is $2L_{2n}$. From Equation (2) we have $\beta(2L_{2n}) \doteq 20^{2n} \cdot 0^{2n-1}2 \doteq 20^{2n} \cdot 0^{2n-1}1001$, so $\gamma^-(2L_{2n}) = 0^{2n-1}10$, giving $\mathrm{C}(2L_{2n}) = 2$. It is clear also that the second number $2L_{2n}+1$ in $\Lambda^{(b)}_{2n-2}$ has code $\mathrm{C}(2L_{2n}+1) = 2$. As in the previous case, this implies that the difference between the codes of the last Π-essential number in the first translated Λ-interval, and the first Π-essential number in the second translated Λ-interval is equal to $F_{2n} \mod F_{2n+2}$.

$\boxed{\text{II:}}$ The last number in the second interval $\Lambda^{(b)}_{2n-2}$ is the number $L_{2n-1}+L_{2n+1}$. According to the second case of Equation (20), the associated γ^--block is

$$\begin{aligned}\gamma^-(L_{2n-1}+L_{2n+1}) &= \gamma^-(L_{2n-1}+L_{2n+1}-L_{2n+1})\,0010\\ &= \gamma^-(L_{2n-1})\,0010 = [01]^{n-2}\,0010.\end{aligned}$$

But we know from Lemma 10 that $\gamma^-(L_{2n-1})\,0101 = [01]^n = \gamma^-(L_{2n+3})$.

By Lemma 10, we have that $\mathrm{C}(L_{2n+3}) = F_{2n+1}-1$. To obtain the code of $N = L_{2n-1}+L_{2n+1}$, we have to subtract the number $F_3+F_1 = 3$ with Zeckendorf expansion 0101, and add the number $F_2 = 2$ with Zeckendorf expansion 0010. This gives the code

$$\mathrm{C}(L_{2n-1}+L_{2n+1}) = F_{2n+1}-1-3+1 = F_{2n+1}-3.$$

The first number in the third interval $\Lambda^{(c)}_{2n-1}$ is the number $L_{2n-1}+L_{2n+1}+1$. According to the third case of Equation (20) the associated γ^--block is

$$\gamma^-(L_{2n-1}+L_{2n+1}+1) = \gamma^-(L_{2n-1}+L_{2n+1}+1-L_{2n+1})\,00 = \gamma^-(L_{2n-1}+1)\,00.$$

But we know from Lemma 10 that $\gamma^-(L_{2n-1}+1)10 = [10]^n = \gamma^-(L_{2n+1}+1)$.

By Lemma 10, we have that $\mathrm{C}(L_{2n+1}+1) = F_{2n+2}-1$. To obtain the code of $N = L_{2n-1}+L_{2n+1}+1$, we have to subtract the number $F_2 = 2$ with Zeckendorf expansion 10, from this code. This gives the code

$$\mathrm{C}(L_{2n-1} + L_{2n+1} + 1) = F_{2n+2} - 1 - 2 = F_{2n+2} - 3.$$

The conclusion is that $L_{2n-1} + L_{2n+1}$ and $N = L_{2n-1} + L_{2n+1} + 1$ are Π-essential, with difference in codes $F_{2n+2} - 3 - (F_{2n+1} - 3) = F_{2n}$.

$\boxed{\text{IV:}}$ The last number in the fourth interval $\Lambda^{(c)}_{2n}$ is the number $L_{2n+1} + L_{2n+1} = 2L_{2n+1}$. According to the third case of Equation (21) the associated γ^{-}-block is

$$\gamma^{-}(2L_{2n+1}) = \gamma^{-}(2L_{2n+1} - L_{2n+1})\,00 = \gamma^{-}(L_{2n+1})\,00 = [01]^{n-1}\,00.$$

But we know from Lemma 10 that $\gamma^{-}(L_{2n+1})\,01 = [01]^{n} = \gamma^{-}(L_{2n+3})$.

By Lemma 10, we have that $\mathrm{C}(L_{2n+3}) = F_{2n+1} - 1$. To obtain the code of $N = 2L_{2n+1}$, we have to subtract the number $F_1 = 1$ with Zeckendorf expansion 01. This gives the code

$$\mathrm{C}(2L_{2n+1}) = F_{2n+1}) - 1 - 1 = F_{2n+1} - 2.$$

The first number in the fifth interval $\Lambda^{(b)}_{2n-1}$ is the number $L_{2n-1} + 1 + L_{2n+2}$. According to the second case of Equation (20) the associated γ^{-}-block is

$$\gamma^{-}(L_{2n-1} + 1 + L_{2n+2}) = \gamma^{-}(L_{2n-1} + 1 + L_{2n+2} - L_{2n+2})\,01 = \gamma^{-}(L_{2n-1} + 1)\,01.$$

But we know from Lemma 10 that $\gamma^{-}(L_{2n-1} + 1)10 = [10]^{n} = \gamma^{-}(L_{2n+1} + 1)$.

By Lemma 10, we have that $\mathrm{C}(L_{2n+1} + 1) = F_{2n+2} - 1$. To obtain the code of $N = L_{2n-1} + 1 + L_{2n+2}$, we have to subtract the number $F_2 = 2$ with Zeckendorf expansion 10, and add the number $F_1 = 1$ with Zeckendorf expansion 01 to this code. This gives the code

$$\mathrm{C}(L_{2n-1} + L_{2n+1} + 1) = F_{2n+2} - 1 - 2 + 1 = F_{2n+2} - 2.$$

The conclusion is that $2L_{2n+1}$ and $L_{2n-1}+1+L_{2n+2}$ are Π-essential, with difference in codes $F_{2n+2} - 2 - (F_{2n+1} - 2) = F_{2n}$. $\square$

We now explain the connection with a rotation on a circle mentioned at the beginning of this section. Note that with this point of view all the cyclic groups of Theorem 11 are represented by a single object: the rotation on the circle.

Theorem 12. *For all natural numbers n, the permutations Π^{β}_{2n} are given by the order in which the first F_{2n} iterates of the rotation $z \to \exp(2\pi i(z - \varphi))$ occur on the circle.*

We sketch a proof of this result based on the paper [10]. In the literature one does not find the rotation $z \to \exp(2\pi i(z - \varphi))$, but several papers treat the rotation $z \to \exp(2\pi i(z + \tau))$, where τ is the algebraic conjugate of φ. Note that this rotation has exactly the same orbits as $z \to \exp(2\pi i(z+\varphi))$, and replacing φ by $-\varphi$ amounts to reversing the permutation. In the literature the origin is usually added to the orbit. For instance in [10], the N ordered iterates are given by the permutation $\big(u_1\, u_2\, \ldots\, u_N\big)$, which for *all* N gives a permutation starting trivially with $u_1 = 0$. Lemma 2.1 in [10] states that for $j = 1, \ldots, N$ one has $u_j = (j-1)u_2 \bmod N$. Next, Theorem 3.3 in [10] states that $u_2 = u_2(N) = F_{2n-1}$ in the case that $N = F_{2n}$, $n \geq 1$. We illustrate this for the case $n = 3$. We have $N = F_6 = 8$, and $0 < \{5\tau\} < \{2\tau\} < \{7\tau\} < \{4\tau\} < \{\tau\} < \{6\tau\} < \{3\tau\}$, so $\big(u_1\, u_2\, \ldots\, u_N\big) = \big(0\,5\,2\,7\,4\,1\,6\,3\big)$. As $\{8\tau\}$ is the largest number in the rotation orbit of the first 9 iterations, $\big(u_{N+1}\, u_N\, \ldots\, u_2\big) = \big(8\,3\,6\,1\,4\,7\,2\,5\big)$. After subtraction of 1 in all entries, one obtains the permutation Π_6^β.

7.2. Digit Blocks $w = d_{-1} \cdots d_{-m}(N)$

For all digit blocks w we will try to determine the sequence R_w of those numbers N with w as the prefix of $\beta^-(N)$. The tridents introduced in the previous section give occurrence sequences R_w which are unions of three consecutive generalized Beatty sequences. We write for short

$$V(p, q, [r, r+1, r+2]) := V(p, q, r) \sqcup V(p, q, r+1) \sqcup V(p, q, r+2).$$

As before, we order the w in a Fibonacci tree. The first four levels of this tree are depicted in Figure 3.

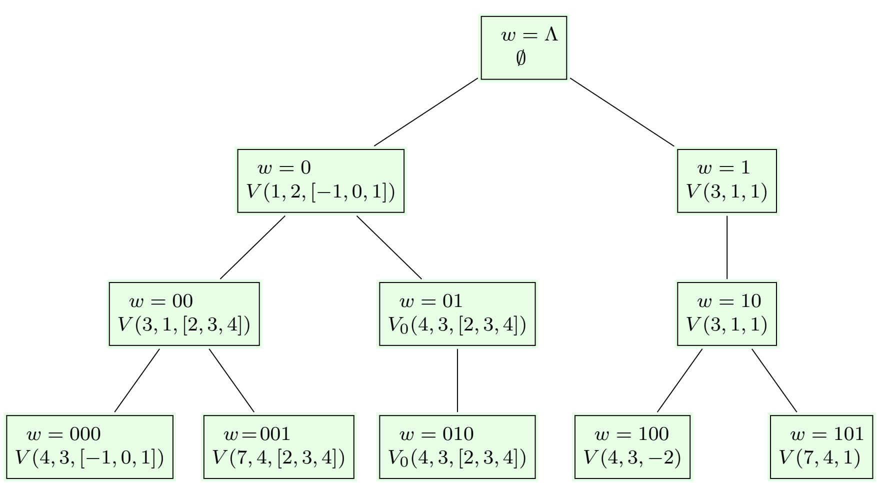

Figure 3: Tree of digit blocks $w = d_{-1} \cdots d_{-m}(N)$.

We start with the words w on this tree. We write $R_{\cdot w}$ for the occurrence sequences of words w occurring as a prefix of the words $\beta^-(N)$, to emphasize the positions of these words in the expansion $\beta(N)$.

Proposition 3. *Let $\beta(N) = \beta^+(N) \cdot \beta^-(N)$ be the base phi expansion of N. Let w be a word of length m. Then the sequence of occurrences of numbers N such that the first m digits of $\beta^-(N)$ are equal to w, i.e., $d_{-1} \cdots d_{-m}(N) = w$, is given for the words w of length at most 3, by*
(a) $R_{\cdot 0} = V(2,1,-1) \sqcup V(2,1,0) \sqcup V(2,1,1)$,
(b) $R_{\cdot 1} = R_{\cdot 10} = V(3,1,1)$,
(c) $R_{\cdot 00} = V(3,1,2) \sqcup V(3,1,3) \sqcup V(3,1,4)$,
(d) $R_{\cdot 01} = R_{\cdot 010} = V_0(4,3,2) \sqcup V_0(4,3,3) \sqcup V_0(4,3,4)$,
(e) $R_{\cdot 000} = V(4,3,-1) \sqcup V(4,3,0) \sqcup V(4,3,1)$,
(f) $R_{\cdot 001} = V(7,4,2) \sqcup V(7,4,3) \sqcup V(7,4,4)$,
(g) $R_{\cdot 100} = V(4,3,-2)$,
(h) $R_{\cdot 101} = V(7,4,1)$.

Proof. (a) $w = \cdot 0$: In Section 5 of the paper [5] the tridents are coded by triples (A, B, C). It follows from Theorem 5.1 of [5] that the first elements (coded A) of the tridents are all members of $V(2,1,-1)$. This implies the statement in (a).

(b) $w = \cdot 1$: We already know from Proposition 1 that $R_{\cdot 1} = V(3,1,1)$.

(c) $w = \cdot 00$: Using the Propagation Principle, we see that a digit block $\cdot 10$ is always followed directly by the first element of a trident of $\cdot 00$'s and vice versa. This implies the statement in (c), because of (b).

(d) $w = \cdot 01$: This result is given in Remark 6.2 in the paper [5].

(e) $w = \cdot 000$: Using the Propagation Principle, we see that a $\cdot 100$ is always followed directly by the first element of a trident of $\cdot 000$'s and vice versa. So (e) is implied by (g).

(f) $w = \cdot 001$: Take the first sequence $V(3,1,2)$ of $R_{\cdot 00}$, and put $p = 3, q = 1, r = 2$. Then the first sequence of $R_{\cdot 000}$ is equal to $V(4,3,-1) = V(p+q, p, r-p)$. It then follows from Lemma 4 that the first sequence of $R_{\cdot 001}$ is equal to $V(2p+q, p+q, r) = V(7,4,2)$.

(g) $w = \cdot 100$: For the first 17 numbers we check that $\cdot 100$ occurs as the prefix of $\beta^-(N)$ if and only if 1000 occurs as the suffix of $\beta^+(N)$. The result then follows from Theorem 7: $R_w = V(L_{m-1}, L_{m-2}, \gamma_w)$ if $w_{m-1} = 0$, where here $m = 4$, so $R_{1000} = V(L_3, L_2, \gamma_{1000}) = V(4,3,-2)$. Here γ_{1000} is determined by noting that $N = 5$ is the first number in R_{1000}.

(h) $w = \cdot 101$: Take the sequence $R_{\cdot 10} = V(3,1,1)$, and put $p = 3, q = 1, r = 1$. Then $R_{\cdot 100}$ is equal to $V(4,3,-2) = V(p+q, p, r-p)$. It then follows from Lemma 4 that the sequence $R_{\cdot 101}$ is equal to $V(2p+q, p+q, r) = V(7,4,1)$. □

The reader might think that we can now proceed, as we did earlier, from these cases to words w with larger lengths m, using the same tools. However, this does not work. The reason is that the $\beta^-(N)$ words do not occur in lexicographical order, in contrast with the $\beta^+(N)$ words. Some occurrence sequences are Lucas-Wythoff, some are not — but still close to Lucas-Wythoff sequences.

Recall the three (Sturmian) morphisms f, g and h from Equation (8). Note that f equals the square of the Fibonacci morphism $a \mapsto ab$, $b \mapsto a$, so f has fixed point x_{F}, the Fibonacci word. The fixed point x_{G} of g is given by $x_{\mathrm{G}} = b\,x_{\mathrm{F}}$, and the fixed point x_{H} of h is given by $x_{\mathrm{H}} = a\,x_{\mathrm{F}}$ — see [3, Theorem 3.1].

Let $V_{\mathrm{F}}, V_{\mathrm{G}}, V_{\mathrm{H}}$ denote the families of sequences having $x_{\mathrm{F}}, x_{\mathrm{G}}, x_{\mathrm{H}}$ as first differences, with first element an arbitrary integer. Then, by definition, one example is $V = V_{\mathrm{F}}$, if we take $V_{\mathrm{F}}(1) = p+q+r$. We have also already encountered a V_{G}, since $V_0 = V_{\mathrm{G}}$, if we take $V_{\mathrm{G}}(1) = r$. This follows from $V_0(p,q,r) = r, p+q+r, \dots = r, b+r, \dots$, which gives $\Delta V_0 = bx_{\mathrm{F}} = x_{\mathrm{G}}$. We mention that one can show that there do not exist α, p, q, and r such that V_{H} is a generalized Beatty sequence $V = (p\lfloor n\alpha \rfloor + qn + r)$.

We conjecture that the following holds.

Conjecture. Let $\beta(N) = \beta^+(N) \cdot \beta^-(N)$ be the base phi expansion of N. Let w be a word of length m. Let $R_{\cdot w}$ be the sequence of occurrences of numbers N such that the first m digits of $\beta^-(N)$ are equal to w, i.e., $d_{-1} \cdots d_{-m}(N) = w$. Then there exist two Lucas numbers a and b such that either $R_{\cdot w} = V_{\mathrm{F}}$, $R_{\cdot w} = V_{\mathrm{G}}$, or $R_{\cdot w} = V_{\mathrm{H}}$. A second possibility is that $R_{\cdot w}$ is a union of three of such sequences.

In all cases in Proposition 3 the sequence $R_{\cdot w}$ is a V_{F}, except $R_{\cdot 010}$, which is a union of three V_{G}'s, the middle one being $V_{\mathrm{G}}(4,3,-4)$. The first case where a V_{G} as $R_{\cdot w}$ occurs, is for $w = \cdot 1001$, where $a = 29, b = 18$. The first case where V_{H} as $R_{\cdot w}$ occurs, is as the first element of the trident for the digit block $w = \cdot 0100$, where $a = 18, b = 11$.

References

[1] J.-P. Allouche and F. M. Dekking, Generalized Beatty sequences and complementary triples, *Moscow J. Comb. Number Th.* **8** (2019), 325–341.

[2] G. Bergman, A number system with an irrational base, *Math. Mag.* **31** (1957), 98–110.

[3] J. Berstel and P. Séébold, A remark on morphic Sturmian words, *RAIRO Theor. Inform. Appl.* **28** (1994), 255–263.

[4] R. M. Capocelli, A note on Fibonacci trees and the Zeckendorf representation of integers, *Fibonacci Quart.* **26** (1988), 318–324.

[5] F. M. Dekking, Base phi representations and golden mean beta-expansions, *Fibonacci Quart.* **58** (2020), 38–48.

[6] M. Dekking, The structure of Zeckendorf expansions, *Integers* **21**, #A6 (2021), 10 pp.

[7] M. Dekking, How to add two natural numbers in base phi, *Fibonacci Quart.* **59** (2021), 19–22.

[8] C. Frougny and J. Sakarovitch, Automatic conversion from Fibonacci representation to representation in base φ and a generalization, *Int. J. Algebra Comput.* **9** (1999), 351–384.

[9] P. J. Grabner, I. Nemes, A. Pethö and R. F. Tichy, Generalized Zeckendorf decompositions, *Appl. Math. Lett.* **7** (1994), 25–28.

[10] T. van Ravenstein, The Three Gap Theorem (Steinhaus Conjecture). *J. Austral. Math. Soc. Ser. A.* **45** (1988), 360–370.