#A27 INTEGERS 24 (2024)

ON THE 2-ADIC VALUATION OF DIFFERENCES OF HARMONIC NUMBERS

Tamás Lengyel[1]
Mathematics Department, Occidental College, Los Angeles, California
lengyel@oxy.edu

Received: 5/28/23, Revised: 11/13/23, Accepted: 1/30/24, Published: 3/15/24

Abstract

We explicitly determine the exact 2-adic valuation of differences of harmonic sums, $H_n - H_m$, $0 \le m \le n-1$, which also yields the 2-adic valuation of the products of these differences. Sharp lower and upper bounds on the average 2-adic orders of the differences follow. We present an application to obtain lower bounds on the 2-adic valuations of products of binomial coefficients and differences of harmonic numbers, and lacunary sums involving binomial coefficients.

1. Introduction

With the harmonic numbers $H_n = \sum_{k=1}^{n} 1/k$ and $H_0 = 0$, we define their differences:

$$H_n - H_m = \sum_{k=m+1}^{n} \frac{1}{k}, \text{ with } 0 \le m \le n-1.$$

The harmonic numbers and their differences have been extensively studied, and congruential identities have been established and their p-adic valuations have been investigated under different settings, e.g., when the valuation is a positive integer (see [7, 1]), in particular.

The most basic result on the 2-adic valuations of harmonic numbers states that $\nu_2(H_n) = -k$ if $2^k \le n < 2^{k+1}$. We note that the author investigated the p-adic properties of differences of other combinatorial quantities, e.g., central binomial coefficients and Catalan numbers in [13], Motzkin numbers in [11, 12], and Stirling numbers of the second kind in [14] by various methods. In general, answering these questions are significantly more difficult for odd primes; cf. [7, 17, 19, 22].

DOI: 10.5281/zenodo.10821695

[1]Professor Lengyel died on February 6, 2024. The Editors wish to express their appreciation of his long association with *Integers* and his significant contribution to the journal in the form of both published papers and refereeing.

Our focus is on the 2-adic valuation of $H_n - H_m$ which is never positive if $0 \leq m \leq n-1$; cf. the proof of Theorems 1 and 2. The next two theorems are of historical significance.

Theorem 1. *For all $n \geq 2$, we have $H_n \notin \mathbb{Z}$.*

This theorem on integrality is attributed to Theisinger who proved it in 1915. A 2-adic proof is based on the fact that $H_n \notin \mathbb{Z}_2$, the ring of 2-adic integers, so $H_n \notin \mathbb{Z}$. In fact, Kürschák was the first one to use a 2-adic approach to prove Theorem 1. In 1918 (see [5]) he strengthened Theorem 1 and proved the following improvement.

Theorem 2. *For all $0 \leq m \leq n-2$, we have $H_n - H_m \notin \mathbb{Z}_2$ and, in particular, $H_n - H_m \notin \mathbb{Z}$.*

This theorem includes the prior one with the settings $m = 1$ and $n \geq 3$ but Theorem 2 does not apply if $m = n-1$ and n is odd, since $H_n - H_m = 1/n \in \mathbb{Z}_2$; cf. [5], although still $H_n - H_m \notin \mathbb{Z}$ if $1 \leq m \leq n-1$. The proofs of Theorems 1 and 2 are based on the observation that there is a unique term in the sum $\sum_{k=m+1}^{n} 1/k$ with the most negative 2-adic valuation. With $r = \max_{m+1 \leq k \leq n} \nu_2(k)$ we get that $r \geq 1$ since $H_n - H_m$, $m \leq n-2$, has at least two terms. The proof follows by noting that now there is a unique term $1/k$ with $m+1 \leq k \leq n$ and 2-adic valuation r. Otherwise, there are at least two terms with $k_1 = 2^r c$ and $k_2 = 2^r d$, with $c < d$ and both being odd. However, then $k_3 = 2^r(c+1)$ is even and it is between k_1 and k_2, and $\nu_2(k_3) > r$, which is a contradiction.

We note that from our perspective it is less relevant, but the integrality problem has been extended in different directions, e.g., to the r^{th} elementary symmetric functions of $1, 1/2, \ldots, 1/n$ (see Erdős and Niven [6], and Chen and Tang [3]), and to variations of multiple harmonic sums, e.g., Pilehrood et al. [17]. Density related results, in which the density of positive integers n for which $H_n = u_n/v_n$, with $u_n, v_n \in \mathbb{Z}$, $(u_n, v_n) = 1$ and $v_n > 0$, and either $p \mid u_n$ or $p \nmid v_n$, have been obtained recently in [19, 21, 22] and were initiated in [7].

We are interested in certain divisibility properties of differences of harmonic numbers. We note that some interesting properties have been obtained for these differences, e.g., it is known that no two differences of harmonic numbers can be equal; cf. [6, Theorem 2].

In Sections 2 and 3 we present theorems on determining the exact 2-adic order of the difference of harmonic numbers (Theorems 3 and 6) and their products (Corollary 2 and Theorem 4). They yield sharp lower and upper bounds on the average of the 2-adic orders of these differences. We use the results to obtain lower bounds on the 2-adic valuations of terms involving products of binomial coefficients and differences of harmonic numbers. We obtain lower bounds on the 2-adic valuations of related elementary symmetric functions in Section 4, e.g., in Lemma 2, Corollary 3, and Remark 6. Finally, in Section 5 we discuss an application involving the determination of the 2-adic order of some lacunary binomial sums; see Theorem 7 and Remark 7.

2. The Main Results on the Exact 2-Adic Valuation of Differences of Harmonic Numbers

2.1. Differences

Our first observation is straightforward. With $r = \lfloor \log_2(2n) \rfloor$, the proof of Theorem 2 easily implies the following lemma.

Lemma 1. *For $n \geq 1$, we have that $\nu_2(H_{2n} - H_n) = -\lfloor \log_2(2n) \rfloor$.*

Here $m = 2n/2$, and it turns out that many more values of m result in the same valuation as it is explored in Theorem 3 and noted in Remark 2. One of our main results of this note considers arbitrary differences.

Theorem 3. *We write the base 2 expansion of n as $n = \sum_{i=1}^{t} 2^{a_i}$ with $a_i \in \mathbb{N}$, $0 \leq a_1 < a_2 < \cdots < a_t$. For $n \geq 2$ there are 2^{a_t}, $2^{a_{t-1}}$, $\ldots$, 2^{a_1} 2-adic values of $\nu_2(H_n - H_m)$, $m = 0, 1, \ldots, n-1$, that are equal to $-a_t$, $-a_{t-1}$, $\ldots$, $-a_1$, respectively, in increasing order of m.*

More specifically, with $M_j = \sum_{i=t-j+1}^{t} 2^{a_i}$, $1 \leq j \leq t$, and $M_0 = 0$, we have $\nu_2(H_n - H_m) = -a_{t-j+1}$ if $M_{j-1} \leq m < M_j$, $j = 1, 2, \ldots, t$.

Note that for $t \geq 1$ we have

$$a_j = \lfloor \log_2(n - M_{t-j}) \rfloor, \; j = 1, 2, \ldots, t. \tag{2.1}$$

Visually, we can use a ruler (see Figure 1) to read out the proper value of $-a_j$.

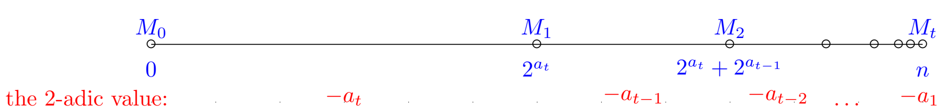

Figure 1: The ruler

Note that for the gap size between consecutive markers we obtain

$$\frac{M_{t-j+1} - M_{t-j}}{2} = \frac{2^{a_j}}{2} = 2^{a_j - 1} \geq 2^{a_{j-1}}.$$

The following corollary follows from Theorem 3.

Corollary 1. *We have $\nu_2(H_n - H_{n-m}) = -a_t$, $1 \leq m \leq n$, if $n = 2^{a_t}$, with $a_t \geq 0$, is a power of 2.*

Remark 1. We can easily generalize the theorem for the differences of the so-called harmonic numbers of order r: $H_n^{(r)} = \sum_{k=1}^{n} 1/k^r$ with $r \in \mathbb{Z}$.

Remark 2. Lemma 1 immediately follows from Theorem 3 since the first case with $j = t$ applies in Theorem 3 and (2.1). Therefore, exactly $2^{\lfloor \log_2(2n) \rfloor}$ values of m (starting with 0 and increasing) result in $\nu_2(H_{2n} - H_m) = -\lfloor \log_2(2n) \rfloor = -1 - \lfloor \log_2(n) \rfloor = -\lceil \log_2(n+1) \rceil$.

Proof of Theorem 3. Note that for any particular $j = 1, 2, \ldots, t$, we deal with the sequence

$$\Big(\frac{1}{2^{a_t} + \cdots + 2^{a_{j-1}} + 1} + \cdots + \frac{1}{n}, \frac{1}{2^{a_t} + \cdots + 2^{a_{j-1}} + 2} + \cdots + \frac{1}{n}, \ldots, \\ \frac{1}{2^{a_t} + \cdots + 2^{a_{j+1}} + 2^{a_j}} + \cdots + \frac{1}{n}\Big),$$

which gives $\max_{1 \le i \le 2^{a_j}} \nu_2(i) = a_j$, i.e., the last element of the sequence has the unique most negative 2-adic order, $-a_j$. □

Remark 3. Theorem 3 provides the Newton polygon of the polynomial $\prod_{m=0}^{n-1}(x - (H_n - H_m))$. The polygon describes the set of 2-adic valuations of its roots, $\{H_n - H_m\}_{0 \le m \le n-1}$. Moreover, the theorem identifies the individual 2-adic valuations of the actual roots.

The Newton polygon conveys information about the poles of a rational polynomial. If $\nu_p(c) = -m$, we say that c has a *pole of order* m. For any rational polynomial $f(x)$ we define the *maximum pole* of $f(x)$ as the highest order pole of its coefficients, which is the highest power of p in the denominator of any coefficient. For a rational polynomial of degree n, $f(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_n$, we plot the lattice points $(i, \nu_p(c_i))$, $0 \le i \le n$. The Newton polygon of $f(x)$ is the lower boundary of the convex hull of the set of these lattice points.

2.2. Product of Differences and the Average of 2-Adic Orders

Inspired by Remark 3, we note that Theorem 3 implies the following corollary.

Corollary 2. *If the base 2 expansion of* n *is* $n = \sum_{i=1}^{t} 2^{a_i}$*, then*

$$\nu_2\left(\prod_{m=0}^{n-1}(H_n - H_m)\right) = -\sum_{i=1}^{t} a_i 2^{a_i}.$$

The sum $a(n) = \sum_{i=1}^{t} a_i 2^{a_i}$ is a weighted digit sum of n in base 2. Generally, digit sum functions behave rather irregularly, e.g., with $b(n) = t$ we have $b(2^k - 1) = k$, and $b(2^k) = 1$, $k \ge 0$, for the digit sum, while their sum $\sum_{k=1}^{n-1} b(k)$ is more smooth and $\sum_{k=1}^{n-1} b(k) = \sum_{i=1}^{t} 2^{a_i}(\frac{a_i}{2} + i - 1) = \frac{1}{2}a(n) + \sum_{i=1}^{t}(i-1)2^{a_i} =$

$\frac{n}{2}\log_2(n) + nF(\log_2(n))$ where $F(x)$ is a continuous, one-periodic, nowhere differentiable function according to Delange; cf. [9, 4].

The sequence $a(n)$ behaves more regularly. We mention different ways to calculate $a(n)$ and derive sharp bounds on it in Theorem 4. The exponent of y in the two-variable generating function

$$[x^n]\prod_{k=0}^{\infty}\left(1 + y^{k\,2^k}x^{2^k}\right)$$

determines this sum for $n \geq 1$.

For faster calculations we can use the recurrence

$$a(2n) = 2a(n) + 2n; \quad a(2n+1) = a(2n),$$

with $a(0) = 0$ and $a(1) = 0$. The first few values of the sequence $a(n)$ are: 0, 0, 2, 2, 8, 8, 10, 10, 24, 24, 26, 26, 32, 32, 34, 34. The `FindRegularSequenceRecurrence` function of the IntegerSequences *Mathematica* package (cf. [18]) confirms that the sequence $a(n)$ is 2-regular with its 2-kernel contained in a 3-dimensional vector space generated by $a(2n)_{n\geq 0}, a(2n+1)_{n\geq 0}$, and $a(4n+2)_{n\geq 0}$; cf. [20, Corollaries 1 and 2].

For the generating function $A(x) = \sum_{n\geq 1} a(n)x^n$ we obtain the functional equation

$$A(x^2) = \left(\frac{A(x)}{2(1+x)} - \frac{x^2}{(1-x^2)^2}\right).$$

It is a linear functional equation of substitution type and it can be solved in an iterative fashion; cf. [8, (61) in III.7].

We have $a(2^k - 1) = \sum_{t=0}^{k-1} t2^t < \sum_{t=0}^{k-1} k2^t < a(2^k) = k2^k$, $k \geq 1$, and the additive relation $a(n_1+n_2) = a(n_1)+a(n_2)$ if $2^{\nu_2(n_1)} > n_2 \geq 0$. It can be repeatedly used to determine $a(n_1 + n_2)$ if n_1 is the highest 2-power in $n_1 + n_2$ and to prove that $a(n)$ is an increasing sequence. It also helps in the inductive step in the proof of Theorem 4. For the asymptotics we can prove the next theorem by the above additivity and induction on $m \geq 1$, assuming that $2^{m-1} < n \leq 2^m$.

Theorem 4. *For $n \geq 1$, we have*

$$(n+1)\log_2(n+1) - 2n \leq a(n) \leq n\log_2 n. \tag{2.2}$$

Moreover, $a(n) = (n+1)\log_2(n+1) - 2n$ exactly if n is one less than a power of 2, and $a(n) = n\log_2 n$ exactly if n is a power of 2. For the average $\sum_{m=0}^{n-1}\nu_2(H_n - H_m)/n$ of the 2-adic orders of the differences we have the sharp bounds

$$-\log_2(n) \leq \frac{1}{n}\sum_{m=0}^{n-1}\nu_2(H_n - H_m) \leq -\frac{n+1}{n}\log_2(n+1) + 2.$$

Of course, we can take the respective upper and lower integer parts on the sides of (2.2). Theorem 4 improves the result of Brown who studied $a(n)$ in the context of the analysis of binomial queues in [4].

3. More Results for the Differences of Harmonic Sums

We have the following consequences of Theorem 3. In Corollary 1, n was a power of 2, while in Theorem 5 we assume that m is a power of 2.

Theorem 5. *With $m = 2^r$, $r \in \mathbb{N}$, and $1 \le m \le n$, we have that*

$$\nu_2(H_n - H_{n-m}) = -r - \nu_2(\lfloor n/m \rfloor).$$

The next theorem further generalizes Theorem 5.

Theorem 6. *Let $M = \lceil \log_2(m+1) \rceil$, $j = \lfloor (n-m)/2^M \rfloor$, where $1 \le m \le n$. Then we have*

$$\nu_2(H_n - H_{n-m}) = \begin{cases} -M+1, & \text{if } 2^M j + m \le n \le 2^M(j+1) - 1, \\ -M - \nu_2(j+1), & \text{if } 2^M(j+1) \le n \le 2^M(j+1) + m - 1, \end{cases} \tag{3.1}$$

and the lower bound

$$\nu_2\left(\binom{n}{m}(H_n - H_{n-m})\right) \ge -M + 1, \tag{3.2}$$

that does not depend on n. Equality holds in (3.2) if m is a power of 2.

Remark 4. Theorem 6 also implies Lemma 1.

Proof of Theorem 5. The proof can be done by inspection of the 2-adic orders of the denominators. We present a proof based on Theorem 3. The statement is clear if $n = 2^t$, $r \le t \in \mathbb{N}$, by Theorem 3. On the other hand, according to whether $l = \lfloor n/m \rfloor$ is odd or even, the binary expansion of $n = \sum_{i=1}^{t} 2^{a_i}$ has or does not have 2^r in it, respectively. Note that it means that $n-m$ does not have or does have the term 2^r at the same time. We add the 2-powers in n from high to low powers, as we move on the ruler (see Figure 1) from left to right, until we just pass $n-m$. If l is odd then $n - m < 2^{a_t} + 2^{a_{t-1}} + \cdots + 2^r$, and thus, $A = \nu_2(H_n - H_{n-m}) = -r$ since we stop right after including the 2-power 2^r. On the other hand, if l is even then n does not have, while m and $n-m$ do have, the term 2^r; and therefore, the last added term is 2^{s+r} with $s = \nu_2(l)$, and then $A = -(s + r)$. □

Proof of Theorem 6. The proof of Equation (3.1) is a straightforward follow up to that of Theorem 5. In fact, if $2^{M-1}(2j) + 1 \le n - m + 1$ and $n \le 2^{M-1}(2j+2) - 1$, with the given choice of j, then the unique most negative 2-power is $1/(2^{M-1}(2j+1))$ in the summation $H_n - H_{n-m}$. On the other hand, if $2^M(j+1)-m+1 \leq n-m+1$ and $n \leq 2^M(j+1)+m-1$, then $1/(2^M(j+1))$ is the corresponding term.

To prove (3.2) we find a non-trivial lower bound on $\nu_2(\binom{n}{m})$. If $m = 2^r$ and thus, $M = r+1$, then

$$\nu_2\left(\binom{n}{m}\right) = \begin{cases} 0, & \text{if } \ 2^M j + m \leq n \leq 2^M(j+1) - 1, \\ \nu_2(j+1)+1, & \text{if } \ 2^M(j+1) \leq n \leq 2^M(j+1)+m-1, \end{cases} \tag{3.3}$$

which guarantees equality in (3.2). If m is not a power of 2 then the right-hand side provides only a lower bound on $\nu_2(\binom{n}{m})$ in (3.3). □

4. Lower Bounds on the 2-Adic Valuations of Related Elementary Symmetric Functions

The purpose of this section is to obtain a fairly simple estimate in Remark 6 for the proof of Theorem 7. We define

$$E_k(n, am) = \sum \frac{1}{i_1 i_2 \ldots i_k}, \tag{4.1}$$

where the summation uses indices so that $n - am + 1 \leq i_1 < i_2 \cdots < i_k \leq n$, i.e., it is the k^{th} elementary symmetric function of $1/(n-am+1), 1/(n-am+2), \ldots, 1/n$. For $k = 1$ it simplifies to $E_1(n, am) = H_n - H_{n-am}$. We note that

$$E_k(n, am) = \begin{bmatrix} n+1 \\ n+1-k \end{bmatrix}_{n-am}$$

is also referred to as the r-Stirling number of the first kind (see [2]), and

$$E_k(n, n) = \frac{|s(n+1, k+1)|}{n!}$$

where $s(n, k)$ is the (signed) Stirling number of the first kind.

We obtain a lower bound (4.4) on $\nu_2(E_k(n, am))$ in Remark 6, based on (4.2) when $1 \leq k \leq am \leq n$. Note that, trivially, we have

$$\nu_2(E_k(n, n)) \geq -k\lfloor \log_2 n \rfloor. \tag{4.2}$$

As a sidenote, we add that this inequality can be improved in various ways. For instance, we have the following lemma.

Lemma 2. *For $1 \leq k \leq n$ we have*

$$\begin{aligned} \nu_2(E_k(n, n)) &\geq \sum_{i=1}^{k} (-\lfloor \log_2 n \rfloor + \lfloor \log_2 i \rfloor) \\ &= -k\lfloor \log_2 n \rfloor + \Big((k+1)\lfloor \log_2 k \rfloor - 2(2^{\lfloor \log_2 k \rfloor} - 1)\Big). \end{aligned} \tag{4.3}$$

Furthermore, similar and often stronger and more general estimates can be found on $\nu_p(E_k(n,n))$ with a prime p, e.g., in [10], although they are not needed here.

Corollary 3. *For all $k \geq 1$ there exists $n \geq 1$ such that*

$$\nu_2(E_k(n,n)) = \sum_{i=1}^{k}(-\lfloor \log_2 n \rfloor + \lfloor \log_2 i \rfloor)$$

.

Remark 5. We can easily derive the generating function $\frac{1}{1-x}\sum_{i=1}^{\infty} x^{2^i}$ of $\lfloor \log_2 n \rfloor$, $n \geq 1$. Therefore, the generating function of the correcting term $\sum_{i=1}^{k}\lfloor \log_2 i \rfloor$ in (4.3) is $\frac{1}{(1-x)^2}\sum_{i=1}^{\infty} x^{2^i}$; cf. [16, A061168].

Remark 6. Clearly, $\nu_2(E_k(n,am)) \geq \nu_2(E_k(n,n)) \geq -k\lfloor \log_2 n \rfloor$ if $1 \leq k \leq am \leq n$ by using the argument on the corresponding terms with the smallest 2-adic valuations and (4.2). Therefore,

$$\nu_2(N^k E_k(n,am)) \geq k(\nu_2(N) - \lfloor \log_2 n \rfloor), \tag{4.4}$$

which is non-negative if $\nu_2(N) \geq \lfloor \log_2 n \rfloor$. Note that equality holds in (4.4) if $n = 2^t$ and $k = 1$ by Corollary 1.

5. An Application

Our goal is to study the 2-adic properties of the lacunary binomial sum

$$\sum_{m=1}^{\lfloor n/a \rfloor}\left(\binom{n+N}{am} - \binom{n}{am}\right)2^{bm}$$

if $a, b \in \mathbb{Z}^+$ and $\nu_2(N) \geq 1$. For the terms of the sum we have the lower bound $\nu_2(N) - \nu_2((am)!) + bm$, which is not helpful if $a > b$ and m is large. To avoid this problem, under some conditions we find a non-trivial lower bound by applying Theorem 6.

Theorem 7. *With $a, b \in \mathbb{Z}^+$ and $\nu_2(N) \geq \lfloor \log_2 n \rfloor$ we get that*

$$\nu_2\left(\sum_{m=1}^{\lfloor n/a \rfloor}\left(\binom{n+N}{am} - \binom{n}{am}\right)2^{bm}\right) \geq \nu_2(N) - \lfloor \log_2 n \rfloor + b. \tag{5.1}$$

Proof. Note that

$$\left(\binom{n+N}{am} - \binom{n}{am}\right)2^{bm} = \sum_{k=1}^{am} N^k \binom{n}{am} E_k(n,am)2^{bm} \tag{5.2}$$

with the k^{th} elementary symmetric function $E_k(n, am)$ of

$$1/(n-am+1), 1/(n-am+2), \ldots, 1/n,$$

defined in (4.1). For $k = 1$ the first term on the right-hand side of (5.2) simplifies to $N\binom{n}{am}(H_n - H_{n-am})2^{bm}$. A lower bound on its 2-adic valuation is given in (3.2) of Theorem 6. Note that $M = \lceil \log_2(am+1) \rceil \le \lfloor \log_2 n \rfloor + 1$.

An application of (4.4) completes the proof. □

Remark 7. It turns out that the lower bound in (5.1) can be improved, even without the assumption $\nu_2(N) \ge \lfloor \log_2 n \rfloor$, and typically, the exact 2-adic order of the sum can be determined by evaluating the terms with small values of m. For instance, with $d, n \in \mathbb{N}$ we get in [15] that

$$\nu_2\left(\sum_{m=1}^{\lfloor (16n+9)/3 \rfloor} \left(\binom{16n+9+2^{d+4}}{3m} - \binom{16n+9}{3m}\right) 2^m\right)$$
$$= \nu_2\left(\sum_{m=1}^{3} \left(\binom{16n+9+2^{d+4}}{3m} - \binom{16n+9}{3m}\right) 2^m\right) = d+5.$$

Acknowledgement. The author thanks the referee and Eszter Neuman for careful reading and useful suggestions.

References

[1] D.W. Boyd, A p-adic study of the partial sums of the harmonic series, *Experiment. Math.* **3** (4) (1994), 287–302.

[2] A.Z. Broder, The r-Stirling numbers, *Discrete Math.* **49** (3) (1984), 241–259.

[3] Y. Chen and M. Tang, On the elementary symmetric functions of $\{1, 1/2, \ldots, 1/n\}$, *Amer. Math. Monthly* **119** (10) (2012), 862-867.

[4] Y.K. Cheung, Ph. Flajolet, M. Golin, and C.Y. James Lee, Multidimensional divide-and-conquer and weighted digital sums, preprint `arXiv:1003.0150`

[5] K. Conrad, The p-adic growth of harmonic sums, https://kconrad.math.uconn.edu/blurbs/gradnumthy/padicharmonicsum.pdf, 1-5.

[6] P. Erdős and I. Niven, Some properties of partial sums of the harmonic series, *Bull. Amer. Math. Soc.* **52** (4) (1946), 248-251.

[7] A. Eswarathasan and E. Levine, p-integral harmonic sums, *Discrete Math.* **91** (3) (1991), 249–257.

[8] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.

[9] R. Hofer, G. Larcher, G., and F. Pillichshammer, Average growth-behavior and distribution properties of generalized weighted digit-block-counting functions, *Monatsh. Math.* **154** (3) (2008), 199–230.

[10] T. Komatsu and P. Young, Exact p-adic valuations of Stirling numbers of the first kind, *J. Number Theory* **177** (2017), 20–27.

[11] T. Lengyel, On divisibility properties of some differences of Motzkin numbers, *Ann. Math. Inform.* **41** (2013), 121-136 (in Proceedings of the 15th International Conference on Fibonacci Numbers and Their Applications, Institute of Mathematics and Informatics, Eszterházy Károly College, Eger, Hungary, June 25–30, 2012).

[12] T. Lengyel, Exact p-adic orders for differences of Motzkin numbers, *Int. J. Number Theory* **10** (3) (2014), 653-667.

[13] T. Lengyel, On divisibility properties of some differences of the central binomial coefficients and Catalan numbers, *Integers* **13** (2013), #A10, 20 pp.

[14] T. Lengyel, On the 2-adic order of Stirling numbers of the second kind and their differences, *Discrete Math. Theoret. Comput. Sci. Proceedings* 2009, vol. AK, 561-572 (in 21st International Conference on Formal Power Series and Algebraic Combinatorics, FPSAC 2009, Hagenberg, Austria), https://doi.org/10.46298/dmtcs.2694.

[15] T. Lengyel, On the p-adic properties of 2-sected sums involving binomial coefficients, *p-Adic Numbers Ultrametric Anal. Appl.* **15** (1) (2023), 23–47.

[16] OEIS Foundation Inc. (2021), The On-Line Encyclopedia of Integer Sequences, https://oeis.org.

[17] Kh. H. Pilehrood, T. H. Pilehrood and R. Tauraso, Multiple harmonic sums and multiple harmonic star sums are (nearly) never integers, *Integers* **17** (2017), #A10, 12 pp.

[18] E. Rowland, IntegerSequences, https://people.hofstra.edu/Eric_Rowland/packages.html#IntegerSequences or https://ericrowland.github.io/packages.html.

[19] C. Sanna, On the p-adic valuation of harmonic numbers, *J. Number Theory* **166** (2016), 41-46.

[20] Z. Shu, and J.-Y. Yao, Analytic functions over $\mathbb{Z}_p$ and p-regular sequences, *C. R. Acad. Sci. Paris, Ser. I* **349** (2011), 947–952.

[21] B-L. Wu and Y-G. Chen, On certain properties of harmonic numbers, *J. Number Theory* **175** (2017), 66-86.

[22] B-L. Wu and Y-G. Chen, On the denominators of harmonic numbers, *C. R. Math. Acad. Sci. Paris* **356** (2) (2018), 129-132.