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THE GENERAL DIVISOR PROBLEM OF CUSP FORM COEFFICIENTS OVER ARITHMETIC PROGRESSIONS

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Abstract

Let f be a normalized primitive holomorphic cusp form of even integral weight k for the full modular group $\Gamma = SL(2,\mathbb{Z})$, and denote by $\lambda_f(n)$ the nth normalized Fourier coefficient of f. Let $\lambda_{f\times f}(n)$ be the nth normalized coefficient of the Dirichlet expansion of the Rankin-Selberg L-function $L(f\times f,s)$ associated with f. In this paper, we investigate the asymptotic behavior of the sum

$$\sum_{\substack{n\leq x\\ n\equiv l(\bmod q)}} \lambda^2_{f\times f}(n),$$

where q is a prime with $(l,q)=1$. In a similar manner, we also consider the general divisor problem of the coefficients of Rankin-Selberg L-functions associated with f over the set of arithmetic progressions.

1. Introduction

The Fourier coefficients of modular forms are important and interesting objects in number theory. Let H_k^* be the set of all normalized primitive holomorphic cusp forms of even integral weight $k\geq 2$ for the full modular group $\Gamma = SL(2,\mathbb{Z})$. Then the Hecke eigenform $f(z)\in H_k^*$ has the Fourier expansion at the cusp ∞:

$$f(z)=\sum_{n=1}^{\infty}\lambda_f(n)n^{\frac{k-1}{2}}e(nz), \quad \Im(z)>0,$$

where $e(z)=e^{2\pi iz}$, and $\lambda_f(n)$ is the n-th normalized Fourier coefficient (Hecke eigenvalue) such that $\lambda_f(1)=1$. Then $\lambda_f(n)$ is real and satisfies the multiplicative

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property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right), \tag{1}$$

where $m \geq 1$ and $n \geq 1$ are positive integers. In 1974, P. Deligne [8] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leq d(n), \tag{2}$$

where $d(n)$ is the classical divisor function. By Equation (2), Deligne's bound is equivalent to the fact that there exist $\alpha_f(p), \beta_f(p) \in \mathbb{C}$ satisfying

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{3}$$

More generally, for all positive integers $l \geq 1$, one has

$$\lambda_f(p^l) = \alpha_f(p)^l + \alpha_f(p)^{l-1}\beta_f(p) + \cdots + \alpha_f(p)\beta_f(p)^{l-1} + \beta_f(p)^l.$$

In 1927, Hecke [11] proved that

$$\sum_{n \leq x} \lambda_f(n) \ll x^{\frac{1}{2}}. \tag{4}$$

Later, the upper bound in Equation (4) was improved by several authors (see, for example, [8, 14, 46]). In particular, Wu [49] has shown that

$$\sum_{n \leq x} \lambda_f(n) \ll x^{\frac{1}{3}} \log^{\rho} x,$$

where

$$\rho = \frac{102 + 7\sqrt{21}}{210}\left(\frac{6 - \sqrt{21}}{5}\right)^{\frac{1}{2}} + \frac{102 - 7\sqrt{21}}{210}\left(\frac{6 + \sqrt{21}}{5}\right)^{\frac{1}{2}} - \frac{33}{35} = -0.118\cdots.$$

Using the Sato-Tate conjecture proved by T. Barnet-Lamb et al. [1], one has the best result to date that

$$\sum_{n \leq x} \lambda_f(n) \ll x^{\frac{1}{3}} (\log x)^{-(1 - \frac{8}{3\pi})}.$$

In the 1930s, Rankin [45] and Selberg [47] independently proved the following asymptotic formula,

$$\sum_{n \leq x} \lambda_f^2(n) = c_f x + O(x^{3/5}), \tag{5}$$

where $c_f > 0$ is a positive constant depending on f, and $\varepsilon > 0$ is an arbitrarily small positive number. Recently, the exponent in Equation (5) was improved to $\frac{3}{5} - \delta$ in place of $\frac{3}{5}$ by Huang [16] , where $\delta \leq 1/560$. More recently, motivated by the work of Kowalski, Lin, and Michel [28] concerning the Rankin-Selberg coefficients in large arithmetic progressions, Huang [17] further sharpened the exponent of the result to $\frac{3}{5} - \delta_2$ with $\delta_2 = \frac{3}{305}$. This remains the best possible result in this direction.

Later, based on the work about symmetric power L-functions, Moreno and Shahidi [40] were able to prove

$$\sum_{n\leq x} \tau_0^4(n) \sim c_1 x \log x, \quad x \to \infty,$$

where $\tau_0(n) = \tau(n)/n^{\frac{11}{2}}$ is the normalized Ramanujan tau-function, and $c_1 > 0$ is a positive constant. Obviously, Moreno and Shahidi's result also holds true if we replace $\tau_0(n)$ with the normalized Fourier coefficient $\lambda_f(n)$.

Let $f \in H_k^*$ be a Hecke eigenform and denote its nth normalized Fourier coefficient by $\lambda_f(n)$. Define

$$S_j(f;x) = \sum_{n\leq x} \lambda_f^j(n),$$

where $j \in \mathbb{Z}^+$ and $x \geq 1$.

Based on the work of Moreno and Shahidi concerning the symmetric power L-functions $L(\mathrm{sym}^j f, s)$ for $j = 1, 2, 3, 4$, Fomenko [9] established the estimate

$$S_3(f;x) \ll_{f,\varepsilon} x^{5/6+\varepsilon}, \quad S_4(f;x) = c_f x \log x + d_f x + O_{f,\varepsilon}(x^{9/10+\varepsilon}),$$

where $c_f > 0$ and d_f are suitable constants depending on f; here ε is an arbitrarily small positive number. Later, Lü (see, e.g., [29, 30, 31]) considered higher moments $S_j(f;x)$ for $3 \leq l \leq 8$, which improved and generalized the work of Fomenko. Later, Lau, Lü, and Wu [32] proved that

$$S_j(f;x) = xP_j^*(\log x) + O_{f,\varepsilon}\big(x^{\theta_j+\varepsilon}\big), \quad 3 \leq j \leq 8,$$

where $P_j^*(t) \equiv 0$ are the constant functions for $j = 3, 5, 7$, and $P_4^*(t), P_6^*(t), P_8^*(t)$ are polynomials of degree $1, 4, 13$, respectively, and

$$\begin{aligned} \theta_3 &= \frac{7}{10}, \qquad \theta_5 = \frac{40}{43}, \qquad \theta_7 = \frac{176}{179}, \\ \theta_4 &= \frac{151}{175}, \qquad \theta_6 = \frac{175}{181}, \qquad \theta_8 = \frac{2933}{2957}. \end{aligned}$$

Lau and Lü [33] derived the general results for $S_j(f;x)$ for all $j \geq 2$ under the assumption that $L(\mathrm{sym}^l f, s)$ is automorphic cuspidal for some positive l. Now we know that $L(\mathrm{sym}^j f, s)$ is automorphic for all $j \geq 1$ due to the recent celebrated works of Newton and Thorne [42, 43].

Andrianov and Fomenko [3] firstly considered the second power sum of $\lambda_f(n)$ over arithmetic progressions for holomorphic cusp forms. Later, Andrianov [2] improved the error term. Ichihara [19, 20] has investigated $\lambda_f^2(n)$ over arithmetic progresssions for holomorphic cusp forms for $x \ll q^2$:

$$\begin{aligned}\sum_{\substack{n\le x\\ n\equiv l(\bmod q)}} \lambda_f^2(n) &= \frac{c}{\varphi(q)}\prod_{p|q}(1-\alpha_f(p)^2p^{-1})(1-p^{-1})(1-\beta_f(p)^2)(1+p^{-1})^{-1}x\\ &\quad +O_{f,\varepsilon}(x^{\frac{3}{5}}q^{\frac{4}{5}+\varepsilon}),\end{aligned}$$

where c is some suitable constant depending on f, and $\alpha_f(p), \beta_f(p)$ are the Satake parameters given by Equation (3). Later, Jiang and Lü [22] considered the sum of $\lambda_f^{2j}(n)$ over arithmetic progressions for $j = 2, 3, 4$. In a similar manner, they also established the corresponding results for the normalized Hecke-Maass cusp form with respect to $SL(2, \mathbb{Z})$ for $j = 2, 3, 4$.

Very recently, Zou et al. [52], by using the existence of automorphic cuspidal self-dual representation $\mathrm{sym}^j\pi_f$ for all $j \ge 1$ due to Newton and Thorne [42, 43] in combination with some nice properties of the corresponding automorphic L-functions, established the following result.

Theorem 1.1 ([52, Theorem 1]). *Let $f \in H_k^*$ be a Hecke eigenform. Let q be a prime with $(q, l) = 1$. For $j \ge 2$ and $q \le x^{\frac{3}{4}\delta_j}$, one has*

$$\sum_{\substack{n\le x\\ n\equiv l(\bmod q)}} \lambda_f^2(n^j) = \frac{c_j x}{\varphi(q)} + O_{f,\varepsilon}\big(qx^{1-\frac{3}{2}\delta_j+\varepsilon}\big)$$

for any $\varepsilon > 0$, where $c_j > 0$ are some suitable constants, and $\delta_2 = \frac{92}{597}$ and $\delta_j = \frac{92}{69(j-1)(j+3)+247}$ for $j \ge 3$.

Let $f \in H_k^*$ be a Hecke eigenform, and the Rankin-Selberg L-function $L(f \times f, s)$ associated with f is defined as

$$L(f\times f,s)=\prod_p\left(1-\frac{\alpha_f(p)^2}{p^s}\right)^{-1}\left(1-\frac{1}{p^s}\right)^{-2}\left(1-\frac{\beta_f(p)^2}{p^s}\right)^{-1}.$$

Then we can rewrite $L(f \times f, s)$ as

$$L(f\times f,s)=\zeta(2s)\sum_{n=1}^{\infty}\frac{\lambda_f(n)^2}{n^s} := \sum_{n=1}^{\infty}\frac{\lambda_{f\times f}(n)}{n^s}.$$

For any given integer $w \ge 1$, we write

$$L(f\times f,s)^w=\sum_{n=1}^{\infty}\frac{\lambda_{w,f\times f}(n)}{n^s}$$

for $\Re(s) > 1$. Then

$$\lambda_{w,f\times f}(n) = \sum_{n=n_1\ldots n_w} \lambda_{f\times f}(n_1)\ldots\lambda_{f\times f}(n_w).$$

In particular, $\lambda_{1,f\times f}(n) = \lambda_{f\times f}(n)$.

Define

$$U_w(x) := \sum_{n\leq x} \lambda_{w,f\times f}(n).$$

A classical problem is to investigate the asymptotic behaviour of $U_w(x)$, which can be regarded as the general divisor problem considered by Kanemitsu, Sankaranarayanan, and Tanigawa [24]. When $w = 1$, Rankin [45] and Selberg [47] independently proved that

$$U_1(x) = C_f x + O_f(x^{\frac{3}{5}}),$$

where C_f is some suitable constant depending on f. For $w \geq 2$, Kanemitsu, Sankaranarayanan, and Tanigawa [24] showed that

$$U_w(x) = M_w(x) + O_{f,\varepsilon}\big(x^{1-\frac{1}{2w}+\varepsilon}\big), \tag{6}$$

where $M_w(x)$ denotes the residue of the function $\frac{L(f\times f,s)^w}{s}x^s$ at $s = 1$, which takes the form of $xP_{w-1}(\log x)$, where $P_{w-1}(t)$ denotes a polynomial of t with degree $w - 1$.

In [36], Liu and Zhang established the asymptotic formula

$$\sum_{n\leq x} \lambda_{f\times f}(n)^2 = xP(\log x) + O_{f,\varepsilon}\big(x^{\frac{6}{7}+\varepsilon}\big),$$

where $P(t)$ is a polynomial of t with degree 1, and they also proved that

$$U_w(x) = \begin{cases} M_w(x) + O_{f,\varepsilon}\big(x^{1-\frac{84}{131w+42}+\varepsilon}\big), & \text{if } 2 \leq w \leq 5, \\ M_w(x) + O_{f,\varepsilon}\big(x^{1-\frac{84}{131w+33}+\varepsilon}\big), & \text{if } 6 \leq w \leq 11, \\ M_w(x) + O_{f,\varepsilon}\big(x^{1-\frac{84}{131w+24}+\varepsilon}\big), & \text{if } w \geq 12, \end{cases}$$

where $M_w(x)$ is defined as in Equation (6). For more results in this direction, the interested readers can refer to [34, 35, 50, 38, 51].

Inspired by the above results, for $j \geq 1$, in this paper we firstly consider the distribution of the coefficients $\lambda^2_{f\times f}(n)$ over arithmetic progressions by adopting the similar approach given by Zou et al. [52] and the celebrated work of Newton and Thorne [42, 43]. More precisely, we establish the following result.

Theorem 1.2. *Let $f \in H_k^*$ be a Hecke eigenform, and let q be a prime with $(q,l) = 1$. For $j \geq 1$ and $q \ll x^\eta$, one has*

$$\sum_{\substack{n\leq x\\ n\equiv l(\bmod q)}} \lambda^2_{f\times f}(n) = \frac{x}{\varphi(q)}P(\log x) + O_{f,\varepsilon}\big(qx^{1-\frac{3}{2}\eta+\varepsilon}\big)$$

for any $\varepsilon > 0$*, where* $\eta = \frac{23}{261}$£¬ *and* $P(t)$ *is a polynomial in* t *of degree* 1 *with leading positive coefficient.*

By using a similar approach as that of Theorem 1.2, along with some nice properties of the associated L-functions, we also establish the following theorem.

Theorem 1.3. *Let* $f \in H_k^*$ *be a Hecke eigenform, and let* q *be a prime with* $(q,l) = 1$*. For* $w \geq 2$ *and* $q \ll x^{\vartheta_w}$*, one has*

$$\sum_{\substack{n \leq x \\ n \equiv l (\bmod q)}} \lambda_{w, f \times f}(n) = \frac{x}{\varphi(q)} P_{w-1}(\log x) + O_{f,\varepsilon}\big(q x^{1 - \frac{3}{2}\vartheta_w + \varepsilon}\big)$$

for any $\varepsilon > 0$*, where* $\vartheta_w = \frac{92}{247w+10}$*, and* $P_{w-1}(t)$ *is a polynomial in* t *of degree* $w-1$ *with positive leading coefficient.*

Throughout the paper, we always assume that $f \in H_k^*$ is a Hecke eigenform. Let $\varepsilon > 0$ be an arbitrarily small positive constant that may vary in different occurrence. The symbols p and q always denotes prime numbers.

2. Preliminaries

In this section, we introduce some background on the analytic properties of automorphic L-functions and give some useful lemmas which play an important role in the proof of the main results in this paper.

Let $f \in H_k^*$ be a Hecke eigenform, and let $\lambda_f(n)$ denote its n-th normalized Fourier coefficient. It is natural to define the *Hecke* L*-function* $L(f,s)$ associated to f by

$$\begin{aligned} L(f,s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p (1 - \lambda_f(p) p^{-s} + p^{-2s})^{-1} \\ &= \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}, \ \Re(s) > 1, \end{aligned}$$

where $\alpha_f(p), \beta_f(p)$ are the local parameters satisfying (3). The j*-th symmetric power* L*-function associated with* f is defined by

$$L(\mathrm{sym}^j f, s) = \prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)^{j-m} \beta_f(p)^m p^{-s})^{-1}, \ \Re(s) > 1.$$

We may expand it into a Dirichlet series:

$$
\begin{aligned}
L(\mathrm{sym}^j f, s) &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s}, \\
&= \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}} + \cdots\right), \ \Re(s) > 1. \quad (7)
\end{aligned}
$$

Apparently, $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. In particular, for $j = 1$, we have $L(\mathrm{sym}^1 f, s) = L(f, s)$.

It is standard to find that

$$
\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \frac{\alpha_f(p)^{j+1} - \beta_f(p)^{j+1}}{\alpha_f(p) - \beta_f(p)} = \sum_{m=0}^{j} \alpha_f(p)^{j-m} \beta_f(p)^m,
$$

which can be rewritten as

$$
\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = U_j(\lambda_f(p)/2),
$$

where $U_j(x)$ is the j-th Chebyshev polynomial of the second kind. It is not hard to find that

$$
|\lambda_{\mathrm{sym}^j f}(n)| \leq d_{j+1}(n),
$$

where $d_{j+1}(n)$ denotes the number of representations of n as the product of $j + 1$ positive integers, which can also be regarded as the Dirichlet coefficients of $\zeta(s)^{j+1}$; here, as usual, $\zeta(s)$ denotes the classical Riemann zeta function.

Let χ be a Dirichlet character modulo q. Then, we can define the twisted jth symmetric power L-function by the Euler product representation with degree $j + 1$

$$
\begin{aligned}
L(\mathrm{sym}^j f \otimes \chi, s) &= \prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)^{j-m} \beta_f(p)^m \chi(p) p^{-s})^{-1} \\
&= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n) \chi(n)}{n^s}
\end{aligned}
$$

for $\Re(s) > 1$.

As is well-known, to a primitive form f is associated an automorphic cuspidal representation π_f of $GL_2(\mathbb{A}_{\mathbb{Q}})$, and hence an automorphic L-function $L(\pi_f, s)$ which coincides with $L(f, s)$. It is predicted that π_f gives rise to a symmetric power lift–an automorphic representation whose L-function is the symmetric power L-function attached to f.

For $1 \leq j \leq 8$, the Langlands functoriality conjecture, which states that $\mathrm{sym}^j f$ is automorphic cuspidal, was proven by a series of important work of Gelbart and

Jacquet [10], Kim [27], Kim and Shahidi [26, 25], Shahidi [48], and Clozel and Thorne [5, 6, 7]. Very recently, Newton and Thorne [42, 43] proved that $\mathrm{sym}^j f$ corresponds with a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ for all $j \geq 1$ (with f being a holomorphic cusp form). Then we know that $L(\mathrm{sym}^j f, s)$, $j \geq 1$, has the analytic continuation to the whole complex plane as an entire function and satisfies a certain Riemann-type functional equation. We refer the interested readers to [21, Chapter 5] for a more comprehensive treatment.

Lemma 2.1. *Let $f \in H_k^*$ be a distinct Hecke eigenform, and let χ be a primitive character modulo a prime q. Then the complete L-function*

$$\Lambda(sym^i f \otimes \chi, s) := q^{(i+1)s/2}\gamma(s)L(sym^i f \otimes \chi, s)$$

can be extended to the whole complex plane as an entire function, and satisfies the functional equation

$$\Lambda(sym^i f \otimes \chi, s) = \varepsilon(f, \chi)\Lambda(sym^i f \otimes \bar{\chi}, 1 - s),$$

where $i \geq 1$ and $|\varepsilon(f, \chi)| = 1$. Here, $\gamma(s)$ denotes the product of some gamma functions $\Gamma((s + \kappa_n))/2, n = 1, 2, \dots, (i + 1)$, with κ_n depending on the weight of f and the parity of the character χ and $\Re(\kappa_n) \geq 0$.

Proof. This can be deduced by a similar argument as done by Zou et al. [52]. □

Lemma 2.2. ([36, Lemma 2.2]) *For $\Re(s) > 1$, define*

$$F_1(s) = \sum_{n=1}^{\infty} \frac{\lambda_{f\times f}^2(n)}{n^s}.$$

Then we have

$$F_1(s) = \zeta^2(s)L^3(sym^2 f, s)L(sym^4 f, s)U(s),$$

where the function $U(s)$ admits the Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \geq \frac{1}{2} + \varepsilon$, and $U(s) \neq 0$ for $\Re(s) = 1$.

Let $f \in H_k^*$ be a Hecke eigenform, and let χ be the Dirichlet character modulo q. Define

$$F_1(s, \chi) = \sum_{n=1}^{\infty} \frac{\lambda_{f\times f}^2(n)\chi(n)}{n^s}. \tag{8}$$

Lemma 2.3. *Let $F_1(s, \chi)$ be defined by (8), then*

$$F_1(s, \chi) = L^2(s, \chi)L^3(sym^2 f \otimes \chi, s)L(sym^4 f \otimes \chi, s)\widetilde{U}(s, \chi),$$

where $\widetilde{U}(s, \chi)$ admits the Dirichlet series which converges absolutely for $\Re(s) \geq \frac{1}{2}+\varepsilon$, and the convergence for all cases is uniform in q.

Proof. This follows a similar approach as that of [52, Lemma 10] by using Lemma 2.2. □

Lemma 2.4. *For any $\varepsilon > 0$, we have*

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \ll T^{2+\varepsilon}, \tag{9}$$

uniformly for $T \geq 1$, and

$$\zeta(\sigma+it) \ll \big(1+|t|\big)^{\max\{\frac{13}{42}(1-\sigma),0\}+\varepsilon} \tag{10}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1+\varepsilon$ and $|t| \geq 1$.

Proof. The first result is given by Heath-Brown [12] and the second result is the recent breakthrough due to Bourgain [4, Theorem 5]. □

Lemma 2.5. *For any $\varepsilon > 0$, we have*

$$L(sym^2 f, \sigma+it) \ll \big(1+|t|\big)^{\max\{\frac{6}{5}(1-\sigma),0\}+\varepsilon} \tag{11}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1+\varepsilon$ and $|t| \geq 1$.

Proof. From the result given by Lin et al. [39, Corollary 1.2], we can easily deduce that

$$L\left(\mathrm{sym}^2 f, \frac{1}{2}+it\right) \ll \big(1+|t|\big)^{\frac{3}{5}+\varepsilon}. \tag{12}$$

We can obtain the required result from the Phragmén-Lindelöf principle for a strip [21, Theorem 5.53] and the Equation (12). □

Lemma 2.6. *Let χ be a primitive character modulo q. For $T \geq 1$ and $q \ll T^2$,*

$$L(\sigma+iT,\chi) \ll_\varepsilon (q(1+|T|))^{\max\{\frac{1}{3}(1-\sigma),0\}+\varepsilon} \tag{13}$$

$$L(\sigma+iT, sym^2 f \otimes \chi) \ll_\varepsilon (q(1+|T|))^{\max\{\frac{67}{46}(1-\sigma),0\}+\varepsilon}, \tag{14}$$

and further for q is a prime,

$$\int_0^T |L(\sigma+it,\chi)|^{12} dt \ll_\varepsilon q^{4(1-\sigma)} T^{3-2\sigma+\varepsilon}. \tag{15}$$

Proof. The results follow from the work of Heath-Brown [13], Huang [15], and Motohashi [41], together with the Phragmén-Lindelöf principle for a strip, respectively. □

From above, we note that the automorphic L-functions $L(\mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^j f \otimes \chi, s)$ are the general L-functions in the sense of Perelli [44]. For these L-functions, we have the following individual or averaged convexity bounds.

Lemma 2.7. *Let χ be a primitive character modulo q. For the general L-functions $\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(s,\chi)$ of degree $2A$ indicated above, we have*

$$\int_T^{2T} |\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(\sigma+it,\chi)|^2 dt \ll (qT)^{2A(1-\sigma)+\varepsilon} \tag{16}$$

uniformly for $\frac{1}{2} \le \sigma \le 1$ and $T \ge 1$. Furthermore,

$$\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(\sigma+it,\chi) \ll (q(|t|+1))^{\max\{A(1-\sigma),0\}+\varepsilon} \tag{17}$$

uniformly for $-\varepsilon \le \sigma \le 1+\varepsilon$.

Proof. This can be derived by following a similar argument as in Zou et al. [52], which was originally deduced from Jiang and Lü [22]. □

Remark 2.8. For the automorphic L-functions $L(\mathrm{sym}^j f, s), j \ge 1$ we can regard the modulus q to be 1.

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we firstly consider the sum $\sum_{n\le x} \lambda^2_{f\times f}(n)\chi(n)$, where χ is a primitive character modulo a prime q.

Proposition 3.1. *Let $f \in H_k^*$ be a Hecke eigenform, and let χ be a primitive character modulo a prime q. For any $\varepsilon > 0$ and $q \ll x^\eta$, we have*

$$\sum_{n\le x} \lambda^2_{f\times f}(n)\chi(n) = O_{f,\varepsilon}\big(qx^{1-\frac{3}{2}\eta+\varepsilon}\big), \tag{18}$$

where $\eta = \frac{23}{261}$.

Proof. Applying Perron's formula [21, Proposition 5.54] to the generating function $F_1(s,\chi)$ appearing in Lemma 2.3, and using Deligne's bound, we obtain

$$\sum_{n\le x} \lambda^2_{f\times f}(n)\chi(n) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} F_1(s,\chi)\frac{x^s}{s}ds + O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where $s = \sigma + it$, and $1 \le T \le x$ is a parameter to be chosen later.

By shifting the line of integration to the parallel line with $\Re(s) = \frac{1}{2} + \varepsilon$, and invoking Cauchy's residue theorem, we get

$$\begin{aligned}\sum_{n\le x}\lambda^2_{f\times f}(n)\chi(n) &= \frac{1}{2\pi i}\left\{\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}+\int_{1+\varepsilon-iT}^{\frac{1}{2}+\varepsilon-iT}+\int_{\frac{1}{2}+\varepsilon+iT}^{1+\varepsilon+iT}\right\}F_1(s,\chi)\frac{x^s}{s}ds\\ &\quad +O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right)\\ &:= I_{1,1}+I_{1,2}+I_{1,3}+O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right). \qquad (19)\end{aligned}$$

For the integral over the vertical segment, by Equations (14), (15) and (16), together with Hölder's inequality, for $q \ll T^2$, it follows that

$$\begin{aligned}I_{1,1} &\ll x^{\frac{1}{2}+\varepsilon}\log T\sup_{1\le T_1\le T/2}\left\{\frac{1}{T_1}\int_{T_1}^{2T_1}\left|F_1(\sigma+it,\chi)\right|dt\right\}\\ &\ll x^{\frac{1}{2}+\varepsilon}\log T\sup_{1\le T_1\le T/2}\left\{\frac{1}{T_1}\left(\int_{T_1}^{2T_1}\left|L^2\left(\frac{1}{2}+it,\chi\right)\right|^6dt\right)^{\frac{1}{6}}\right.\\ &\quad\times\left(\int_{T_1}^{2T_1}\left|L^3\left(\mathrm{sym}^2f\otimes\chi,\frac{1}{2}+it\right)\right|^3dt\right)^{\frac{1}{3}}\\ &\quad\left.\times\left(\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^4f\otimes\chi,\frac{1}{2}+it\right)\right|^2dt\right)^{\frac{1}{2}}\right\}\\ &\ll x^{\frac{1}{2}+\varepsilon}\log T\sup_{1\le T_1\le T/2}\left\{\frac{1}{T_1}\left(\int_{T_1}^{2T_1}\left|L^2\left(\frac{1}{2}+it,\chi\right)\right|^6dt\right)^{\frac{1}{6}}\right.\\ &\quad\times\left(\max_{T_1\le t\le 2T_1}\left|L\left(\mathrm{sym}^2f\otimes\chi,\frac{1}{2}+it\right)\right|^7\right.\\ &\quad\left.\times\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^2f\otimes\chi,\frac{1}{2}+it\right)\right|^2dt\right)^{\frac{1}{3}}\\ &\quad\left.\times\left(\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^4f\otimes\chi,\frac{1}{2}+it\right)\right|^2dt\right)^{\frac{1}{2}}\right\}\\ &\ll x^{\frac{1}{2}+\varepsilon}q^{\frac{87}{23}+\varepsilon}T^{\frac{64}{23}+\varepsilon}. \qquad (20)\end{aligned}$$

For the integrals over the horizontal segments, by Equations (13), (14) and (17), we have

$$\begin{aligned}I_{1,2}+!`!`I_{1,3} &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon}x^\sigma\left|F_1(\sigma+it,\chi)\right|T^{-1}d\sigma\\ &\ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon}x^\sigma(qT)^{(\frac{1}{3}\times 2+3\times\frac{67}{46}+\frac{5}{2})(1-\sigma)+\varepsilon}T^{-1}\\ &\ll \frac{x^{1+\varepsilon}}{T}+x^{\frac{1}{2}+\varepsilon}q^{\frac{260}{69}}T^{\frac{191}{69}}. \qquad (21)\end{aligned}$$

Combining Equations (19)-(21) and taking $T = x^{\frac{23}{174}}/q$, we get (18). Since $q \ll T^2$, we have $q \ll x^{\frac{23}{261}}$. This proves the desired result. $\square$

Proposition 3.2. *Let $f \in H_k^*$ be a Hecke eigenform, and let χ be a principal character modulo a prime q. For any $\varepsilon > 0$ and $q \ll x$, we have*

$$\sum_{n \leq x} \lambda_{f\times f}^2(n)\chi(n) = xP(\log x) + O_{f,\varepsilon}\big(x^{\tilde{\eta}+\varepsilon}\big), \tag{22}$$

where $\tilde{\eta} = \frac{179}{209}$, and $P(t)$ is a polynomial in t of degree 1 with leading positive coefficient.

Proof. From Lemmas 2.2-2.3, we know that

$$\begin{aligned} F_1(s,\chi_0) &:= \sum_{n=1}^{\infty} \frac{\lambda_{f\times f}^2(n)\chi_0(n)}{n^s} \\ &= F_1(s)H_1(s,\chi), \end{aligned} \tag{23}$$

where $H_1(s,\chi)$ is a Dirichlet series which converges absolutely for $\Re(s) \geq \frac{1}{2}+\varepsilon$ and uniformly in q. Applying Perron's formula and invoking Cauchy's residue theorem, we obtain

$$\begin{aligned} \sum_{n \leq x} \lambda_{f\times f}^2(n)\chi(n) &= \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} F_1(s,\chi_0)\frac{x^s}{s}ds + O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right) \\ &= \frac{1}{2\pi i}\left\{\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} + \int_{1+\varepsilon-iT}^{\frac{1}{2}+\varepsilon-iT} + \int_{\frac{1}{2}+\varepsilon+iT}^{1+\varepsilon+iT}\right\}F_1(s,\chi_0)\frac{x^s}{s}ds \\ &\quad +\mathrm{Res}_{s=1}\left(F_1(s,\chi_0)\frac{x^s}{s}\right) + O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right) \\ &:= I_{2,1} + I_{2,2} + I_{2,3} + xP(\log x) + O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right), \end{aligned} \tag{24}$$

where $P(t)$ is a polynomial in t of degree 1 with leading positive coefficient. Here, due to the holomorphy of $L(\mathrm{sym}^2 f, s)$ and $L(\mathrm{sym}^4, s)$ at $s = 1$, the main term $xP(\log x)$ is derived from the residue of $F_1(s,\chi_0)\frac{x^s}{s}$ at the pole $s = 1$ of order 2, coming from the factor $\zeta(s)^2$.

Now we begin to handle the three terms $I_{2,1}, I_{2,2}$ and $I_{2,3}$. For $I_{2,1}$, by Equations

(9), (17), and Lemma 2.5, and Hölder's inequality, we have

$$\begin{aligned}
I_{2,1} &\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\le T_1\le T/2}\left\{\frac{1}{T_1}\int_{T_1}^{2T_1}\big|F_1(\sigma+it)\big|dt\right\}\\
&\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\le T_1\le T/2}\left\{\frac{1}{T_1}\left(\int_{T_1}^{2T_1}\left|\zeta^2\left(\frac{1}{2}+it\right)\right|^6dt\right)^{\frac{1}{6}}\right.\\
&\quad\times\left(\max_{T_1\le t\le 2T_1}\left|L\left(\mathrm{sym}^2 f,\frac{1}{2}+it\right)\right|^7\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^2 f,\frac{1}{2}+it\right)\right|^2dt\right)^{\frac{1}{3}}\\
&\quad\left.\times\left(\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^4 f,\frac{1}{2}+it\right)\right|^2dt\right)^{\frac{1}{2}}\right\}\\
&\ll x^{\frac{1}{2}+\varepsilon}T^{\frac{149}{60}+\varepsilon}. \qquad (25)
\end{aligned}$$

For $I_{2,2}$ and $I_{2,3}$, by Equations (10), (11) and (17), we have

$$\begin{aligned}
I_{2,2}+!‘!‘I_{2,3} &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^{\sigma}\big|F_1(\sigma+it)\big|T^{-1}d\sigma\\
&\ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} x^{\sigma}T^{(\frac{13}{42}\times 2+3\times\frac{6}{5}+\frac{5}{2})(1-\sigma)+\varepsilon}T^{-1}\\
&\ll \frac{x^{1+\varepsilon}}{T}+x^{\frac{1}{2}+\varepsilon}T^{\frac{991}{420}}. \qquad (26)
\end{aligned}$$

Putting together Equations (24)-(26), we obtain

$$\sum_{n\le x}\lambda^2_{f\times f}(n)\chi(n)=xP(\log x)+O_{f,\varepsilon}\big(x^{\frac{1}{2}+\varepsilon}T^{\frac{149}{60}+\varepsilon}\big)+O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right), \qquad (27)$$

On taking $T=x^{\frac{30}{209}}$ in Equation (27), we get

$$\sum_{n\le x}\lambda^2_{f\times f}(n)\chi(n)=xP(\log x)+O_{f,\varepsilon}\big(x^{\frac{179}{209}+\varepsilon}\big).$$

This completes the proof of the proposition. □

Proof of Theorem 1.2 Let χ be a Dirichlet character modulo a prime q. By the orthogonality of Dirichlet character, we get

$$\begin{aligned}
\sum_{\substack{n\le x\\ n\equiv l(\bmod q)}}\lambda^2_{f\times f}(n) &= \frac{1}{\varphi(q)}\sum_{\chi(\bmod q)}\bar{\chi}(l)\sum_{n\le x}\lambda^2_{f\times f}(n)\chi(n)\\
&= \frac{1}{\varphi(q)}\sum_{n\le x}\lambda^2_{f\times f}(n)\chi_0(n)+O\left(\sum_{n\le x}\lambda^2_{f\times f}(n)\chi(n)\right),
\end{aligned}$$

where $\varphi(q)$ is the Euler function and $\varphi(q)=q-1$.

From Equations (18) and (22), and noting that $1-\frac{3}{2}\eta>\tilde{\eta}$, we have

$$\sum_{\substack{n\le x\\ n\equiv l(\bmod q)}}\lambda^2_{f\times f}(n)=\frac{x}{\varphi(q)}P(\log x)+O_{f,\varepsilon}\big(qx^{1-\frac{3}{2}\eta+\varepsilon}\big),$$

where $P(t)$ is a polynomial in t of degree 1 with leading positive coefficient. This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

Now we are at the stage where we are able to give the proof of Theorem 1.3. Let $f\in H_k^*$ and let χ be a Dirichlet character modulo q. For $\Re(s)>1$, by [36, Lemma 2.1], we define

$$\begin{aligned}F_2(s,\chi) &:= \sum_{n=1}^{\infty}\frac{\lambda_{w,f\times f}(n)\chi(n)}{n^s}\\ &= L(f\times f\otimes\chi,s)^w G_1(s,\chi)\\ &= L(s,\chi)^w L(\mathrm{sym}^2 f\otimes\chi,s)^w G_1(s,\chi),\end{aligned}$$

where $G_1(s,\chi)$ is a Dirichlet series which converges absolutely for $\Re(s)\ge\frac{1}{2}+\varepsilon$ and uniformly in q.

Proposition 4.1. *Let $f\in H_k^*$ a Hecke eigenform, and let χ be a primitive character modulo a prime q. For any $\varepsilon>0, q\ll x^{\vartheta_w}$ and $w\ge 2$, we have*

$$\sum_{n\le x}\lambda_{w,f\times f}(n)\chi(n)=O_{f,\varepsilon}\big(qx^{1-\frac{3}{2}\vartheta_w+\varepsilon}\big),\tag{28}$$

where $\vartheta_w=\frac{92}{247w+10}$.

Proof. Applying Perron's formula to the generating function $F_2(s,\chi)$, shifting the line of integration to the parallel line with $\Re(s)=\frac{1}{2}+\varepsilon$, and invoking Cauchy's residue theorem, we obtain

$$\begin{aligned}\sum_{n\le x}\lambda_{w,f\times f}(n)\chi(n) &= \frac{1}{2\pi i}\bigg\{\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}+\int_{1+\varepsilon-iT}^{\frac{1}{2}+\varepsilon-iT}+\int_{\frac{1}{2}+\varepsilon+iT}^{1+\varepsilon+iT}\bigg\}F_2(s,\chi)\frac{x^s}{s}ds\\ &\quad +O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right)\\ &:= J_{1,1}+J_{1,2}+J_{1,3}+O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right),\end{aligned}\tag{29}$$

where $s=\sigma+it$, and $1\le T\le x$ is some parameter to be chosen later.

For $J_{1,1}$, by Lemma 2.6, Equation (16), and Hölder's inequality, we have

$$\begin{aligned}
J_{1,1} &\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\le T_1\le T/2}\left\{\frac{1}{T_1}\left(\max_{T_1\le t\le 2T_1}\left|L\left(\frac{1}{2}+it,\chi\right)\right|^{6w-12}\right.\right.\\
&\quad\times\int_{T_1}^{2T_1}\left|L\left(\frac{1}{2}+it,\chi\right)\right|^{12}dt\right)^{\frac{1}{6}}\left(\max_{T_1\le t\le 2T_1}\left|L\left(\mathrm{sym}^2 f\otimes\chi,\frac{1}{2}+it\right)\right|^{\frac{3}{2}w-2}\\
&\quad\times\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^2 f\otimes\chi,\frac{1}{2}+it\right)\right|^{2}dt\right)^{\frac{1}{3}}\\
&\quad\times\left(\max_{T_1\le t\le 2T_1}\left|L\left(\mathrm{sym}^2 f\otimes\chi,\frac{1}{2}+it\right)\right|^{w-2}\right.\\
&\quad\left.\left.\times\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^2 f\otimes\chi,\frac{1}{2}+it\right)\right|^{2}dt\right)^{\frac{1}{2}}\right\}\\
&\ll x^{\frac{1}{2}+\varepsilon}q^{\frac{247}{276}w+\frac{5}{138}+\varepsilon}T^{\frac{247}{276}w-\frac{133}{138}+\varepsilon}. \qquad (30)
\end{aligned}$$

For the integrals over the horizontal segments $J_{1,2}$ and $J_{1,3}$, by Equations (13) and (14), it follows that

$$\begin{aligned}
J_{1,2}+J_{1,3} &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^{\sigma}\big|F_2(\sigma+it,\chi)\big|T^{-1}d\sigma\\
&\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^{\sigma}\big|L(\sigma+it,\chi)L(\mathrm{sym}^2 f\otimes\chi,\sigma+it)\big|^{w}T^{-1}d\sigma\\
&\ll \frac{x^{1+\varepsilon}}{T}+x^{\frac{1}{2}+\varepsilon}q^{\frac{247}{276}w+\varepsilon}T^{\frac{247}{276}w-1+\varepsilon}. \qquad (31)
\end{aligned}$$

Combining Equations (29)-(31), we obtain (28) by setting $T=x^{\frac{138}{247w+10}}/q$. Since $q\ll T^2$, we have $q\ll x^{\frac{92}{247w+10}}$. □

Proposition 4.2. *Let $f\in H_k^*$ a Hecke eigenform, and let χ be a principal character modulo a prime q. For any $\varepsilon>0, q\ll x$ and $w\ge 2$, we have*

$$\sum_{n\le x}\lambda_{w,f\times f}(n)\chi_0(n)=xP_{w-1}(\log x)+O_{f,\varepsilon}\big(x^{\tilde{\vartheta}_w+\varepsilon}\big), \qquad (32)$$

where $\tilde{\vartheta}_w=1-\frac{210}{317w+115}$, and $P_{w-1}(t)$ is a polynomial of t with degree $w-1$.

Proof. For $\chi_0(n)$ being a principal character modulo q, we have

$$\begin{aligned}
F_2(s,\chi_0) &= \sum_{n=1}^{\infty}\frac{\lambda_{w,f\times f}(n)\chi_0(n)}{n^s}\\
&= \zeta(s)^w L(\mathrm{sym}^2 f,s)^w G_2(s,\chi),
\end{aligned}$$

where $G_2(s,\chi)$ is a Dirichlet series which converges absolutely for $\Re(s) \geq \frac{1}{2} + \varepsilon$ and uniformly in q.

Applying Perron's formula, moving the line of integration to the parallel line $\Re(s) = \frac{1}{2} + \varepsilon$, and invoking Cauchy's residue theorem, we obtain

$$\begin{aligned}\sum_{n\leq x} \lambda_{w,f\times f}(n)\chi_0(n) &= \frac{1}{2\pi i}\left\{\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} + \int_{1+\varepsilon-iT}^{\frac{1}{2}+\varepsilon-iT} + \int_{\frac{1}{2}+\varepsilon+iT}^{1+\varepsilon+iT}\right\} F_2(s,\chi_0)\frac{x^s}{s}ds \\ &\quad + \mathrm{Res}_{s=1}\left(F_2(s,\chi_0)\frac{x^s}{s}\right) + O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right) \\ &:= J_{2,1} + J_{2,2} + J_{2,3} + xP_{w-1}(\log x) + O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right), \quad (33)\end{aligned}$$

where $P_{w-1}(t)$ is a polynomial in t of degree $w-1$.

For the integral over the vertical segment $J_{2,1}$, by Lemmas 2.4-2.5, Equation (16), and Hölder's inequality, we get

$$\begin{aligned}J_{2,1} &\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\leq T_1\leq T/2}\left\{\frac{1}{T_1}\int_{T_1}^{2T_1}\left|F_2\left(\frac{1}{2}+it,\chi_0\right)\right|dt\right\} \\ &\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\leq T_1\leq T/2}\left\{\frac{1}{T_1}\left(\int_{T_1}^{2T_1}\left|\zeta\left(\frac{1}{2}+it\right)^w\right|^6 dt\right)^{\frac{1}{6}}\right. \\ &\quad \times\left(\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^2 f,\frac{1}{2}+it\right)^{\frac{w}{2}}\right|^3 dt\right)^{\frac{1}{3}} \\ &\quad \left.\times\left(\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^2 f,\frac{1}{2}+it\right)^{\frac{w}{2}}\right|^2 dt\right)^{\frac{1}{2}}\right\} \\ &\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\leq T_1\leq T/2}\left\{\frac{1}{T_1}\left(\max_{T_1\leq t\leq 2T_1}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{6w-12}\right.\right. \\ &\quad \left.\times\int_{T_1}^{2T_1}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{12}dt\right)^{\frac{1}{6}} \\ &\quad \times\left(\max_{T_1\leq t\leq 2T_1}\left|L\left(\mathrm{sym}^2 f,\frac{1}{2}+it\right)\right|^{\frac{3}{2}w-2}\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^2 f,\frac{1}{2}+it\right)\right|^2 dt\right)^{\frac{1}{3}} \\ &\quad \left.\times\left(\max_{T_1\leq t\leq 2T_1}\left|L\left(\mathrm{sym}^2 f,\frac{1}{2}+it\right)\right|^{w-2}\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^2 f,\frac{1}{2}+it\right)\right|^2 dt\right)^{\frac{1}{2}}\right\} \\ &\ll x^{\frac{1}{2}+\varepsilon}T^{\frac{317}{420}w-\frac{61}{84}+\varepsilon}. \quad (34)\end{aligned}$$

The estimates for the integrals $J_{2,2}$ and $J_{2,3}$ can be treated similarly. By Equations (10) and (11), we have

$$\begin{aligned} J_{2,2}+J_{2,3} &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^{\sigma}\big|F_2(\sigma+it,\chi_0)\big|T^{-1}d\sigma \\ &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^{\sigma}\big|\zeta(\sigma+it)L(\mathrm{sym}^2 f,\sigma+it)\big|^{w}T^{-1}d\sigma \\ &\ll \frac{x^{1+\varepsilon}}{T}+x^{\frac{1}{2}+\varepsilon}T^{\frac{317}{420}w-1+\varepsilon}. \end{aligned} \tag{35}$$

Putting together Equations (33)-(35), we obtain

$$\sum_{n\le x}\lambda_{w,f\times f}(n)\chi_0(n)=xP_{w-1}(\log x)+O_{f,\varepsilon}\big(x^{\frac{1}{2}+\varepsilon}T^{\frac{317}{420}w-\frac{61}{84}+\varepsilon}\big)+O_{f,\varepsilon}\bigg(\frac{x^{1+\varepsilon}}{T}\bigg). \tag{36}$$

On taking $T=x^{\frac{210}{317w+115}}$ in Equation (36), we get

$$\sum_{n\le x}\lambda_{w,f\times f}(n)\chi_0(n)=xP_{w-1}(\log x)+O_{f,\varepsilon}\big(x^{1-\frac{210}{317w+115}+\varepsilon}\big).$$

This proves the proposition. □

Proof of Theorem 1.3 Let χ be a Dirichlet character modulo a prime q. By the orthogonality of Dirichlet character, we get

$$\begin{aligned} \sum_{\substack{n\le x\\ n\equiv l(\bmod q)}}\lambda_{w,f\times f}(n) &= \frac{1}{\varphi(q)}\sum_{\chi(\bmod q)}\bar{\chi}(l)\sum_{n\le x}\lambda^2_{f\times f}(n)\chi(n) \\ &= \frac{1}{\varphi(q)}\sum_{n\le x}\lambda_{w,f\times f}(n)\chi_0(n)+O\bigg(\sum_{n\le x}\lambda_{w,f\times f}(n)\chi(n)\bigg), \end{aligned}$$

where $\varphi(q)$ is the Euler function and $\varphi(q)=q-1$.

From Equations (28) and (32), and noting $1-\frac{3}{2}\vartheta_w>\tilde{\vartheta}_w$, we have

$$\sum_{\substack{n\le x\\ n\equiv l(\bmod q)}}\lambda_{w,f\times f}(n)=\frac{x}{\varphi(q)}P_{w-1}(\log x)+O_{f,\varepsilon}\big(qx^{1-\frac{3}{2}\vartheta_w+\varepsilon}\big),$$

where $P_{w-1}(t)$ is a polynomial in t of degree $w-1$ with leading positive coefficient. This completes the proof. □

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