PRIMITIVE PYTHAGOREAN TRIANGLES WITH SIDES OF CERTAIN FORMS

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Received: 6/12/23, Revised: 10/2/23, Accepted: 3/5/24, Published: 3/15/24

Abstract

We show that there exist infinitely many primitive Pythagorean (resp. rational right) triangles with sides of certain forms, such as the sum of squares, the sum of two cubes, the sum (resp. difference) of the reciprocal of two positive integers.

1. Introduction

A *right triangle* is a triangle whose sides (X, Y, Z) satisfy the Diophantine equation

$$X^2 + Y^2 = Z^2.$$

A *primitive triangle* is an integral triangle such that the greatest common divisor of the lengths of its sides is 1. A right triangle is a *Pythagorean* (resp. *rational right*) *triangle* if the right triangle has integral (resp. rational) sides. A primitive Pythagorean triangle has sides

$$(X, Y, Z) = (m^2 - n^2,\ 2mn,\ m^2 + n^2),$$

where $m > n > 0$, $\gcd(m, n) = 1$, and $m + n \equiv 1 \pmod 2$.

DOI: 10.5281/zenodo.10821715

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[2]This research was supported by the National Natural Science Foundation of China (Grant No. 11501052), the Natural Science Foundation of Education Department of Hunan Province (Project No. 21C0211), the Natural Science Foundation of Hunan Province (Project No. 2021JJ30699 and No. 2022JJ40464), and Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering (Changsha University of Science and Technology).

From Fermat's Last Theorem, there is no Pythagorean triangle whose sides are positive integer powers. In 1961, Sierpiński [4] proved that there exist infinitely many Pythagorean triangles whose two legs (not the hypotenuse) are triangular numbers t_n, where $t_n = n(n+1)/2$. In 1964, Sierpiński [5] introduced a Pythagorean triangle (given by Zarankiewicz)

$$(X, Y, Z) = (t_{132}, t_{143}, t_{164}) = (8778, 10296, 13530), \tag{1}$$

whose three sides are triangular numbers.

In 2010, He, Togbé, and Ulas [3] considered the rational solutions of the Diophantine equations

$$z^2 = f(x)^2 \pm f(y)^2 \tag{2}$$

for some particular polynomials $f(x)$, which lead to right triangles with two sides of lengths $f(x)$ and $f(y)$. In [8, 9, 10, 12], many authors have done further research on the solutions of Equation (2).

In 1783, Euler [2, p. 167] gave sufficient conditions for a right triangle with sides $(x + 1/x, y + 1/y, z)$. In 2019, Zhang and Zargar [13] considered the non-trivial rational (parametric) solutions of Equation (2) for some simple Laurent polynomials, such as

$$f(x) = x + b + \frac{c}{x}, \quad \frac{(x+1)(x+b)(x+c)}{x},$$

with $b, c \in \mathbb{Z}\backslash\{0\}$. Further study of the solutions of Equation (2) for Laurent polynomials is included in [11].

In 2017, Tengely and Ulas [7] studied the integer solutions of the Diophantine equations

$$z^2 = f(x)^2 \pm g(y)^2$$

for some particular polynomials $f(x)$ and $g(y)$.

Sierpiński [6] gave the parametric solutions of a right triangle whose sides are $(1/x, 1/y, 1/z)$. In [2, p. 188], Turrière discussed the Pythagorean triangle $(9, 40, 41)$ each of whose sides is the sum of two squares, i.e.,

$$9 = 3^2 + 0^2, \quad 40 = 2^2 + 6^2, \quad 41 = 4^2 + 5^2.$$

In this paper, we generalize the example observed by Turrière and obtain the following theorem.

Theorem 1. *There exist infinitely many primitive Pythagorean triangles whose sides are*

$$(X, Y, Z) = \left(x^2 + y^2, p^2 + q^2, r^2 + s^2\right),$$

where x, y, p, q, r, s are positive integers.

Corollary 1. *There exist infinitely many primitive Pythagorean triangles whose sides are*

$$(X,Y,Z)=\left(x^2+y^2,p^2,r^2+s^2\right),$$

where x,y,p,r,s are positive integers.

Remark 1. Fermat proved that the Diophantine equation $x^4+y^4=z^2$ has no integer solutions, which means that there is no Pythagorean triangle whose sides are

$$(X,Y,Z)=\left(x^2,y^2,r^2+s^2\right),$$

where r^2+s^2 is the hypotenuse, and x,y,r,s are positive integers.

Remark 2. According to the identity

$$\left(\frac{A^2+B}{2A}\right)^2=\left(\frac{A^2-B}{2A}\right)^2+B, \tag{3}$$

one can easily see that every primitive Pythagorean triple $(m^2-n^2,2mn,m^2+n^2)$ is of the form

$$(X,Y,Z)=\left(x^2-y^2,p^2-q^2,r^2-s^2\right),$$

where $x>y,\ p>q$, and $r>s$ are positive integers. Indeed,

$$2mn=\left(\frac{mn}{2}+1\right)^2-\left(\frac{mn}{2}-1\right)^2,$$

$$m^2+n^2=\left(\frac{m^2+n^2+1}{2}\right)^2-\left(\frac{m^2+n^2-1}{2}\right)^2,$$

where $m>n>0,\ \gcd(m,n)=1$, and $m+n\equiv 1 \pmod 2$.

We further investigate other forms for the sides of a Pythagorean triangle and obtain the following theorems.

Theorem 2. *There are infinitely many primitive Pythagorean triangles whose sides are*

$$(X,Y,Z)=\left(x^3+y^3,p^3+q^3,r^3+s^3\right),$$

where x,y,p,q,r,s are positive integers.

Theorem 3. *There exist infinitely many rational right triangles whose sides are*

$$(X,Y,Z)=\left(\frac{1}{x}+\frac{1}{y},\frac{1}{p}+\frac{1}{q},\frac{1}{r}+\frac{1}{s}\right),$$

and there exist infinitely many rational right triangles whose sides are

$$(X,Y,Z)=\left(\frac{1}{x}-\frac{1}{y},\frac{1}{p}-\frac{1}{q},\frac{1}{r}-\frac{1}{s}\right),$$

where x,y,p,q,r,s are positive integers.

2. Proofs of the Theorems

In this section, we will present the proofs of the theorems and corollaries, and after each proof, we will provide a concrete example.

Proof of Theorem 1. We first consider the case of a rational right triangle with sides $(x^2+y^2, p^2+q^2, r^2+s^2)$, where x, y, p, q, r, s are positive rational numbers. By the Pythagorean theorem, we have

$$m^2 - n^2 = x^2 + y^2, \tag{4}$$

$$2mn = p^2 + q^2, \tag{5}$$

$$m^2 + n^2 = r^2 + s^2, \tag{6}$$

where $m > n$ are positive rational numbers.

We start by constructing a family of positive rational solutions of this system. For Equation (6), one can take $m = r = u$ and $n = s = v$. Then from Equation (4), we have

$$u^2 = x^2 + y^2 + v^2. \tag{7}$$

In order to find a solution of Equation (7), we use the identity

$$(2t^2+1)^2 = (2t^2)^2 + (2t)^2 + 1. \tag{8}$$

So one family of rational solutions of Equation (7) is given by setting

$$u = 2t^2+1, \quad v = 1, \quad x = 2t^2, \quad y = 2t,$$

where t is a positive rational number. Equation (5) now becomes

$$2(2t^2+1) = p^2 + q^2.$$

It follows from Equation (3) that we obtain

$$p = \frac{d^2+2-k^2}{2d}, \quad q = k, \quad t = \frac{d^2-2+k^2}{4d},$$

where d and k are integers. We thus have

$$\begin{aligned} u &= \frac{d^4+2d^2k^2+4d^2+k^4-4k^2+4}{8d^2}, \\ x &= \frac{(d^2-2+k^2)^2}{8d^2}, \\ y &= \frac{d^2-2+k^2}{2d}. \end{aligned}$$

To get sides of integral length, we multiply through by the least common denominator $8d^2$ to get the following family of solutions:

$$\begin{aligned} &m = d^4 + 2(k^2+2)d^2 + (k^2-2)^2, \quad n = 8d^2, \\ &x = (d^2 - 2 + k^2)^2, \quad y = 4d(d^2 - 2 + k^2), \\ &p = 4d(d^2 + 2 - k^2), \quad q = 8kd^2, \\ &r = d^4 + 2(k^2+2)d^2 + (k^2-2)^2, \quad s = 8d^2, \end{aligned} \tag{9}$$

where k and d are integers. Therefore, the sides of the Pythagorean triangles are given by

$$\begin{aligned} X &= (d^2+k^2-2)^2\left(d^4 + 2(k^2+6)d^2 + (k^2-2)^2\right), \\ Y &= 16d^2\left(d^4 + 2(k^2+2)d^2 + (k^2-2)^2\right), \\ Z &= d^8 + 4(k^2+2)d^6 + 2(3k^4+4k^2+44)d^4 + 4(k^2+2)(k^2-2)^2d^2 + (k^2-2)^4. \end{aligned}$$

We now specify conditions on d and k such that the Pythagorean triangles are primitive, that is, $m > n > 0$, $\gcd(m,n) = 1$, and $m + n \equiv 1 \pmod 2$. We have

$$m - n = (d^2 + k^2 - 2)^2,$$

$$m + n = d^4 + 2(k^2+6)d^2 + (k^2-2)^2 \equiv d^4 + k^4 \equiv d + k \pmod 2.$$

Therefore, if $d + k > 1$ and $d + k \equiv 1 \pmod 2$, then $m > n > 0$ and $m + n \equiv 1 \pmod 2$. Furthermore,

$$\begin{aligned} \gcd(m,n) &= \gcd\left(d^4 + 2(k^2+2)d^2 + (k^2-2)^2, 8d^2\right) \\ &= \gcd\left(d^4 + 2(k^2+2)d^2 + (k^2-2)^2, d^2\right) \\ &= \gcd\left((k^2-2)^2, d^2\right) \\ &= \left(\gcd\left(k^2-2, d\right)\right)^2. \end{aligned}$$

Thus, if

$$d + k > 1,\ d + k \equiv 1 \pmod 2,\ \gcd\left(k^2 - 2, d\right) = 1,\ d, k \in \mathbb{Z}, \tag{10}$$

then there exist infinitely many primitive Pythagorean triangles whose sides are

$$(X, Y, Z) = \left(x^2 + y^2, p^2 + q^2, r^2 + s^2\right),$$

where x, y, p, q, r, s are given by (9). Conditions (10) are possible, such as in each of the following cases:

(i) $k = 0,\ d > 1,\ d \equiv 1 \pmod 2$;

(ii) $d = 1,\ k > 0,\ k \equiv 0 \pmod 2$;

(iii) $d = 2^\alpha,\ \alpha > 0,\ k > 0,\ k \equiv 1 \pmod 2$. □

Example 1. When $d = 2,\ k = 1$, we get the primitive Pythagorean triangle with sides $(657, 2624, 2705)$ satisfying

$$657 = 9^2 + 24^2, \quad 2624 = 40^2 + 32^2, \quad 2705 = 41^2 + 32^2.$$

This provides a concrete example of Theorem 1.

Proof of Corollary 1. By setting $k = 0$ in (9), it follows that $q = 0$. The corresponding parameters now become

$$\begin{aligned} &m = d^4 + 4d^2 + 4, \quad n = 8d^2, \\ &x = (d^2 - 2)^2, \quad y = 4(d^2 - 2)d, \\ &p = 4(d^2 + 2)d, \quad q = 0, \\ &r = d^4 + 4d^2 + 4, \quad s = 8d^2. \end{aligned}$$

Therefore, the sides of the primitive Pythagorean triangles are given by

$$\begin{aligned} &X = (d^4 + 12d^2 + 4)(d^2 - 2)^2, \\ &Y = 16(d^2 + 2)^2 d^2, \\ &Z = d^8 + 8d^6 + 88d^4 + 32d^2 + 16, \end{aligned}$$

where $d > 1,\ d \in \mathbb{Z}$, and $d \equiv 1 \pmod 2$. □

Example 2. When $d = 3$, we get the primitive Pythagorean triangle with sides $(9457, 17424, 19825)$ satisfying

$$9457 = 49^2 + 84^2, \quad 17424 = 132^2 + 0^2, \quad 19825 = 121^2 + 72^2.$$

This gives a concrete example of Corollary 1.

Remark 3. The identity (8) can be derived from Equation (3) with $A = B = 4t^2 + 1$.

Proof of Theorem 2. For a primitive Pythagorean triangle with sides $(x^3 + y^3, p^3 + q^3, r^3 + s^3)$, where x, y, p, q, r, s are positive integers, by the Pythagorean theorem, we have

$$m^2 - n^2 = x^3 + y^3, \tag{11}$$

$$2mn = p^3 + q^3, \tag{12}$$

$$m^2 + n^2 = r^3 + s^3, \tag{13}$$

where $m > n$ are positive integers.

We start by constructing a family of positive integer solutions of this system. First, we consider Equation (13). One infinite family of solutions of Equation (13) is given by

$$m = t^3, \quad n = k^3, \quad r = t^2, \quad s = k^2,$$

where t and k are positive integers.

From Equation (11), we get

$$(t^2)^3 = x^3 + y^3 + (k^2)^3. \tag{14}$$

To find a solution of Equation (14), we use the following identity [1]:

$$(9a^4)^3 = (9a^4 - 3a)^3 + (9a^3 - 1)^3 + 1.$$

Then

$$t = 3a^2, \quad k = 1, \quad x = 9a^4 - 3a, \quad y = 9a^3 - 1,$$

where a is a positive integer.

For Equation (12), we have $54a^6 = p^3 + q^3$. Letting $p = q$, we obtain $p = q = 3a^2$. Further, by the values of t and k, we get

$$m = 27a^6, \quad n = 1, \quad r = 9a^4, \quad s = 1.$$

Therefore, the sides of the Pythagorean triangles are given by

$$(X, Y, Z) = \left(729a^{12} - 1, 54a^6, 729a^{12} + 1\right).$$

The Pythagorean triangles are primitive if $a \equiv 0 \pmod 2$. Hence, there are infinitely many primitive Pythagorean triangles whose sides are $(x^3 + y^3, p^3 + q^3, r^3 + s^3)$, where x, y, p, q, r, s are given above. □

Example 3. When $a = 2$, we get the primitive Pythagorean triangle with sides $(2985983, 3456, 2985985)$ satisfying

$$2985983 = 138^3 + 71^3, \quad 3456 = 12^3 + 12^3, \quad 2985985 = 144^3 + 1^3.$$

This provides a concrete example of Theorem 2.

Proof of Theorem 3. We prove this in two cases.

`Case 1`. For a rational right triangle with sides $\left(1/x + 1/y, 1/p + 1/q, 1/r + 1/s\right)$, where x, y, p, q, r, s are positive integers, by the Pythagorean theorem, we have

$$m^2 - n^2 = \frac{1}{x} + \frac{1}{y}, \tag{15}$$

$$2mn = \frac{1}{p} + \frac{1}{q}, \tag{16}$$

$$m^2 + n^2 = \frac{1}{r} + \frac{1}{s}, \tag{17}$$

where $m > n$ are positive rational numbers.

We start by constructing a family of positive rational solutions of this system. First we consider Equation (17), and set

$$m = \frac{1}{m_1}, \quad n = \frac{1}{n_1}, \quad r = m_1^2, \quad s = n_1^2,$$

where m_1 and n_1 are positive integers.

From Equation (15), we have

$$\left(\frac{1}{m_1}\right)^2 - \left(\frac{1}{n_1}\right)^2 = \frac{1}{x} + \frac{1}{y}.$$

Letting $x = x_1^2,\ y = y_1^2,\ x_1, y_1 \in \mathbb{Z}^+$, we get

$$\left(\frac{1}{m_1}\right)^2 = \left(\frac{1}{n_1}\right)^2 + \left(\frac{1}{x_1}\right)^2 + \left(\frac{1}{y_1}\right)^2.$$

From the identity (8), we get

$$m_1 = \frac{n_1}{2t^2+1}, \quad x_1 = \frac{n_1}{2t^2}, \quad y_1 = \frac{n_1}{2t}.$$

Taking $n_1 = 2t^2(2t^2+1)$, we have

$$m_1 = 2t^2, \quad x_1 = 2t^2+1, \quad y_1 = t(2t^2+1), \quad t \in \mathbb{Z}^+.$$

Equation (16) now becomes

$$\frac{1}{2t^4(2t^2+1)} = \frac{1}{p} + \frac{1}{q},$$

that is,

$$(q - 4t^6 - 2t^4)p = 2t^4(2t^2+1)q.$$

Letting $q - 4t^6 - 2t^4 = 1$, we have

$$\begin{aligned} p &= 2t^4(2t^2+1)(4t^6+2t^4+1), \\ q &= 4t^6+2t^4+1. \end{aligned}$$

Furthermore,

$$\begin{aligned} &m = \frac{1}{2t^2}, \quad n = \frac{1}{2t^2(2t^2+1)}, \\ &x = (2t^2+1)^2, \quad y = t^2(2t^2+1)^2, \\ &p = 2t^4(2t^2+1)(4t^6+2t^4+1), \quad q = 4t^6+2t^4+1, \\ &r = 4t^4, \quad s = 4t^4(2t^2+1)^2, \end{aligned} \tag{18}$$

where t is a positive integer.

Therefore, the sides of the rational right triangles are given by

$$(X, Y, Z) = \left(\frac{t^2+1}{t^2(2t^2+1)^2}, \frac{1}{2t^4(2t^2+1)}, \frac{2t^4+2t^2+1}{2t^4(2t^2+1)^2}\right).$$

Hence, there are infinitely many rational right triangles whose sides are $\big(1/x + 1/y, 1/p + 1/q, 1/r + 1/s\big)$, where x, y, p, q, r, s are given by (18).

`Case 2`. For a rational right triangle with sides $(1/x - 1/y, 1/p - 1/q, 1/r - 1/s)$, where $x < y,\ p < q$, and $r < s$ are positive integers, by the Pythagorean theorem, we have

$$m^2 - n^2 = \frac{1}{x} - \frac{1}{y}, \tag{19}$$

$$2mn = \frac{1}{p} - \frac{1}{q}, \tag{20}$$

$$m^2 + n^2 = \frac{1}{r} - \frac{1}{s}. \tag{21}$$

First, we consider Equation (19), and set

$$m = \frac{1}{m_1},\ n = \frac{1}{n_1}, \quad x = m_1^2, \quad y = n_1^2,$$

where m_1 and n_1 are positive integers. From Equation (21), we have

$$\left(\frac{1}{m_1}\right)^2 + \left(\frac{1}{n_1}\right)^2 = \frac{1}{r} - \frac{1}{s}.$$

Letting $r = r_1^2,\ s = s_1^2,\ r_1, s_1 \in \mathbb{Z}^+$, we obtain

$$\left(\frac{1}{r_1}\right)^2 = \left(\frac{1}{s_1}\right)^2 + \left(\frac{1}{m_1}\right)^2 + \left(\frac{1}{n_1}\right)^2.$$

From the identity (8), we get

$$r_1 = \frac{n_1}{2t^2+1}, \quad m_1 = \frac{n_1}{2t^2}, \quad s_1 = \frac{n_1}{2t}.$$

Taking $n_1 = 2t^2(2t^2+1)$, we have

$$m_1 = 2t^2+1, \quad r_1 = 2t^2, \quad s_1 = t(2t^2+1), \quad t \in \mathbb{Z}^+.$$

Equation (20) now becomes

$$\frac{1}{(2t^2+1)^2 t^2} = \frac{1}{p} - \frac{1}{q},$$

that is,

$$(4t^6 + 4t^4 + t^2 - p)q = (2t^2+1)^2 t^2 p.$$

Letting $4t^6+4t^4+t^2-p=1$, we get

$$\begin{aligned} p &= 4t^6+4t^4+t^2-1,\\ q &= (2t^2+1)^2t^2(4t^6+4t^4+t^2-1). \end{aligned}$$

Furthermore,

$$\begin{aligned} &m=\frac{1}{2t^2+1}, \quad n=\frac{1}{2t^2(2t^2+1)},\\ &x=(2t^2+1)^2, \quad y=4t^4(2t^2+1)^2,\\ &p=4t^6+4t^4+t^2-1, \quad q=(2t^2+1)^2t^2(4t^6+4t^4+t^2-1),\\ &r=4t^4, \quad s=(2t^2+1)^2t^2, \end{aligned} \tag{22}$$

where t is a positive integer.

Therefore, the sides of the rational right triangles are given by

$$(X,Y,Z)=\left(\frac{2t^2-1}{4(2t^2+1)t^4}, \frac{1}{(2t^2+1)^2t^2}, \frac{(2t^2-2t+1)(2t^2+2t+1)}{4t^4(2t^2+1)^2}\right).$$

Hence, there are infinitely many rational right triangles whose sides are $\big(1/x-1/y, 1/p-1/q, 1/r-1/s\big)$, where x,y,p,q,r,s are given by (22). □

Example 4. We provide the following two concrete examples of Theorem 3.

1) When $t=1$, we get the rational right triangle with sides $(2/9, 1/6, 5/18)$ satisfying

$$\frac{2}{9}=\frac{1}{9}+\frac{1}{9}, \quad \frac{1}{6}=\frac{1}{42}+\frac{1}{7}, \quad \frac{5}{18}=\frac{1}{4}+\frac{1}{36}.$$

2) When $t=1$, we get the rational right triangle with sides $(1/12, 1/9, 5/36)$ satisfying

$$\frac{1}{12}=\frac{1}{9}-\frac{1}{36}, \quad \frac{1}{9}=\frac{1}{8}-\frac{1}{72}, \quad \frac{5}{36}=\frac{1}{4}-\frac{1}{9}.$$

3. Some Related Questions

In Theorem 2, we get infinitely many primitive Pythagorean triangles whose sides are

$$(X,Y,Z)=\left(x^3+y^3, p^3+q^3, r^3+s^3\right),$$

with $p=q$. By some numerical calculations, we find all the solutions:

$$\begin{aligned} (x,y,p,q,r,s) = {} & (13,2,9,3,11,10),\ (26,4,18,6,22,20),\\ & (39,6,27,9,33,30),\ (48,8,13,11,46,24),\\ & (32,13,22,20,32,19),\ (38,22,28,8,41,7),\\ & (43,29,31,8,41,34) \end{aligned}$$

in the range $(x, y, p, q, r, s) \in [1, 50]^6$ with $x > y,\ p > q$, and $r > s$.

Question 1. For a Pythagorean triangle whose sides are in the form of

$$(X, Y, Z) = \left(x^3 + y^3, p^3 + q^3, r^3 + s^3\right),$$

where x, y, p, q, r, s are positive integers, are there infinitely many positive integer solutions such that $x \neq y$ and $p \neq q$?

Question 2. For a Pythagorean triangle whose sides are in the form of

$$(X, Y, Z) = \left(x^3 - y^3, p^3 - q^3, r^3 - s^3\right),$$

where $x > y,\ p > q$, and $r > s$ are positive integers, are there infinitely many positive integer solutions?

For Question 2, by some numerical calculations, we find all the solutions

$$\begin{aligned}(x, y, p, q, r, s) = {} & (10, 6, 20, 17, 17, 12),\ (15, 12, 45, 41, 41, 36),\\ & (20, 12, 40, 34, 34, 24),\ (31, 19, 18, 2, 29, 9),\\ & (31, 19, 24, 20, 29, 9)\end{aligned}$$

in the range $(x, y, p, q, r, s) \in [1, 50]^6$ with $x > y,\ p > q$, and $r > s$.

Question 3. For a Pythagorean triangle whose sides are in the form of

$$(X, Y, Z) = (x^n + y^n, p^n + q^n, r^n + s^n),$$

or

$$(X, Y, Z) = (x^n - y^n, p^n - q^n, r^n - s^n),$$

where x, y, p, q, r, s, n are positive integers and $n \geq 4$, are there infinitely many positive integer solutions?

For Question 3, we find all the solutions when $n = 4$ with sign "$-$":

$$(x, y, p, q, r, s) = (32, 4, 19, 9, 33, 19),\ (47, 1, 83, 9, 84, 38),\ (64, 8, 38, 18, 66, 38)$$

in the range $(x, y, p, q, r, s) \in [1, 100]^6$ with $x > y,\ p > q$, and $r > s$.

In 1964, Sierpiński [5] introduced a Pythagorean triangle (see (1), given by Zarankiewicz), whose three sides are triangular numbers. We have the following question.

Question 4. For a Pythagorean triangle whose sides are in the form of

$$(X, Y, Z) = (t_x, t_y, t_z + t_w),$$

where x, y, z, w are positive integers, are there infinitely many positive integer solutions?

For Question 4, we find all the solutions:

$$(x,y,z,w) = (5,8,2,8),\ (8,14,3,14),\ (8,14,9,11),\ (18,32,15,29),\ (29,64,9,64),\ (32,44,33,33),\ (49,63,28,62),\ (64,143,26,142),\ (77,143,75,125),\ (95,189,69,179),\ (104,121,1,135),\ (104,121,16,134),\ (104,121,36,130),\ (104,121,53,124),\ (104,121,67,117),\ (104,121,82,107)$$

in the range $(x,y,z,w) \in [1,200]^4$ with $x \le y$ and $z \le w$.

Acknowledgements. The authors thank the referee for his/her comments and suggestions and helping improve the quality of this paper. Especially, the results of Remark 2 are given by the referee.

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