EXPRESSING AN INTEGER AS A SUM OF CUBES OF POLYNOMIALS

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Abstract

In this paper we prove that there exist infinitely many integers which can be expressed as a sum of four cubes of polynomials with integer coefficients. We give several identities that express the integers 1 and 2 as a sum of four cubes of polynomials. We also show that every integer can be expressed as a sum of five cubes of polynomials with integer coefficients.

1. Introduction

This paper is concerned with the problem of expressing an integer as a sum of cubes of polynomials with integer coefficients. As mentioned by Mordell [6], the first such result was obtained in 1908 by Werebrusow who proved the identity

$$(1 + 6t^3)^3 + (1 - 6t^3)^3 + (-6t^2)^3 = 2.$$

Subsequently, in 1936, Mahler [5] proved the identity

$$(9t^4)^3 + (3t - 9t^4)^3 + (1 - 9t^3)^3 = 1.$$

If a is any arbitrary integer, we can multiply these identities by a^3 to express the integers $2a^3$ and a^3, respectively, as a sum of three cubes of polynomials. In this context it might be pertinent to mention here the considerable interest in the problem of finding representations of integers as sums of three cubes of integers. Except for integers not expressible as a sum of three cubes because of congruence considerations, representations of all other integers less than 100 are now known ([1], [2], [3], [4]). However, regarding the problem of expressing an integer as a sum of three cubes of polynomials, except for integers of the types a^3 and $2a^3$ mentioned above, it is not known whether there is any other integer that can be expressed as a sum of three cubes of polynomials with integer coefficients. Identities expressing

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an integer as a sum of four or five cubes of polynomials have apparently not been published.

We show in this paper that there exist infinitely many integers that are expressible as a sum of four cubes of polynomials with integer coefficients. Specifically, we give several identities expressing the integers 1 and 2 as a sum of four cubes of polynomials. We also prove that every integer is expressible as a sum of five cubes of polynomials with integer coefficients.

2. Integers Expressed as Sums of Four Cubes of Polynomials

2.1. Two Polynomial Identities

The theorem below gives identities that describe infinitely many integers which can be written as a sum of four cubes of polynomials.

Theorem 1. *If p and q are arbitrary integers, any integer expressible either as p^3+q^3 or $2(p^6-q^6)$ is expressible as a sum of four cubes of univariate polynomials with integer coefficients.*

Proof. The proof is based on the following two identities in both of which t is an arbitrary parameter:

$$\begin{aligned}p^3+q^3 &= \{2(p+q)t^2+4qt+q)\}^3+\{2(p+q)t^2+4qt-p+2q\}^3\\ &\quad+\{-2(p+q)t^2+(p-3q)t+p\}^3+\{-2(p+q)t^2-(p+5q)t+p-2q\}^3, \quad (1)\end{aligned}$$

and

$$2(p^6-q^6)=(pt-q^2)^3+(-pt-q^2)^3+(qt+p^2)^3+(-qt+p^2)^3. \quad (2)$$

To obtain Identity (1), we solve the equation

$$x_1^3+x_2^3+x_3^3+x_4^3=p^3+q^3, \quad (3)$$

by writing,

$$x_1=-y+p, \quad x_2=-y+q, \quad x_3=y+m, \quad x_4=y-m, \quad (4)$$

when (3) is readily solved for y to yield a nonzero solution given by $y=-(2m^2-p^2-q^2)/(p+q)$. On writing $m=(p+q)t+q$, we get $y=-2(p+q)t^2-4qt+p-q$, and on substituting the values of m and y in (4), we get the values of $x_i, i=1,\ldots,4$, and we thus get Identity (1).

To prove Identity (2), we consider

$$(pt+u)^3+(-pt+u)^3+(qt+v)^3+(-qt+v)^3$$

as a cubic polynomial in t. It is evident that the coefficients of t^3 and t in this polynomial vanish. We now choose $u = -q^2, v = p^2$, so that the coefficient of t^2 also vanishes, and the polynomial reduces to $2(p^6 - q^6)$, and we thus get Identity (2).

It immediately follows from Identities (1) and (2) that integers that may be written as $p^3 + q^3$ or $2(p^6 - q^6)$ can be written as a sum of four cubes of univariate polynomials with integer coefficients. $\square$

As numerical applications of Identity (1), we give two examples by taking $(p, q) = (2, -1)$ and $(p, q) = (2, 1)$ when we obtain the following two identities expressing the integers 7 and 9, respectively, as a sum of four cubes of polynomials with integer coefficients:

$$(2t^2 - 4t - 1)^3 + (2t^2 - 4t - 4)^3 + (-2t^2 + 5t + 2)^3 + (-2t^2 + 3t + 4)^3 = 7,$$
$$(6t^2 + 4t + 1)^3 + (6t^2 + 4t)^3 + (-6t^2 - t + 2)^3 + (-6t^2 - 7t)^3 = 9.$$

Similarly, on taking $(p, q) = (2, 1)$ in Identity (2), we get the following identity which expresses the integer 126 as a sum of four cubes of polynomials:

$$(2t - 1)^3 + (-2t - 1)^3 + (t + 4)^3 + (-t + 4)^3 = 126.$$

2.2. Expressing 1 as a Sum of Four Cubes of Polynomials

We give below three identities that express 1 as a sum of four cubes of polynomials:

$$(2t^2)^3 + (2t^2 - 1)^3 + (-2t^2 - t + 1)^3 + (-2t^2 + t + 1)^3 = 1, \tag{5}$$

and

$$(3t^6 + 3t^3 + 1)^3 + \{-3t^3(t^3 + 1)\}^3 + (-3t^4 - 2t)^3 + (-t)^3 = 1, \tag{6}$$

and

$$\begin{aligned}(8t^3 - 2t^2 - 4t + 1)^3 &+ (8t^3 - 6t^2 - 3t + 2)^3 \\ &+ (-8t^3 + 2t^2 + 3t)^3 + (-8t^3 + 6t^2 + 4t - 2)^3 = 1.\end{aligned} \tag{7}$$

To obtain Identity (5), we solve the equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 1, \tag{8}$$

by writing $x_1 = p + 1, x_2 = p, x_3 = -p - t, x_4 = -p + t$. Equation (8) is now readily solved and we get $p = 2t^2 - 1$, which yields the values of $x_i, i = 1, \ldots, 4$, that leads to Identity (5).

To obtain Identity (6), we write $x_1 = -t^2p + 1, x_2 = t^2p, x_3 = p + t, x_4 = -t$. Equation (8) is now readily solved and we get $p = -3t(t^3 - 1)$, which yields the values of $x_i, i = 1, \ldots, 4$, that leads to Identity (6).

To obtain Identity (6), we write $x_1 = pm + 1, x_2 = qm, x_3 = -pm - t$ and $x_4 = -qm + t$. Equation (8) now reduces to

$$((q^2 - p^2)t + p^2)m - (p + q)t^2 + p = 0, \tag{9}$$

and on writing

$$p = 2t, q = 2t - 1, \tag{10}$$

Equation (9) further reduces to

$$m - 4t^2 + t + 2 = 0,$$

and hence we get $m = 4t^2 - t - 2$. Using the values of p, q given by (10), we get the values of $x_i, i = 1, \ldots, 4$, which leads to Identity (7).

2.3. Expressing 2 as a Sum of Four Cubes of Polynomials

The following identity expresses the integer 2 as a sum of of four cubes of polynomials in three variables:

$$\begin{aligned}\{6t^3(g^3 + h^3)^2 + 1\}^3 + \{-6t^3(g^3 + h^3)^2 + 1\}^3 \\ + \{-6gt^2(g^3 + h^3)\}^3 + \{-6ht^2(g^3 + h^3)\}^3 = 2.\end{aligned} \tag{11}$$

To obtain Identity (11), we begin by writing

$$(p + 1)^3 + (-p + 1)^3 + q^3 + r^3 = 2 \tag{12}$$

which reduces to

$$6p^2 + q^3 + r^3 = 0. \tag{13}$$

We solve Equation (13) by writing $p = fm, q = gm, r = hm$, and we get,

$$m = -6f^2/(g^3 + h^3).$$

To obtain a solution with integer coefficients, we write $f = t(g^3 + h^3)$, and we get $m = -6t^2(g^3 + h^3)$ which leads to the solution,

$$p = -6t^3(g^3 + h^3)^2, \quad q = -6gt^2(g^3 + h^3), \quad r = -6ht^2(g^3 + h^3),$$

and on substituting these values in (12), we get Identity (11).

We give below three more identities expressing the integer 2 as a sum of four cubes of univariate polynomials:

$$(t^2)^3 + (t^2)^3 + (-t^2 + t + 1)^3 + (-t^2 - t + 1)^3 = 2, \tag{14}$$

$$(3t^3 + 1)^3 + (-3t^3 + 1)^3 + (-3t^2)^3 + (-3t^2)^3 = 2, \tag{15}$$

$$(18t^3 + 1)^3 + (-18t^3 + 1)^3 + (-6t^2)^3 + (-12t^2)^3 = 2. \tag{16}$$

More such identities can be obtained.

To obtain Identity (14), we solve the equation,

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 2, \tag{17}$$

by writing $x_3 = -m + 1, x_2 = -m + 1, x_3 = m + t, x_4 = m - t$. Equation (17) can now be readily solved for m, and we get $m = 1 - t^2$, which yields the values of $x_i, i = 1, \ldots, 4$, that leads to Identity (14).

Identity (15) may be obtained from Identity (11) by writing $g = 1, h = 1$ and replacing t by $t/2$.

To obtain Identity (16), we solve Equation (17) by writing

$$x_1 = -2tm - 6t^3 + 1, \quad x_2 = 2tm + 6t^3 + 1, \quad x_3 = -6t^2, \quad x_4 = m, \tag{18}$$

when Equation (17) reduces to $m(m + 12t^2)^2 = 0$, which yields $m = -12t^2$, and now the relations (18) yield the values of $x_i, i = 1, \ldots, 4$, that leads to Identity (16).

3. Integers Expressed as Sums of Five Cubes of Polynomials

We will now prove that every integer can be expressed as a sum of five cubes of polynomials.

Theorem 2. *Every integer is expressible as a sum of five cubes of univariate polynomials with integer coefficients.*

Proof. The proof is based on the following six identities:

$$6m = (36t^3 + m + 1)^3 + (36t^3 + m - 1)^3 + 2(-36t^3 - m)^3 + (-6t)^3, \tag{19}$$

$$\begin{aligned} 6m + 1 = {} & (36t^3 - 18t^2 + 3t + m + 1)^3 + (36t^3 - 18t^2 + 3t + m - 1)^3 \\ & + 2(-36t^3 + 18t^2 - 3t - m)^3 + (-6t + 1)^3, \end{aligned} \tag{20}$$

$$\begin{aligned} 6m + 2 = {} & (36t^3 - 36t^2 + 12t + m)^3 + (36t^3 - 36t^2 + 12t + m - 2)^3 \\ & + 2(-36t^3 + 36t^2 - 12t - m + 1)^3 + (-6t + 2)^3, \end{aligned} \tag{21}$$

$$\begin{aligned} 6m + 3 = {} & (36t^3 - 54t^2 + 27t + m - 3)^3 + (36t^3 - 54t^2 + 27t + m - 5)^3 \\ & + 2(-36t^3 + 54t^2 - 27t - m + 4)^3 + (-6t + 3)^3, \end{aligned} \tag{22}$$

$$\begin{aligned} 6m + 4 = {} & (36t^3 + 36t^2 + 12t + m + 3)^3 + (36t^3 + 36t^2 + 12t + m + 1)^3 \\ & + 2(-36t^3 - 36t^2 - 12t - m - 2)^3 + (-6t - 2)^3, \end{aligned} \tag{23}$$

$$6m+5=(36t^3+18t^2+3t+m+2)^3+(36t^3+18t^2+3t+m)^3 \\ +2(-36t^3-18t^2-3t-m-1)^3+(-6t-1)^3, \quad (24)$$

where m and t are arbitrary parameters.

To obtain the above identities, we begin with the following simple, readily verifiable, identity:

$$6r=(r+1)^3+(r-1)^3-2r^3. \quad (25)$$

On adding $(-6t+j)^3$ to both sides, we get the identity

$$6r+(-6t+j)^3=(r+1)^3+(r-1)^3-2r^3+(-6t+j)^3. \quad (26)$$

We now equate the left-hand side of Identity (26) to $6m+j$, where j is an integer, and solve for r. Since $j^3 \equiv j \mod 6$, we get a value of r, say $r=r_0$, which is given by a polynomial, with integer coefficients, in the parameters m and t. On replacing r by r_0 in Identity (26), we get an identity that expresses $6m+j$ as a sum of five cubes of polynomials with integer coefficients. By successively taking $j=0,1,2,3$, we obtained Identities (19)–(22).

While we can obtain identities expressing $6m+4$ and $6m+5$ as a sum of five cubes of polynomials in the same way, we obtained the simpler Identity (23) by adding $(-6t-2)^3$ (instead of $(-6t+4)^3$) to both sides of (25) and proceeding as above, while for Identity (24), we added $(-6t-1)^3$ (instead of $(-6t+5)^3$) to both sides of (25) and followed the same procedure. Identities (19)–(24) can also be readily verified by direct computation.

Since any arbitrary integer n is expressible as $6m+j$ where $j \in \{0,1,\ldots,5\}$, it follows from Identities (19)–(24) that every integer is expressible as the sum of five cubes of univariate polynomials with integer coefficients. □

As numerical examples, taking $m=0$ in Identities (22) and (23), we get the following two identities expressing the integers 3 and 4, respectively, as a sum of five cubes of polynomials:

$$(36t^3-54t^2+27t-3)^3+(36t^3-54t^2+27t-5)^3 \\ +2(-36t^3+54t^2-27t+4)^3+(-6t+3)^3=3,$$
$$(36t^3+36t^2+12t+3)^3+(36t^3+36t^2+12t+1)^3 \\ +2(-36t^3-36t^2-12t-2)^3+(-6t-2)^3=4.$$

4. Some Open Problems

It would be interesting to find identities expressing small integers such as 3 and 4 as a sum of four cubes of polynomials. In fact, it would be useful to determine which integers can be expressed as a sum of four cubes of polynomials and for which integers it becomes necessary to use five polynomials.

It is a far more challenging problem to find new identities expressing an integer as a sum of three cubes of polynomials with integer coefficients. In fact, it seems unlikely that, apart from integers of the type a^3 and $2a^3$ mentioned in the introduction, there are any other integers that can be expressed as a sum of three cubes of polynomials.

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